Lagrangian vs LP Relaxations of Boolean LP

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Outline

1. Boolean LP and LP Relaxation
2. Special Case: No Linear Constraints
3. General Case

Boolean LP and LP Relaxation

Boolean LP

$$(BLP) \quad \begin{array}{rl} b^* := \min & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i = 1, \ldots, n \\ \text{equivalently} & x_i^2 - x_i = 0, \quad i = 1, \ldots, n \end{array}$$

LP Relaxation

$$(BLP_{LPR}) \quad \begin{array}{rl} b^* \geq p^* := \min & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

Special Case $A = 0, b = 0$

Dual of LP Relaxation

$$\begin{aligned} b^* \geq p^* &= \max_{u \geq 0, v \geq 0} \min_x c^T x - \sum_{i=1}^n u_i x_i + \sum_{i=1}^n v_i (x_i - 1) \\ &= \max_{u \geq 0, v \geq 0} \min_x \sum_{i=1}^n (c_i - u_i + v_i) x_i - \sum_{i=1}^n v_i \end{aligned}$$

The inner min occurs at $c_i - u_i + v_i = 0, \forall i$. Therefore:

$$u_i = (c_i)_+ := \max\{0, c_i\}; \quad v_i := (c_i)_- = -\min\{0, c_i\}$$

We get **BOTH** optimal values explicitly as:

$$b^* = p^* = \sum_{i=1}^n \min\{0, c_i\}.$$

Special case: Boolean LP Lagrangian Relaxation

Lagrangian dual with $A = 0, b = 0$

$$
\begin{aligned}
b^* \geq d^* \quad &:= \quad \max_\nu \min_x c^T x + \textstyle\sum_{i=1}^n \nu_i (x_i^2 - x_i) \\
&= \quad \max_{\nu \geq 0} \min_x \textstyle\sum_{i=1}^n (c_i - \nu_i) x_i + \nu_i x_i^2 \\
&\quad\; = \max_{\nu \geq 0} \quad \textstyle\sum_{i=1}^n (c_i - \nu_i) x_i + \nu_i x_i^2 \\
&\quad\;\; \text{subject to} \quad (c_i - \nu_i) + 2\nu_i x_i = 0, \quad \forall i
\end{aligned}
$$

Special case: Boolean LP Lagrangian Relaxation

Three Cases: $c_i = 0, c_i > 0, c_i < 0$

$c_i = 0 \implies \nu_i = 0$ at optimality

($\because$ i-th term of obj. fcn. becomes after the substitution $= -\nu_i/2$)

$c_i \neq 0 \implies \nu_i \neq 0$ (due to the lin. constr.)

Use $x_i = -(c_i - \nu_i)/(2\nu_i)$ and substitute to get:

$\max_{\nu \geq 0} -(c_i - \nu_i)^2/(4\nu_i)$

$c_i > 0 \implies \nu_i = c_i$ max value is 0

$c_i < 0 \implies \nu_i = -c_i$ max value is c_i

Again, as in LP relax., GET: $d^* = \sum_{i=1}^{n} \min\{0, c_i\}.$

General Case with $A \preceq b$

(Split) Dual of LP Relaxation

We isolate the linear constraint dual variable μ.

$$b^* \geq p^* = \max_{\mu \geq 0} \max_{u \geq 0, v \geq 0} \min_x (c + A^T\mu)^T x - u^T x + v^T x - v^T e - \mu^T b$$

We now keep μ fixed and work on the inner $\max\min$ problem. This problem is equivalent to the special case above with the new cost $c_\mu = (c + A^T\mu)$ and the fixed term $-\mu^T b$.

Dual LP Problem is found using c_μ:

$$b^* \geq p^* = \max_{\mu \geq 0} \left\{ \sum_{i=1}^n \min\{0, (c + A^T\mu)_i\} - b^T\mu \right\}$$

General Boolean LP Lagrangian Relaxation

Partial Lagrangian dual; isolate linear constraint

$$b^* \geq d^* \quad := \quad \max_{\mu \geq 0} \max_{\nu} \min_{x} \\ (c + A^T \mu)^T x + \sum_{i=1}^{n} \nu_i (x_i^2 - x_i) - \mu^T b$$

We keep μ fixed.

Same Conclusion; using $c_\mu = (c + A^T \mu)$

Again, as in LP relax., GET:

$$d^* = \max_{\mu \geq 0} \sum_{i=1}^{n} \min\{0, (c + A^T \mu)_i\} - \mu^T b.$$

Thanks for your attention!

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