Generalized Inequalities

- Proper Cone $\mathcal{K}$:
  - $\mathcal{K}$ closed and convex
  - $\mathcal{K}$ *solid* (nonempty interior) and *pointed* (contain no line)
- Examples:
  - Nonnegative orthant $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \,|\, x_i \geq 0,\ i = 1, \ldots, n\}$
  - Positive semidefinite cone $\mathbb{S}^n_+ = \{\mathbf{X} \in \mathbb{S}^n \,|\, \mathbf{X} \succeq 0\}$
- Generalized Inequality:
  - Partial ordering

$$\mathbf{x} \succeq_{\mathcal{K}} \mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} \in \mathcal{K}, \quad \mathbf{x} \succ_{\mathcal{K}} \mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} \in \mathrm{int}\mathcal{K}$$

- Examples:
  - $\mathbf{x} \succeq_{\mathbb{R}^n_+} \mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} \in \mathbb{R}^n_+$ or $\mathbf{x} \geq \mathbf{y}$ (componentwise inequality)
  - $\mathbf{X} \succeq_{\mathbb{S}^n_+} \mathbf{Y} \Leftrightarrow \mathbf{X} - \mathbf{Y} \in \mathbb{S}^n_+$ or $\mathbf{X} \succeq \mathbf{Y}$ (matrix inequality)

Minimum and Minimal Elements

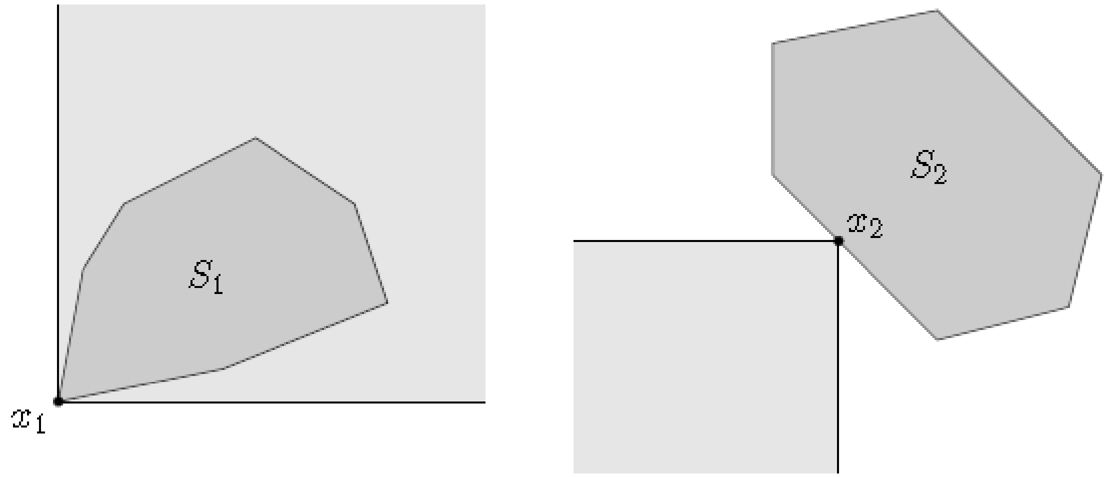
- Minimum Element $\mathbf{x} \in \mathcal{S}$:

$$(\mathbf{y} \in \mathcal{S} \;\Rightarrow\; \mathbf{y} \succeq_{\mathcal{K}} \mathbf{x}) \quad \Leftrightarrow \quad \mathcal{S} \subseteq \mathbf{x} + \mathcal{K}$$

- Minimal Element $\mathbf{x} \in \mathcal{S}$:

$$(\mathbf{y} \in \mathcal{S},\, \mathbf{x} \succeq_{\mathcal{K}} \mathbf{y} \;\Rightarrow\; \mathbf{y} = \mathbf{x}) \quad \Leftrightarrow \quad \mathcal{S} \cap (\mathbf{x} - \mathcal{K}) = \{\mathbf{x}\}$$

- Minimum element is a minimal element
- Example in $\mathbb{R}^2_+$



Dual Generalized Inequalities

- Dual Cone of $\mathcal{K}$:

$$\mathcal{K}^* = \{\mathbf{x} \,|\, \mathbf{x}^T\mathbf{y} \geq 0, \; \forall\, \mathbf{y} \in \mathcal{K}\}$$

- $\mathcal{K}^*$ is a *convex cone*
- Examples
  - $\mathcal{K} = \mathbb{R}^n_+$: $\mathcal{K}^* = \mathcal{K}$ (self-dual cone)
  - $\mathcal{K} = \mathbb{S}^n_+$: $\mathcal{K}^* = \mathcal{K}$
  - $\mathcal{K} = \{(\mathbf{x}, t) \,|\, ||\mathbf{x}||_2 \leq t\}$: $\mathcal{K}^* = \mathcal{K}$
  - $\mathcal{K} = \{(\mathbf{x}, t) \,|\, ||\mathbf{x}||_1 \leq t\}$: $\mathcal{K}^* = \{(\mathbf{x}, t) \,|\, ||\mathbf{x}||_\infty \leq t\}$
- $\mathcal{K}$ is proper $\rightarrow$ $\mathcal{K}^*$ is proper
- Dual Generalized Inequality $\succeq_{\mathcal{K}^*}$

$$\mathbf{x}_1 \succeq_{\mathcal{K}^*} \mathbf{x}_2 \;\Leftrightarrow\; \mathbf{y}^T\mathbf{x}_1 \geq \mathbf{y}^T\mathbf{x}_2, \quad \forall\, \mathbf{y} \succeq_{\mathcal{K}} 0$$

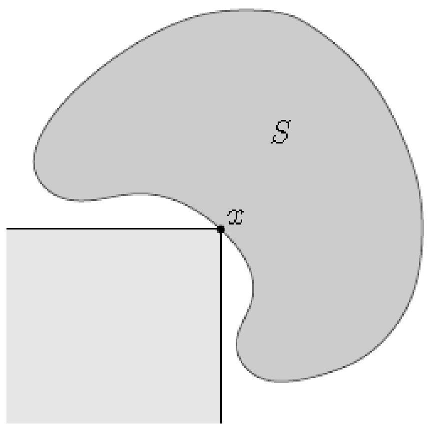
Dual Characterization of Minimum and Minimal Elements

- Minimum Element

  $\mathbf{x}$ is the minimum element of $\mathcal{S}$, with respect to $\succeq_{\mathcal{K}}$, if and only if $\arg\min_{\mathbf{z}\in\mathcal{S}} \mathbf{y}^T\mathbf{z} = \{\mathbf{x}\}, \quad \forall\, \mathbf{y} \succ_{\mathcal{K}^*} 0$

- Minimal Element

  If $\mathbf{x}$ minimizes $\mathbf{y}^T\mathbf{z}$ over $\mathbf{z} \in \mathcal{S}$ for some $\mathbf{y} \succ_{\mathcal{K}^*} 0$, then $\mathbf{x}$ is minimal with respect to $\succeq_{\mathcal{K}}$



If $\mathbf{x}$ is minimal, with respect to $\succeq_{\mathcal{K}}$, for a convex set $\mathcal{S}$, then there exists $\mathbf{y} \succeq_{\mathcal{K}^*} 0$ such that $\mathbf{x}$ minimizes $\mathbf{y}^T\mathbf{z}$ over $\mathbf{z} \in \mathcal{S}$