# 11. Equality constrained minimization

- equality constrained minimization
- eliminating equality constraints
- Newton's method with equality constraints
- infeasible start Newton method
- implementation

# **Equality constrained minimization**

minimize 
$$f(x)$$
 subject to  $Ax = b$ 

- f convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$  with  $\operatorname{rank} A = p$
- ullet we assume  $p^{\star}$  is finite and attained

**optimality conditions:**  $x^{\star}$  is optimal iff there exists a  $\nu^{\star}$  such that

$$\nabla f(x^*) + A^T \nu^* = 0, \qquad Ax^* = b$$

### equality constrained quadratic minimization (with $P \in \mathbf{S}^n_+$ )

minimize 
$$(1/2)x^TPx + q^Tx + r$$
 subject to  $Ax = b$ 

optimality condition:

$$\left[\begin{array}{cc} P & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} x^{\star} \\ \nu^{\star} \end{array}\right] = \left[\begin{array}{c} -q \\ b \end{array}\right]$$

- coefficient matrix is called KKT matrix
- KKT matrix is nonsingular if and only if

$$Ax = 0, \quad x \neq 0 \qquad \Longrightarrow \qquad x^T P x > 0$$

• equivalent condition for nonsingularity:  $P + A^T A \succ 0$ 

### **Eliminating equality constraints**

represent solution of  $\{x \mid Ax = b\}$  as

$$\{x \mid Ax = b\} = \{Fz + \hat{x} \mid z \in \mathbf{R}^{n-p}\}\$$

- $\hat{x}$  is (any) particular solution
- range of  $F \in \mathbf{R}^{n \times (n-p)}$  is nullspace of A (rank F = n p and AF = 0)

#### reduced or eliminated problem

minimize 
$$f(Fz + \hat{x})$$

- an unconstrained problem with variable  $z \in \mathbf{R}^{n-p}$
- from solution  $z^*$ , obtain  $x^*$  and  $\nu^*$  as

$$x^* = Fz^* + \hat{x}, \qquad \nu^* = -(AA^T)^{-1}A\nabla f(x^*)$$

example: optimal allocation with resource constraint

minimize 
$$f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)$$
  
subject to  $x_1 + x_2 + \cdots + x_n = b$ 

eliminate  $x_n = b - x_1 - \cdots - x_{n-1}$ , *i.e.*, choose

$$\hat{x} = be_n, \qquad F = \begin{bmatrix} I \\ -\mathbf{1}^T \end{bmatrix} \in \mathbf{R}^{n \times (n-1)}$$

reduced problem:

minimize 
$$f_1(x_1) + \cdots + f_{n-1}(x_{n-1}) + f_n(b - x_1 - \cdots - x_{n-1})$$

(variables  $x_1, \ldots, x_{n-1}$ )

### **Newton step**

Newton step  $\Delta x_{\rm nt}$  of f at feasible x is given by solution v of

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

#### interpretations

•  $\Delta x_{\rm nt}$  solves second order approximation (with variable v)

minimize 
$$\widehat{f}(x+v) = f(x) + \nabla f(x)^T v + (1/2) v^T \nabla^2 f(x) v$$
 subject to 
$$A(x+v) = b$$

•  $\Delta x_{\rm nt}$  equations follow from linearizing optimality conditions

$$\nabla f(x+v) + A^T w \approx \nabla f(x) + \nabla^2 f(x)v + A^T w = 0, \qquad A(x+v) = b$$

#### **Newton decrement**

$$\lambda(x) = \left(\Delta x_{\rm nt}^T \nabla^2 f(x) \Delta x_{\rm nt}\right)^{1/2} = \left(-\nabla f(x)^T \Delta x_{\rm nt}\right)^{1/2}$$

#### properties

ullet gives an estimate of  $f(x)-p^\star$  using quadratic approximation  $\widehat{f}$ :

$$f(x) - \inf_{Ay=b} \widehat{f}(y) = \frac{1}{2}\lambda(x)^2$$

• directional derivative in Newton direction:

$$\left. \frac{d}{dt} f(x + t\Delta x_{\rm nt}) \right|_{t=0} = -\lambda(x)^2$$

• in general,  $\lambda(x) \neq \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)\right)^{1/2}$ 

### Newton's method with equality constraints

given starting point  $x \in \operatorname{dom} f$  with Ax = b, tolerance  $\epsilon > 0$ . repeat

- 1. Compute the Newton step and decrement  $\Delta x_{\rm nt}$ ,  $\lambda(x)$ .
- 2. Stopping criterion. quit if  $\lambda^2/2 \leq \epsilon$ .
- 3. Line search. Choose step size t by backtracking line search.
- 4. Update.  $x := x + t\Delta x_{\rm nt}$ .

