CO367/CM442 Nonlinear Optimization
Lecture 3

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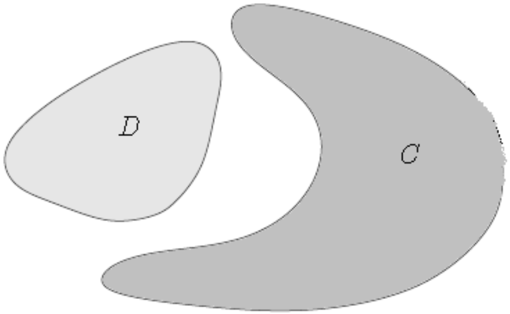
Lecture Outline

- Separating Hyperplane Theorem
- Supporting Hyperplane Theorem
- Generalized Inequalities
- Dual Generalized Inequalities

Separating Hyperplane Theorem

Question

Given two sets $\mathcal{C}$ and $\mathcal{D}$ that do not intersect, $\mathcal{C} \cap \mathcal{D} = \emptyset$, can we separate them using a hyperplane?

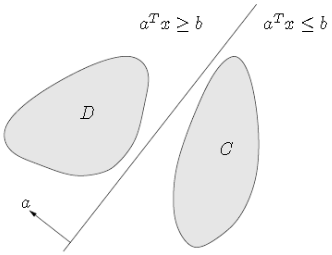


Separating Hyperplane Theorem

Theorem

Suppose $\mathcal{C}$ and $\mathcal{D}$ are two convex sets that do not intersect, $\mathcal{C} \cap \mathcal{D} = \emptyset$. Then there exist $\mathbf{a} \neq \mathbf{0}$ and b such that

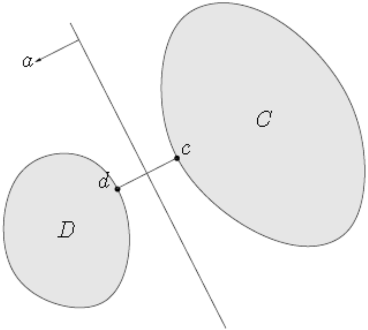
$$\mathbf{a}^T\mathbf{x} \leq b, \quad \forall \mathbf{x} \in \mathcal{C}, \quad \text{and} \quad \mathbf{a}^T\mathbf{x} \geq b, \quad \forall \mathbf{x} \in \mathcal{D}.$$



Separating Hyperplane Theorem

Proof.

- Assume $\text{dist}(\mathcal{C}, \mathcal{D}) := \inf\{||\mathbf{u} - \mathbf{v}||_2 \mid \mathbf{u} \in \mathcal{C}, \mathbf{v} \in \mathcal{D}\}$ is attained: $\exists \mathbf{c} \in \mathcal{C}, \mathbf{d} \in \mathcal{D} : ||\mathbf{c} - \mathbf{d}||_2 = \text{dist}(\mathcal{C}, \mathcal{D}) > 0$
- Possible conditions: $\mathcal{C}$ and $\mathcal{D}$ are *closed* and *bounded*



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Separating Hyperplane Theorem

- Hyperplane perpendicular to line segment between **c** and **d** and pass through the midpoint:

$$\mathbf{a} = \mathbf{d} - \mathbf{c}, \quad b = \frac{1}{2}\mathbf{a}^T(\mathbf{c} + \mathbf{d})$$

- Prove $f(\mathbf{u}) = \mathbf{a}^T\mathbf{u} - \mathbf{b} \geq 0$ for all $\mathbf{u} \in \mathcal{D}$ by *contradiction*
- Assume $f(\mathbf{u}) = (\mathbf{d} - \mathbf{c})^T(\mathbf{u} - \mathbf{d}) + ||\mathbf{c} - \mathbf{d}||_2^2/2 < 0$:

$$(\mathbf{d} - \mathbf{c})^T(\mathbf{u} - \mathbf{d}) < 0, \quad \mathbf{u} \neq \mathbf{d}$$

