CO 602/CM 740: Fundamentals of Optimization
Problem Set 6

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Fall 2016.
Handed out: Thursday 2016-Nov-10.
Due: Thursday 2016-Nov-17 before 10AM lecture starts.

Contents

1 Network Flow and Spanning Tree

Consider the uncapacitated network flow problem defined by costs c_{ij} on the directed arcs from node i to node j given in the following table:

c_{14}	c_{15}	c_{28}	c_{37}	c_{43}	c_{46}	c_{52}	c_{56}	c_{62}	c_{67}	c_{68}	c_{79}	c_{89}
1	2	1	1	3	4	4	1	1	2	2	3	2

There is a supply of 3 at node 1 and demands of $1, 2$ at nodes $2, 9$, respectively.

In addition, consider the spanning tree and associated basic solution defined by the following arcs:

15 37 43 56 62 68 79 89

1. What are the values of the arc flows corresponding to this basic solution? Is this a basic feasible solution?
2. For this basic solution, find the reduced cost of each arc in the network.
3. I this basic solution optimal?
4. Does there exist a nondegenerate optimal basic feasible solution?
5. Find an optimal dual solution.
6. By how much can we increase the arc cost c_{56} and still have the same optimal basic feasible solution?
7. If we increase the supply at node 1 and the demand at node 9 by a small positive amount δ, what is the change in the value of the optimal cost?
8. Does this problem have a special structure that makes it simpler than the general uncapacitated network flow problem?

2 Assignment Problem

Consider the student-University assignment problem, i.e., there are 20 students and 20 universities. Each student has 40 points to allocate for their preferences for the universities. They could allocate all 40 points to one university (as student #8 has done with university number 16. Or they could spread out their points to various universities as student 1 has done with 3 universities.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 4 & 4 \\
0 & 3 & 0 & 15 & 0 & 0 & 0 & 0 & 0 & 22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & 2 & 18 & 3 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 33 & 0 & 0 & 0 & 7 \\
0 & 0 & 14 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 29 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 15 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 5 & 0 & 0 & 0 & 13 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 31 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 23 & 17 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 34 \\
0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 29 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0
\end{pmatrix}
$$

1. The problem is to assign exactly one student to each university while maximizing the total value of the student preferences, i.e., solving the corresponding assignment problem. Model the problem as a network flow problem (with a bipartite graph) with 20 sources and sinks and $b_i = \pm 1$ for each.
2. What is the rank of the associated incidence matrix?
3. Solve the problem using MATLAB linprog with the interior point and simplex options.
4. Now suppose that you have a *balanced transportation problem*. Show how you would transform it into an assignment problem.

3 Minimum-Cost Capacitated Flow

Consider the directed network with $n = 5$ nodes and right-hand side (sources/sinks) $b = \begin{pmatrix} 20 & 0 & 0 & -5 & -15 \end{pmatrix}^T$, i.e., there is one source and two sinks and two transshipment nodes. There are 8 directed arcs with costs and upper bounds $(u, c) = (u_{ij}, c_{ij}) \in \mathbb{R}^{n,2}$ with arcs $ij \in \mathcal{A} = \{12, 13, 23, 24, 25, 34, 35, 45, 53\}$

$$
(u, c) = \begin{pmatrix} (15, \$4) \\ (8, \$4) \\ (\infty, \$2) \\ (4, \$2) \\ (10, \$6) \\ (15, \$1) \\ (5, \$3) \\ (\infty, \$2) \\ (4, \$1) \end{pmatrix}
$$

1. Draw the directed graph.
2. Solve the problem using the network simplex method. At each iteration show the proper/efficient calculations for the dual variable y and the reduced costs $\bar{c}$. Draw the graph for each iteration with the appropriate spanning tree, basic variable values on the arcs, and dual values at the nodes. (Use phase I to find the initial BFS.)
3. Is the optimal solution unique? Why or why not? If it is not unique, then find an alternate optimal spanning tree and BFS.
4. **BONUS:** What can you say about sensitivity analysis for this problem? I.e., how much can you perturb each component of the right-hand side b without changing the optimal basis? How much can you perturb each cost element c_{ij} without losing optimality? What about perturbations of the upper bounds u?