- ullet a feasible descent method:  $x^{(k)}$  feasible and  $f(x^{(k+1)}) < f(x^{(k)})$
- affine invariant

#### Newton's method and elimination

#### Newton's method for reduced problem

minimize 
$$\tilde{f}(z) = f(Fz + \hat{x})$$

- variables  $z \in \mathbf{R}^{n-p}$
- $\hat{x}$  satisfies  $A\hat{x} = b$ ;  $\mathbf{rank} F = n p$  and AF = 0
- Newton's method for  $\tilde{f}$ , started at  $z^{(0)}$ , generates iterates  $z^{(k)}$

#### Newton's method with equality constraints

when started at  $x^{(0)} = Fz^{(0)} + \hat{x}$ , iterates are

$$x^{(k+1)} = Fz^{(k)} + \hat{x}$$

hence, don't need separate convergence analysis

### Newton step at infeasible points

2nd interpretation of page 11–6 extends to infeasible x (i.e.,  $Ax \neq b$ ) linearizing optimality conditions at infeasible x (with  $x \in \text{dom } f$ ) gives

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax - b \end{bmatrix}$$
 (1)

#### primal-dual interpretation

• write optimality condition as r(y) = 0, where

$$y = (x, \nu),$$
  $r(y) = (\nabla f(x) + A^T \nu, Ax - b)$ 

• linearizing r(y) = 0 gives  $r(y + \Delta y) \approx r(y) + Dr(y)\Delta y = 0$ :

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ \Delta \nu_{\rm nt} \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + A^T \nu \\ Ax - b \end{bmatrix}$$

same as (1) with  $w=
u+\Delta 
u_{
m nt}$ 

#### Infeasible start Newton method

given starting point  $x \in \operatorname{dom} f$ ,  $\nu$ , tolerance  $\epsilon > 0$ ,  $\alpha \in (0, 1/2)$ ,  $\beta \in (0, 1)$ . repeat

- 1. Compute primal and dual Newton steps  $\Delta x_{
  m nt}$ ,  $\Delta 
  u_{
  m nt}$ .
- 2. Backtracking line search on  $||r||_2$ .

$$t := 1$$
.

while 
$$||r(x + t\Delta x_{\rm nt}, \nu + t\Delta \nu_{\rm nt})||_2 > (1 - \alpha t)||r(x, \nu)||_2$$
,  $t := \beta t$ .

3. Update.  $x:=x+t\Delta x_{\rm nt},\ \nu:=\nu+t\Delta \nu_{\rm nt}.$ 

until 
$$Ax = b$$
 and  $||r(x, \nu)||_2 \le \epsilon$ .

- not a descent method:  $f(x^{(k+1)}) > f(x^{(k)})$  is possible
- ullet directional derivative of  $\|r(y)\|_2$  in direction  $\Delta y = (\Delta x_{\rm nt}, \Delta \nu_{\rm nt})$  is

$$\left. \frac{d}{dt} \|r(y + t\Delta y)\|_{2} \right|_{t=0} = -\|r(y)\|_{2}$$

# Solving KKT systems

$$\left[\begin{array}{cc} H & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} v \\ w \end{array}\right] = - \left[\begin{array}{c} g \\ h \end{array}\right]$$

#### solution methods

- LDL<sup>T</sup> factorization
- elimination (if *H* nonsingular)

$$AH^{-1}A^Tw = h - AH^{-1}g, \qquad Hv = -(g + A^Tw)$$

ullet elimination with singular H: write as

$$\begin{bmatrix} H + A^T Q A & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} g + A^T Q h \\ h \end{bmatrix}$$

with  $Q \succeq 0$  for which  $H + A^T Q A \succ 0$ , and apply elimination

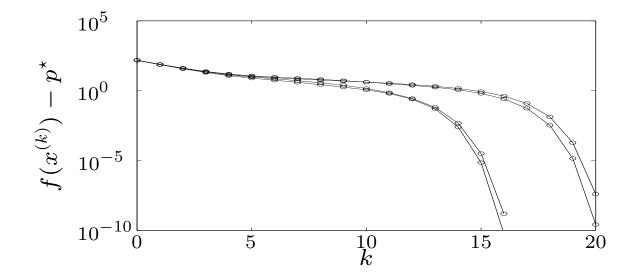
# **Equality constrained analytic centering**

**primal problem:** minimize  $-\sum_{i=1}^{n} \log x_i$  subject to Ax = b

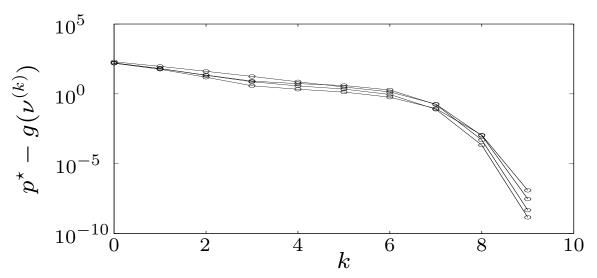
dual problem: maximize  $-b^T \nu + \sum_{i=1}^n \log(A^T \nu)_i + n$ 

three methods for an example with  $A \in \mathbf{R}^{100 \times 500}$ , different starting points

