CO367/CM442 Nonlinear Optimization
Lecture 2

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Lecture Outline

- Affine Sets
- Convex Sets
- Examples
- Convex Operations

Affine Set

- Definition
  - Affine set contains the *line* through any two distinct points in the set
- Line through $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^n$:

$$\mathbf{x} = \theta \mathbf{x}_1 + (1 - \theta)\mathbf{x}_2, \quad \theta \in \mathbb{R}$$

- Example: $\mathcal{A} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$
- Affine combination of $\mathbf{x}_1, \ldots, \mathbf{x}_k$:

$$\mathbf{x} = \sum_{i=1}^{k} \theta_i \mathbf{x}_i, \quad \sum_{i=1}^{k} \theta_i = 1$$

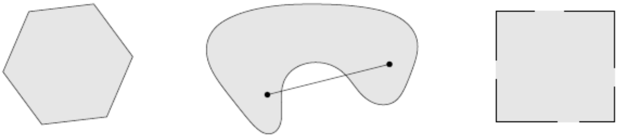
- Exercise
  - Affine set contains every *affine combination* of its points

Convex Set

- Definition
  - Convex set contains the *line segment* between any two points in the set

$$\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C} \;\Rightarrow\; \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in \mathcal{C}, \quad 0 \le \theta \le 1$$

- Example: affine sets are convex
- Simple convex and nonconvex sets

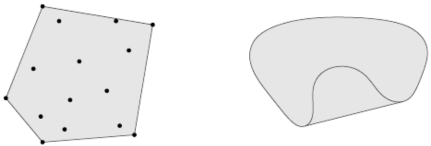


Convex Hull

- Convex combination of $\mathbf{x}_1, \ldots, \mathbf{x}_k$:

$$\mathbf{x} = \sum_{i=1}^{k} \theta_i \mathbf{x}_i, \quad \sum_{i=1}^{k} \theta_i = 1, \; \theta_i \geq 0, \; \forall\, i = 1, \ldots, k$$

- Convex hull of a set $\mathcal{S}$:
  - $\operatorname{conv} \mathcal{S}$ is the set of all *convex combinations* of points in $\mathcal{S}$



- Exercise:
  - If the convex set $\mathcal{C}$ contains $\mathcal{S}$, then $\operatorname{conv} \mathcal{S} \subseteq \mathcal{C}$

Convex Cone

- Cone is a *set* that satisfies:

$$\mathbf{x} \in \mathcal{C} \ \Rightarrow \ \theta \mathbf{x} \in \mathcal{C}, \quad \theta \geq 0$$

- Convex cone is a cone which is convex

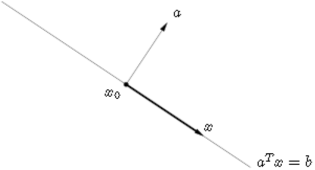
$$\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C} \ \Rightarrow \ \theta_1 \mathbf{x}_1 + \theta_2 \mathbf{x}_2 \in \mathcal{C}, \quad \theta_1, \theta_2 \geq 0$$

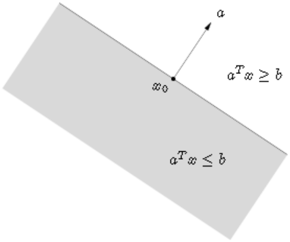
- Example: non-negative orthant in $\mathbb{R}^n$
- Conic combination of $\mathbf{x}_1, \ldots, \mathbf{x}_k$:

$$\mathbf{x} = \sum_{i=1}^{k} \theta_i \mathbf{x}_i, \quad \theta_i \geq 0, \forall\, i = 1, \ldots, k$$

- Exercise:
  - Convex cone contains every *conic combination* of its points

Example: Hyperplanes and Halfspaces





- Normal vector
  - $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{0}$
- Hyperplane
  - $\mathcal{H} = \{\mathbf{x} \,|\, \mathbf{a}^T \mathbf{x} = b\}$
  - Affine
  - Convex
- Halfspace
  - $\mathcal{H} = \{\mathbf{x} \,|\, \mathbf{a}^T \mathbf{x} \leq b\}$
  - Convex

Example: Euclidean Balls and Ellipsoids

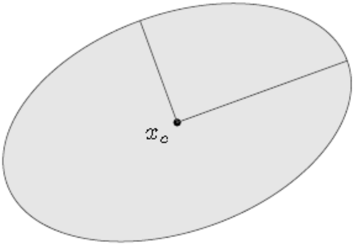
- Euclidean ball with center $\mathbf{x}_c$ and radius r

$$\mathcal{B}(\mathbf{x}_c, r) = \{\mathbf{x}_c + r\mathbf{u} \mid \|\mathbf{u}\|_2 \leq 1\}$$

- Ellipsoid

$$\mathcal{E} = \{\mathbf{x}_c + \mathbf{A}\mathbf{u} \mid \|\mathbf{u}\|_2 \leq 1\},$$

  where $\mathbf{A}$ is a full-rank square matrix
- Euclidean ball is an ellipsoid with $\mathbf{A} = r\mathbf{I}$