- Consider *line segment* between **d** and **u**: $\mathbf{x} = \mathbf{d} + \theta(\mathbf{u} - \mathbf{d}) \in \mathcal{D}$ for $0 \leq \theta \leq 1$
- $||\mathbf{x} - \mathbf{c}||_2^2 = ||\mathbf{d} - \mathbf{c}||_2^2 + 2\theta(\mathbf{d} - \mathbf{c})^T(\mathbf{u} - \mathbf{d}) + \theta^2\,||\mathbf{u} - \mathbf{d}||_2^2$
- There exists $\theta > 0$:

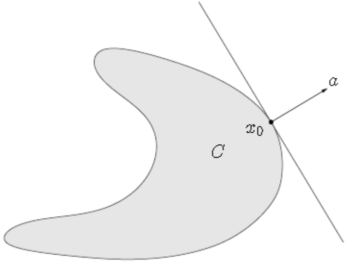
$$||\mathbf{x} - \mathbf{c}||_2^2 < ||\mathbf{d} - \mathbf{c}||_2^2, \text{ (contradiction)}$$

Separating Hyperplane Theoreom

- Strict Separation
  - Additional assumptions ($\mathcal{C}$ closed, $\mathcal{D}$ singleton)
- Application
  - No intersection: related to infeasibility $\rightarrow$ use separation hyperplane theorem to derive *infeasibility condition*
- A Theorem of Alternatives
  The system $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ *is infeasible if and only if there exists* $\mathbf{p}$ *such that* $\mathbf{p}^T\mathbf{A} \geq \mathbf{0}$ *and* $\mathbf{p}^T\mathbf{b} < 0$.
- Convex sets: $\mathcal{C} = \{\mathbf{A}\mathbf{x} \,|\, \mathbf{x} \geq \mathbf{0}\}$ and $\mathcal{D} = \{\mathbf{b}\}$

Supporting Hyperplane Theorem

- Supporting Hyperplane
  - $\{\mathbf{x} \,|\, \mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_0\}$ is a supporting hyperplane of $\mathcal{C}$ at its *boundary point* $\mathbf{x}_0$ if $\quad \mathbf{a} \neq \mathbf{0}, \quad \mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_0, \quad \forall\, \mathbf{x} \in \mathcal{C}$



Theorem

For any nonempty convex set $\mathcal{C}$, and any boundary point $\mathbf{x}_0$, there exists a supporting hyperplane to $\mathcal{C}$ at $\mathbf{x}_0$.

Generalized Inequalities

- Proper Cone $\mathcal{K}$:
  - $\mathcal{K}$ closed and convex
  - $\mathcal{K}$ *solid* (nonempty interior) and *pointed* (contain no line)
- Examples:
  - Nonnegative orthant $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \,|\, x_i \geq 0,\ i = 1, \ldots, n\}$
  - Positive semidefinite cone $\mathbb{S}^n_+ = \{\mathbf{X} \in \mathbb{S}^n \,|\, \mathbf{X} \succeq 0\}$
- Generalized Inequality:
  - Partial ordering

$$\mathbf{x} \succeq_{\mathcal{K}} \mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} \in \mathcal{K}, \quad \mathbf{x} \succ_{\mathcal{K}} \mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} \in \text{int}\mathcal{K}$$

- Examples:
  - $\mathbf{x} \succeq_{\mathbb{R}^n_+} \mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} \in \mathbb{R}^n_+$ or $\mathbf{x} \geq \mathbf{y}$ (componentwise inequality)
  - $\mathbf{X} \succeq_{\mathbb{S}^n_+} \mathbf{Y} \Leftrightarrow \mathbf{X} - \mathbf{Y} \in \mathbb{S}^n_+$ or $\mathbf{X} \succeq \mathbf{Y}$ (matrix inequality)

