Quadratically constrained quadratic program (QCQP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, feasible region is intersection of m ellipsoids and an affine set

Second-order cone programming

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

($A_i \in \mathbf{R}^{n_i \times n}$, $F \in \mathbf{R}^{p \times n}$)

- inequalities are called second-order cone (SOC) constraints:

$$(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbf{R}^{n_i+1}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP
- more general than QCQP and LP

Robust linear programming

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

there can be uncertainty in c, a_i, b_i

two common approaches to handling uncertainty (in a_i, for simplicity)

- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m, \end{array}$$

- stochastic model: a_i is random variable; constraints must hold with probability η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

deterministic approach via SOCP

- choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\} \qquad (\bar{a}_i \in \mathbf{R}^n, \quad P_i \in \mathbf{R}^{n \times n})$$

  center is $\bar{a}_i$, semi-axes determined by singular values/vectors of P_i

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

  is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  (follows from $\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

stochastic approach via SOCP

- assume a_i is Gaussian with mean $\bar{a}_i$, covariance Σ_i ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$)
- $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \Sigma_i x$; hence

$$\mathbf{prob}(a_i^T x \le b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2} \right)$$

  where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2}\, dt$ is CDF of $\mathcal{N}(0,1)$

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \le b_i) \ge \eta, \quad i = 1, \ldots, m, \end{array}$$

  with $\eta \ge 1/2$, is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \le b_i, \quad i = 1, \ldots, m \end{array}$$

Geometric programming

monomial function

$$f(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

with $c > 0$; exponent α_i can be any real number

posynomial function: sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

geometric program (GP)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \ldots, m \\ & h_i(x) = 1, \quad i = 1, \ldots, p \end{array}$$

with f_i posynomial, h_i monomial

Geometric program in convex form

change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \qquad (b = \log c)$$

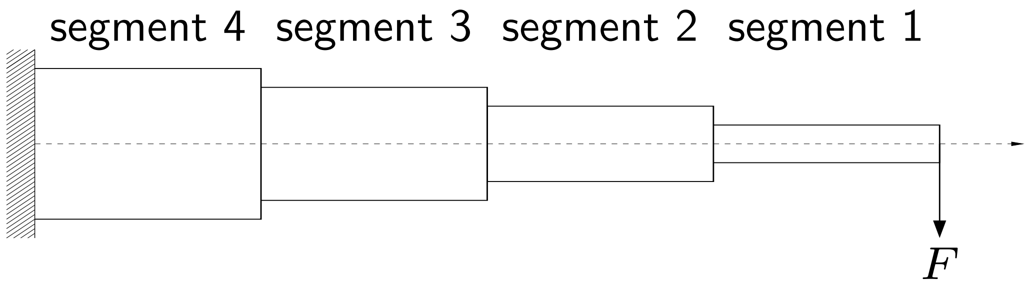
- posynomial $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = \log \left(\sum_{k=1}^K e^{a_k^T y + b_k} \right) \qquad (b_k = \log c_k)$$

- geometric program transforms to convex problem

$$\begin{array}{ll} \text{minimize} & \log \left(\sum_{k=1}^K \exp(a_{0k}^T y + b_{0k}) \right) \\ \text{subject to} & \log \left(\sum_{k=1}^K \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \\ & Gy + d = 0 \end{array}$$

Design of cantilever beam



- N segments with unit lengths, rectangular cross-sections of size $w_i \times h_i$
- given vertical force F applied at the right end

design problem

$$\begin{array}{ll} \text{minimize} & \text{total weight} \\ \text{subject to} & \text{upper \& lower bounds on } w_i,\ h_i \\ & \text{upper bound \& lower bounds on aspect ratios } h_i/w_i \\ & \text{upper bound on stress in each segment} \\ & \text{upper bound on vertical deflection at the end of the beam} \end{array}$$

variables: w_i, h_i for $i = 1, \ldots, N$

objective and constraint functions

- total weight $w_1 h_1 + \cdots + w_N h_N$ is posynomial
- aspect ratio h_i/w_i and inverse aspect ratio w_i/h_i are monomials
- maximum stress in segment i is given by $6iF/(w_i h_i^2)$, a monomial
- the vertical deflection y_i and slope v_i of central axis at the right end of segment i are defined recursively as

