CO 602/CM 740: Fundamentals of Optimization
Problem Set 8

H. Wolkowicz

Fall 2016.
Handed out: Thursday 2016-Nov-24.
Due: Monday Dec. 5, by 10AM (no penalty).

Contents

1 Steepest Descent

1. Apply the steepest descent algorithm (using MATLAB) with constant stepsize α to the function

$$f(x) = \begin{cases} x^2\left(\sqrt{2} - \sin\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the gradient ∇f satisfies the Lipschitz condition

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathbb{R},$$

for some constant L. Write a MATLAB program to verify that the method is a descent method for $\alpha \in (0, 2/L)$. Do you expect to converge to the global minimum $x^* = 0$?

Proof. The first derivative is

$$f'(x) = 2\sqrt{3}\,x\cos\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right) - 2x\left(\sin\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right) - \sqrt{2}\right)$$

This holds at $x \neq 0$. By taking limits we see the derivative is 0 at $x = 0$,

$$f'(0) = 0.$$

f is twice differentiable $\forall x \neq 0$. Its second derivative is

$$f''(x) = 10\sin\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right) + 6\sqrt{3}\cos\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right) + 2\sqrt{2}$$

which is clearly bounded, i.e., $|f''(x)| \leq L := 10 + 6\sqrt{3} + 2\sqrt{2}, \forall x \neq 0$. By using the mean value theorem applied to f', we conclude that

$$|f'(x) - f'(y)| \leq L|x - y|, \qquad \forall x, y \in \mathbb{R}_+.$$

and

$$|f'(x) - f'(y)| \leq L|x - y|, \qquad \forall x, y \in \mathbb{R}_-.$$

Now suppose that $x < 0 < y$. Let $x_n = \frac{1}{n}x, y_n = \frac{1}{n}y$ Then

$$\begin{array}{rcl} |f'(x) - f'(y)| & = & |f'(x) - f'(\frac{1}{n}x) + f'(\frac{1}{n}x) - f'(\frac{1}{n}y) + f'(\frac{1}{n}y) - f'(y)| \\ & \leq & |f'(x) - f'(\frac{1}{n}x)| + |f'(\frac{1}{n}x) - f'(\frac{1}{n}y)| + |f'(\frac{1}{n}y) - f'(y)| \end{array}$$

As $n \to \infty$, we see that the first term is bounded by

$$\lim_{n\to\infty} |f'(x) - f'(\frac{1}{n}x)| \leq L|x|$$

The second term converges to 0

$$\lim_{n\to 0} |f'(\frac{1}{n}x) - f'(\frac{1}{n}y)| \to \infty.$$

As $n \to \infty$, the last term is bounded by

$$\lim_{n\to\infty} |f'(y) - f'(\frac{1}{n}y)| \leq L|y|$$

Adding these three terms together and noting that $x < 0 < y$ implies that $|x - y| = |x| + |y|$, yields the Lipschitz condition with a constant $2L$.

To see the function, one can use:

```
x=linspace(-10,10.1,500);
f=x.^2.*  ( sqrt(2)- sin(  ((5*pi)/6)  - sqrt(3)*log(x.^2) )  );
plot(x,f)
```

□

2. Let $A \in \mathbb{S}^m$, $B \in \mathbb{S}^n$, the vector spaces of $m \times m$, $n \times n$ real symmetric matrices, respectively. Let $C \in \mathcal{M}^{mn}$, the vector space of $m \times n$ real matrices. Let $\gamma \in \mathbb{R}$. Consider the quadratic function

$$q(X) = \operatorname{trace} AXBX^T + \operatorname{trace} CX^T + \gamma, \qquad q : \mathcal{M}^{mn} \to \mathbb{R}.$$

Let $X0 \in \mathcal{M}^{mn}$ be an initial approximation for the minimum.[1]

(a) State the direction of steepest descent d at $X0$. Derive the steplength for exact minimization for d, i.e., explicitly solve

$$\bar{\alpha} \in \operatorname{argmin}_{\alpha>0} q(X0 + \alpha d).$$

(b) Code the steepest descent algorithm with this exact steplength to find the minimum correct to 10 decimals accuracy. How many iterations did it take?

(c) Derive and code Newton's method. How many iterations did the code take? Why?

[1] The values for the data $A, B, C, X0$ are in the file *ABCX0matass8.mat*. Let $\gamma = 1$.

2 Karush-Kuhn-Tucker Optimality Conditions

1. Suppose that $a, b, c \in \mathbb{R}_{++}$. Use Lagrange multipliers to solve

$$\min\{x + y + z : \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1,\ x, y, z \in \mathbb{R}_{++}\}.$$

2. Consider the problem

$$\begin{array}{ll} \min & \frac{1}{2}\alpha(x_1 - 1)^2 - x_1 - x_2 \\ \text{s.t.} & x_1 \geq x_2^2,\ x_1^2 + x_2^2 \leq 1. \end{array}$$

   Find all the points at which both constraints are active. Note that one such point is $\bar{x} = \begin{pmatrix} .618 & .786 \end{pmatrix}^T$ to three decimal places accuracy. Working to this accuracy, find the range of values of α for which $\bar{x}$ satisfies the KKT conditions.