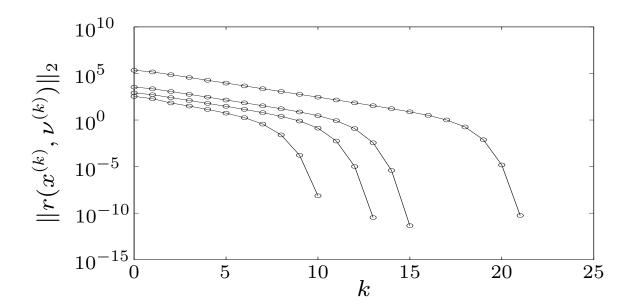
1. Newton method with equality constraints (requires  $x^{(0)} \succ 0$ ,  $Ax^{(0)} = b$ )



# 2. Newton method applied to dual problem (requires $A^T \nu^{(0)} \succ 0$ )



# 3. infeasible start Newton method (requires $x^{(0)} \succ 0$ )



#### complexity per iteration of three methods is identical

1. use block elimination to solve KKT system

$$\begin{bmatrix} \operatorname{\mathbf{diag}}(x)^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ w \end{bmatrix} = \begin{bmatrix} \operatorname{\mathbf{diag}}(x)^{-1} \mathbf{1} \\ 0 \end{bmatrix}$$

reduces to solving  $A \operatorname{diag}(x)^2 A^T w = b$ 

- 2. solve Newton system  $A\operatorname{diag}(A^T\nu)^{-2}A^T\Delta\nu = -b + A\operatorname{diag}(A^T\nu)^{-1}\mathbf{1}$
- 3. use block elimination to solve KKT system

$$\begin{bmatrix} \operatorname{\mathbf{diag}}(x)^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \nu \end{bmatrix} = \begin{bmatrix} \operatorname{\mathbf{diag}}(x)^{-1} \mathbf{1} \\ Ax - b \end{bmatrix}$$

reduces to solving  $A \operatorname{diag}(x)^2 A^T w = 2Ax - b$ 

conclusion: in each case, solve  $ADA^Tw=h$  with D positive diagonal

### **Network flow optimization**

minimize 
$$\sum_{i=1}^{n} \phi_i(x_i)$$
 subject to 
$$Ax = b$$

- directed graph with n arcs, p+1 nodes
- $x_i$ : flow through arc i;  $\phi_i$ : cost flow function for arc i (with  $\phi_i''(x) > 0$ )
- node-incidence matrix  $\tilde{A} \in \mathbf{R}^{(p+1)\times n}$  defined as

$$\tilde{A}_{ij} = \begin{cases} 1 & \text{arc } j \text{ leaves node } i \\ -1 & \text{arc } j \text{ enters node } i \\ 0 & \text{otherwise} \end{cases}$$

- reduced node-incidence matrix  $A \in \mathbf{R}^{p \times n}$  is  $\tilde{A}$  with last row removed
- $b \in \mathbf{R}^p$  is (reduced) source vector
- $\operatorname{rank} A = p$  if graph is connected

#### KKT system

$$\left[\begin{array}{cc} H & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} v \\ w \end{array}\right] = - \left[\begin{array}{c} g \\ h \end{array}\right]$$

- $H = \operatorname{diag}(\phi_1''(x_1), \dots, \phi_n''(x_n))$ , positive diagonal
- solve via elimination:

$$AH^{-1}A^Tw = h - AH^{-1}g, \qquad Hv = -(g + A^Tw)$$

sparsity pattern of coefficient matrix is given by graph connectivity

$$(AH^{-1}A^T)_{ij} \neq 0 \iff (AA^T)_{ij} \neq 0$$
 $\iff$  nodes  $i$  and  $j$  are connected by an arc

### Analytic center of linear matrix inequality

minimize 
$$-\log \det X$$
  
subject to  $\mathbf{tr}(A_iX) = b_i, \quad i = 1, \dots, p$ 

variable  $X \in \mathbf{S}^n$ 

#### optimality conditions

$$X^* \succ 0, \qquad -(X^*)^{-1} + \sum_{j=1}^p \nu_j^* A_i = 0, \qquad \mathbf{tr}(A_i X^*) = b_i, \quad i = 1, \dots, p$$

#### Newton equation at feasible X:

$$X^{-1}\Delta XX^{-1} + \sum_{j=1}^{p} w_j A_i = X^{-1}, \quad \mathbf{tr}(A_i \Delta X) = 0, \quad i = 1, \dots, p$$

- follows from linear approximation  $(X + \Delta X)^{-1} \approx X^{-1} X^{-1} \Delta X X^{-1}$
- n(n+1)/2 + p variables  $\Delta X$ , w

#### solution by block elimination

- eliminate  $\Delta X$  from first equation:  $\Delta X = X \sum_{j=1}^{p} w_j X A_j X$
- ullet substitute  $\Delta X$  in second equation

$$\sum_{j=1}^{p} \mathbf{tr}(A_i X A_j X) w_j = b_i, \quad i = 1, \dots, p$$
 (2)

a dense positive definite set of linear equations with variable  $w \in \mathbf{R}^p$ 

flop count (dominant terms) using Cholesky factorization  $X = LL^T$ :

- form p products  $L^T A_j L$ :  $(3/2)pn^3$
- form p(p+1)/2 inner products  $\mathbf{tr}((L^TA_iL)(L^TA_jL))$ :  $(1/2)p^2n^2$
- solve (2) via Cholesky factorization:  $(1/3)p^3$