$$
\begin{aligned}
v_i &= 12(i-1/2)\frac{F}{E w_i h_i^3} + v_{i+1} \\
y_i &= 6(i-1/3)\frac{F}{E w_i h_i^3} + v_{i+1} + y_{i+1}
\end{aligned}
$$

for $i = N, N-1, \ldots, 1$, with $v_{N+1} = y_{N+1} = 0$ (E is Young's modulus)

v_i and y_i are posynomial functions of w, h

formulation as a GP

$$
\begin{array}{ll}
\text{minimize} & w_1 h_1 + \cdots + w_N h_N \\
\text{subject to} & w_{\max}^{-1} w_i \leq 1, \quad w_{\min} w_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& h_{\max}^{-1} h_i \leq 1, \quad h_{\min} h_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& S_{\max}^{-1} w_i^{-1} h_i \leq 1, \quad S_{\min} w_i h_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& 6iF\sigma_{\max}^{-1} w_i^{-1} h_i^{-2} \leq 1, \quad i = 1, \ldots, N \\
& y_{\max}^{-1} y_1 \leq 1
\end{array}
$$

note

- we write $w_{\min} \leq w_i \leq w_{\max}$ and $h_{\min} \leq h_i \leq h_{\max}$

$$
w_{\min}/w_i \leq 1, \qquad w_i/w_{\max} \leq 1, \qquad h_{\min}/h_i \leq 1, \qquad h_i/h_{\max} \leq 1
$$

- we write $S_{\min} \leq h_i/w_i \leq S_{\max}$ as

$$
S_{\min} w_i/h_i \leq 1, \qquad h_i/(w_i S_{\max}) \leq 1
$$

Minimizing spectral radius of nonnegative matrix

Perron-Frobenius eigenvalue $\lambda_{\rm pf}(A)$

- exists for (elementwise) positive $A \in \mathbf{R}^{n \times n}$
- a real, positive eigenvalue of A, equal to spectral radius $\max_i |\lambda_i(A)|$
- determines asymptotic growth (decay) rate of A^k: $A^k \sim \lambda_{\rm pf}^k$ as $k \to \infty$
- alternative characterization: $\lambda_{\rm pf}(A) = \inf\{\lambda \mid Av \preceq \lambda v \text{ for some } v \succ 0\}$

minimizing spectral radius of matrix of posynomials

- minimize $\lambda_{\rm pf}(A(x))$, where the elements $A(x)_{ij}$ are posynomials of x
- equivalent geometric program:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \sum_{j=1}^n A(x)_{ij} v_j / (\lambda v_i) \leq 1, \quad i = 1, \ldots, n \end{array}$$

variables λ, v, x

Generalized inequality constraints

convex problem with generalized inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- $f_0 : \mathbf{R}^n \to \mathbf{R}$ convex; $f_i : \mathbf{R}^n \to \mathbf{R}^{k_i}$ K_i-convex w.r.t. proper cone K_i
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

conic form problem: special case with affine objective and constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b \end{array}$$

extends linear programming ($K = \mathbf{R}^m_+$) to nonpolyhedral cones

Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

with F_i, $G \in \mathbf{S}^k$

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

$$x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \qquad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

  is equivalent to single LMI

$$x_1 \left[\begin{array}{cc} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{array} \right] + x_2 \left[\begin{array}{cc} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{array} \right] + \cdots + x_n \left[\begin{array}{cc} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{array} \right] + \left[\begin{array}{cc} \hat{G} & 0 \\ 0 & \tilde{G} \end{array} \right] \preceq 0$$

LP and SOCP as SDP

LP and equivalent SDP

LP: minimize $c^T x$
subject to $Ax \preceq b$

SDP: minimize $c^T x$
subject to $\mathbf{diag}(Ax - b) \preceq 0$

(note different interpretation of generalized inequality $\preceq$)

SOCP and equivalent SDP

SOCP: minimize $f^T x$
subject to $\|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m$

SDP: minimize $f^T x$
subject to $\begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m$

Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{S}^k$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

- variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$
- follows from

$$\lambda_{\max}(A) \leq t \quad \Longleftrightarrow \quad A \preceq tI$$

Matrix norm minimization

$$\text{minimize} \quad \|A(x)\|_2 = \left(\lambda_{\max}(A(x)^T A(x))\right)^{1/2}$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{R}^{p \times q}$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} tI & A(x) \\ A(x)^T & tI \end{array}\right] \succeq 0 \end{array}$$

- variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$
- constraint follows from

$$\begin{aligned} \|A\|_2 \leq t \quad &\Longleftrightarrow \quad A^T A \preceq t^2 I, \quad t \geq 0 \\ &\Longleftrightarrow \quad \left[\begin{array}{cc} tI & A \\ A^T & tI \end{array}\right] \succeq 0 \end{aligned}$$

Vector optimization

general vector optimization problem

$$
\begin{array}{ll}
\text{minimize (w.r.t. } K) & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) \leq 0, \quad i = 1, \ldots, p
\end{array}
$$

vector objective $f_0 : \mathbf{R}^n \to \mathbf{R}^q$, minimized w.r.t. proper cone $K \in \mathbf{R}^q$

convex vector optimization problem

$$
\begin{array}{ll}
\text{minimize (w.r.t. } K) & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

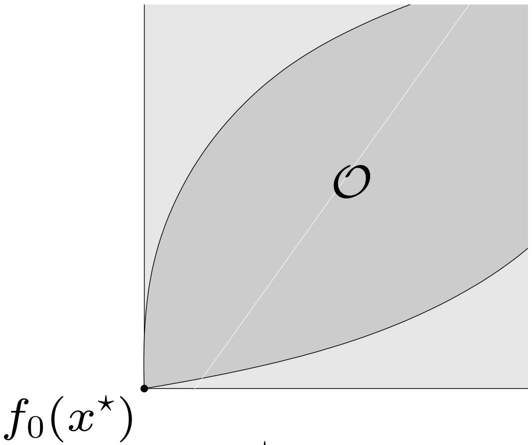
with f_0 K-convex, f_1, . . . , f_m convex

Optimal and Pareto optimal points

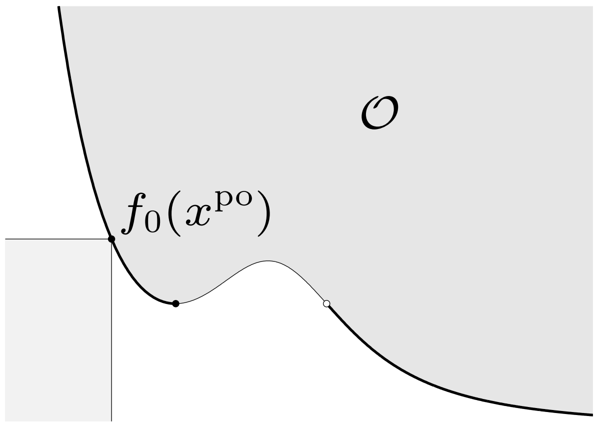
set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- feasible x is **optimal** if $f_0(x)$ is a minimum value of $\mathcal{O}$
- feasible x is **Pareto optimal** if $f_0(x)$ is a minimal value of $\mathcal{O}$



$x^\star$ is optimal



x^{po} is Pareto optimal

Multicriterion optimization

vector optimization problem with $K = \mathbf{R}^q_+$

$$f_0(x) = (F_1(x), \ldots, F_q(x))$$

- q different objectives F_i; roughly speaking we want all F_i's to be small
- feasible $x^\star$ is optimal if

$$y \text{ feasible} \quad \Longrightarrow \quad f_0(x^\star) \preceq f_0(y)$$

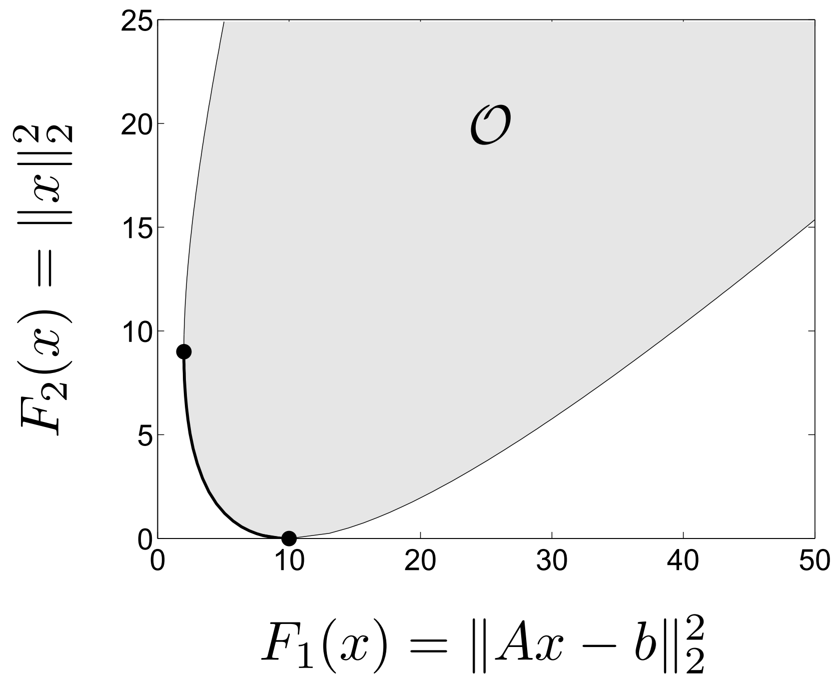
  if there exists an optimal point, the objectives are noncompeting
- feasible x^{po} is Pareto optimal if