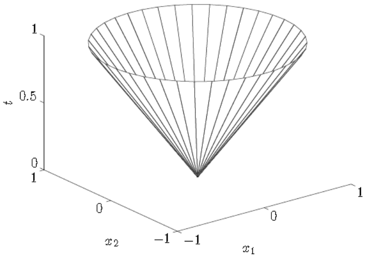


Example: Norm Cones

- Norm function $||.||$:
  - $||\mathbf{x}|| \geq 0$ for all $\mathbf{x}$; $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
  - $||\theta \mathbf{x}|| = |\theta| \, ||\mathbf{x}||$
  - $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$
- Norm cone in $\mathbb{R}^{n+1}$

$$\{(\mathbf{x}, t) \mid ||\mathbf{x}|| \leq t\}$$

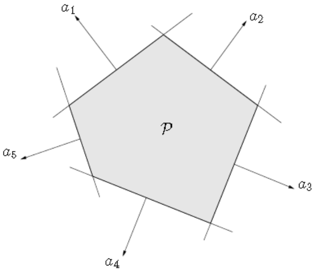
- Norm cone with $||.||_2$
  - *Second-order* cone
- Example
  - Second-order cone in $\mathbb{R}^3$



Example: Polyhedra

- Polyhedron: intersection of *finitely many* halfspaces

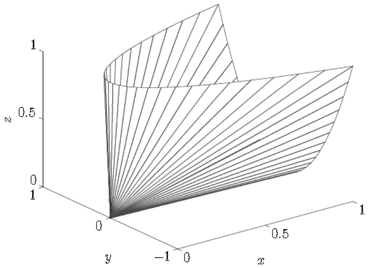
$$\mathcal{P} = \{\mathbf{x} \,|\, \mathbf{a}_i^T\mathbf{x} \le b_i,\ i = 1, \ldots, m\}$$



- $\mathbf{c}^T\mathbf{x} = d \iff \mathbf{c}^T\mathbf{x} \le d,\ -\mathbf{c}^T\mathbf{x} \le -d$
- Exercise
  - Is $\left\{(x, y) \in \mathbb{R}_+^2 \,|\, x\cos\theta + y\sin\theta \le 1,\ \forall\,\theta \in [0, \pi/2]\right\}$ a polyhedron?

Example: Positive Semidefinite Cones

- Notation
  - $\mathbb{S}^n$: set of symmetric $n \times n$ matrices, $\mathbf{X} = \mathbf{X}^T$
  - $\mathbb{S}^n_+ = \{\mathbf{X} \in \mathbb{S}^n \,|\, \mathbf{X} \succeq 0\}$: set of positive semidefinite $n \times n$ matrices,
$$\mathbf{X} \succeq 0 \Leftrightarrow \mathbf{z}^T\mathbf{X}\mathbf{z} \geq 0, \quad \forall\, \mathbf{z} \in \mathbb{R}^n$$
- $\mathbb{S}^n_+$ is a *convex cone*
- Example
$$\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \mathbb{S}^2_+$$



How to Prove Convexity

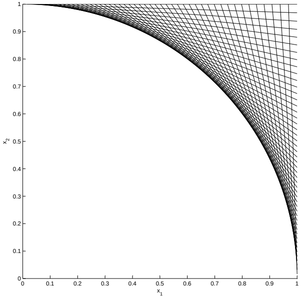
- Definition

$$\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{C} \;\Rightarrow\; \theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in \mathcal{C}, \quad \forall\, 0 \le \theta \le 1$$

- Convex Operations
  - Intersection
  - Affine transformation
  - Linear-fractional function

Operation: Intersection

- Intersection
  - $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are convex $\rightarrow \cap_{i=1}^{k} \mathcal{C}_i$ is convex
- Examples
  - $\mathcal{P} = \{\mathbf{x} \mid \mathbf{a}_i^T \mathbf{x} \leq b_i,\ i = 1, \ldots, m\}$ is convex
  - $\mathcal{S} = \left\{(x, y) \in \mathbb{R}_+^2 \mid x \cos\theta + y \sin\theta \leq 1,\ \forall\, \theta \in [0, \pi/2]\right\}$ is convex



Operation: Affine Transformation

- Affine transformation $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$$

- *Image* of a convex set $\mathcal{C} \subset \mathbb{R}^n$ under f is convex

$$\mathcal{S} = f(\mathcal{C}) = \{f(\mathbf{x}) \,|\, \mathbf{x} \in \mathcal{C}\}$$

- *Inverse image* of a convex set $\mathcal{S} \subset \mathbb{R}^m$ under f is convex

$$\mathcal{C} = f^{-1}(\mathcal{S}) = \{\mathbf{x} \,|\, f(\mathbf{x}) \in \mathcal{S}\}$$

- Examples
  - Projection $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \,|\, \exists\, \mathbf{y} \in \mathbb{R}^m : (\mathbf{x}, \mathbf{y}) \in \mathcal{C} \subset \mathbb{R}^{n \times m}\}$
  - Ellipsoids $\mathcal{E} = \mathbf{x}_c + \mathbf{A}\mathcal{B}(\mathbf{0}, 1)$
  - Linear matrix inequality $\left\{\mathbf{x} \,\middle|\, \mathbf{B} - \sum_{i=1}^{k} x_i \mathbf{A}_i \succeq 0\right\}$, $\mathbf{A}_i, \mathbf{B} \in \mathbb{S}^n$

Operation: Linear-Fractional Functions

- Perspective function $P : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$

$$P(\mathbf{x}, t) = \frac{1}{t}\mathbf{x}$$

- Exercise
  - Both *image* and *inverse image* of a convex set (within function domain) under P are convex
- General linear-fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(\mathbf{x}) = \frac{\mathbf{A}\mathbf{x} + \mathbf{b}}{\mathbf{c}^T\mathbf{x} + d}, \quad \mathrm{dom}(f) = \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{c}^T\mathbf{x} + d > 0\}$$

- Exercise
  - Both *image* and *inverse image* of a convex set (within function domain) under f are convex