Minimum and Minimal Elements

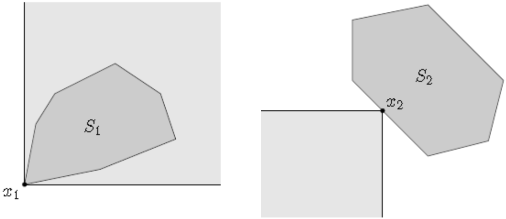
- Minimum Element $\mathbf{x} \in \mathcal{S}$:

$$(\mathbf{y} \in \mathcal{S} \Rightarrow \mathbf{y} \succeq_{\mathcal{K}} \mathbf{x}) \quad \Leftrightarrow \quad \mathcal{S} \subseteq \mathbf{x} + \mathcal{K}$$

- Minimal Element $\mathbf{x} \in \mathcal{S}$:

$$(\mathbf{y} \in \mathcal{S}, \mathbf{x} \succeq_{\mathcal{K}} \mathbf{y} \Rightarrow \mathbf{y} = \mathbf{x}) \quad \Leftrightarrow \quad \mathcal{S} \cap (\mathbf{x} - \mathcal{K}) = \{\mathbf{x}\}$$

- Minimum element is a minimal element
- Example in $\mathbb{R}^2_+$



Dual Generalized Inequalities

- Dual Cone of $\mathcal{K}$:

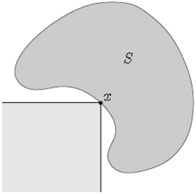
$$\mathcal{K}^* = \{\mathbf{x} \,|\, \mathbf{x}^T \mathbf{y} \geq 0, \,\forall\, \mathbf{y} \in \mathcal{K}\}$$

- $\mathcal{K}^*$ is a *convex cone*
- Examples
  - $\mathcal{K} = \mathbb{R}^n_+$: $\mathcal{K}^* = \mathcal{K}$ (self-dual cone)
  - $\mathcal{K} = \mathbb{S}^n_+$: $\mathcal{K}^* = \mathcal{K}$
  - $\mathcal{K} = \{(\mathbf{x}, t) \,|\, ||\mathbf{x}||_2 \leq t\}$: $\mathcal{K}^* = \mathcal{K}$
  - $\mathcal{K} = \{(\mathbf{x}, t) \,|\, ||\mathbf{x}||_1 \leq t\}$: $\mathcal{K}^* = \{(\mathbf{x}, t) \,|\, ||\mathbf{x}||_\infty \leq t\}$
- $\mathcal{K}$ is proper $\rightarrow$ $\mathcal{K}^*$ is proper
- Dual Generalized Inequality $\succeq_{\mathcal{K}^*}$

$$\mathbf{x}_1 \succeq_{\mathcal{K}^*} \mathbf{x}_2 \;\Leftrightarrow\; \mathbf{y}^T \mathbf{x}_1 \geq \mathbf{y}^T \mathbf{x}_2, \quad \forall\, \mathbf{y} \succeq_{\mathcal{K}} 0$$

Dual Characterization of Minimum and Minimal Elements

- Minimum Element
  $\mathbf{x}$ *is the minimum element of* $\mathcal{S}$, *with respect to* $\succeq_{\mathcal{K}}$, *if and only if* $\quad \arg\min_{\mathbf{z}\in\mathcal{S}} \mathbf{y}^T\mathbf{z} = \{\mathbf{x}\}, \quad \forall\, \mathbf{y} \succ_{\mathcal{K}^*} 0$
- Minimal Element
  If $\mathbf{x}$ *minimizes* $\mathbf{y}^T\mathbf{z}$ *over* $\mathbf{z} \in \mathcal{S}$ *for some* $\mathbf{y} \succ_{\mathcal{K}^*} 0$, *then* $\mathbf{x}$ *is minimal with respect to* $\succeq_{\mathcal{K}}$



If $\mathbf{x}$ *is minimal, with respect to* $\succeq_{\mathcal{K}}$, *for a convex set* $\mathcal{S}$, *then there exists* $\mathbf{y} \succeq_{\mathcal{K}^*} 0$ *such that* $\mathbf{x}$ *minimizes* $\mathbf{y}^T\mathbf{z}$ *over* $\mathbf{z} \in \mathcal{S}$