$$y \text{ feasible}, \quad f_0(y) \preceq f_0(x^{\text{po}}) \quad \Longrightarrow \quad f_0(x^{\text{po}}) = f_0(y)$$

  if there are multiple Pareto optimal values, there is a trade-off between the objectives

Regularized least-squares

$$\text{minimize (w.r.t. } \mathbf{R}_+^2) \quad (\|Ax - b\|_2^2, \|x\|_2^2)$$



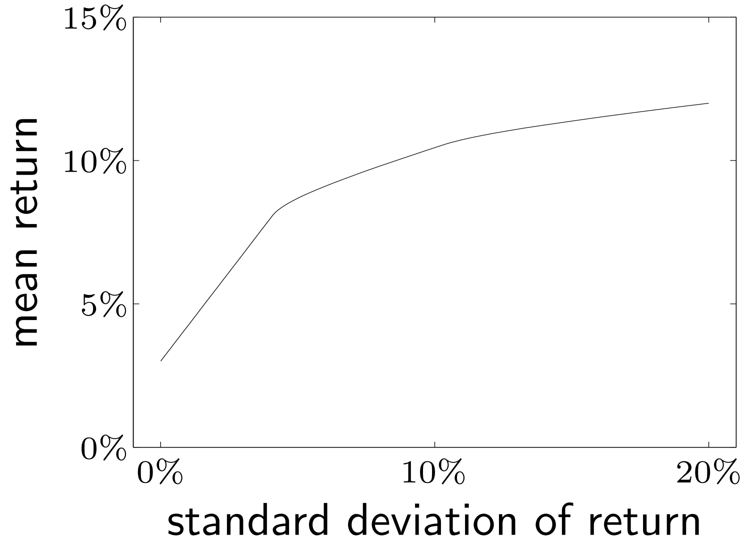
example for $A \in \mathbf{R}^{100 \times 10}$; heavy line is formed by Pareto optimal points

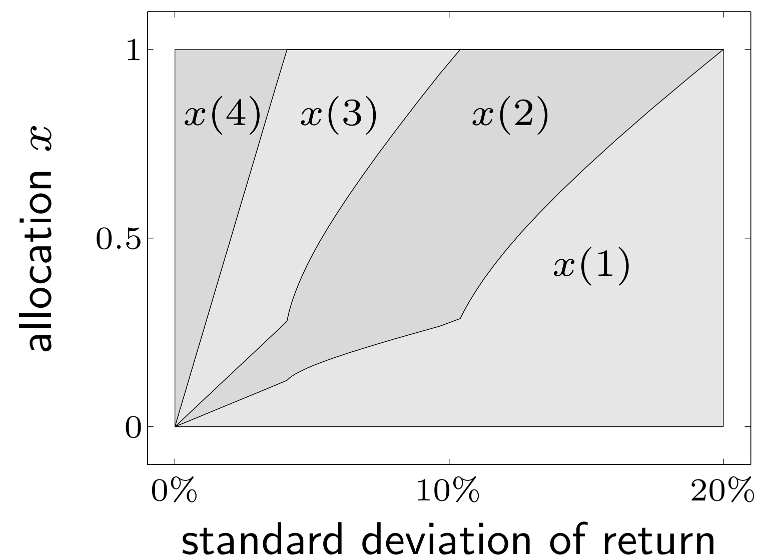
Risk return trade-off in portfolio optimization

$$\begin{array}{ll} \text{minimize (w.r.t. } \mathbf{R}^2_+) & (-\bar{p}^T x, x^T \Sigma x) \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

- $x \in \mathbf{R}^n$ is investment portfolio; x_i is fraction invested in asset i
- $p \in \mathbf{R}^n$ is vector of relative asset price changes; modeled as a random variable with mean $\bar{p}$, covariance Σ
- $\bar{p}^T x = \mathbf{E}\, r$ is expected return; $x^T \Sigma x = \mathbf{var}\, r$ is return variance

example



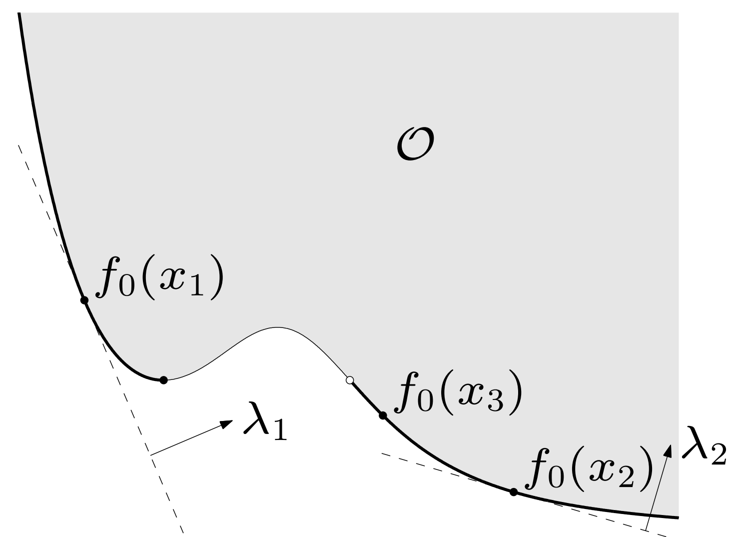


Scalarization

to find Pareto optimal points: choose $\lambda \succ_{K^*} 0$ and solve scalar problem

$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

if x is optimal for scalar problem, then it is Pareto-optimal for vector optimization problem



for convex vector optimization problems, can find (almost) all Pareto optimal points by varying $\lambda \succ_{K^*} 0$

Scalarization for multicriterion problems

to find Pareto optimal points, minimize positive weighted sum

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x)$$

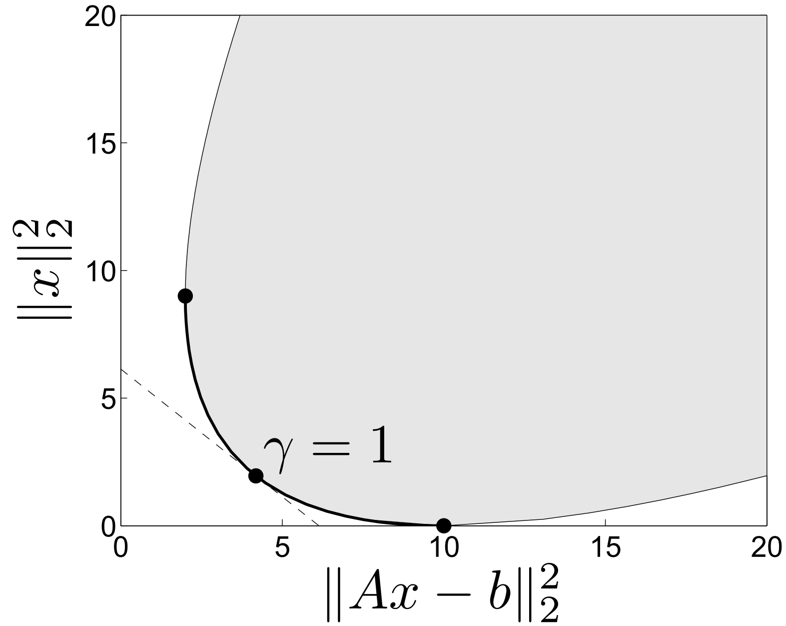
examples

- regularized least-squares problem of page 4–43

take $\lambda = (1, \gamma)$ with $\gamma > 0$

$$\text{minimize} \quad \|Ax - b\|_2^2 + \gamma \|x\|_2^2$$

for fixed γ, a LS problem



- risk-return trade-off of page 4–44

$$\begin{array}{ll} \text{minimize} & -\bar{p}^T x + \gamma x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

for fixed $\gamma > 0$, a quadratic program