CO 602/CM 740: Fundamentals of Optimization
Problem Set 5

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Contents

1 Network and Transportation Problem

Let $G = \{\mathcal{N}, \mathcal{A}\}$ denote a digraph for a network flow problem, first uncapacitated and then capacitated.

1. Consider an uncapacitated network flow problem and assume that the costs on the arcs $c_{ij} \geq 0, \forall ij \in \mathcal{A}$. Let S_+ and S_- be the sets of source and sink nodes, respectively, i.e., $b_i > 0, \forall i \in S_+,\ b_i < 0, \forall i \in S_-$. Let d_{ij} be the length of a shortest path from node $i \in S_+$ to node $j \in S_-$. ($d_{ij} = \infty$ if no such path exists.) Construct an equivalent transportation problem using this data so that the optimal cost from the network flow problem is unchanged.[1]

2. BONUS: Suppose that the problem is now a capacitated flow problem defined by a graph $G = (\mathcal{N}, \mathcal{A})$ and finite valued capacities u_{ij} with supplies and demands in a given vector b. Construct a related transportation problem which has a one-one correspondence between feasible flows and cost of the flows in the two problems.

2 Node-Arc Incidence Matrix

Suppose that $G = (\mathcal{N}, \mathcal{A})$ is a connected digraph and A is the corresponding incidence matrix.

1. Show that A has linearly dependent rows.

2. Show that the rank of A is $|\mathcal{N}| - 1$.

[1] Use the same source i and sink j nodes and the same values for the supplies and demands. Use d_{ij} as a cost between i and j.

3. Suppose that the corresponding network flow problem has balanced supply/sinks, $\sum_i b_i = 0$. Let $\mathcal{T} \subseteq \mathcal{A}$ be a spanning tree of the nodes $\mathcal{N}$ with $|\mathcal{T}| = |\mathcal{N}| - 1$. Let $\tilde{A}, \tilde{b}$ be obtained from A, b, respectively, by deleting the last row. Show that the solution obtained from solving the linear equations $\tilde{A}f = \tilde{b}, f_{ij} = 0, \forall ij \in \mathcal{A} \backslash \mathcal{T}$ is *unique*.

3 The Airline Laundry Problem

A laundry company must provide to an airline r_i clean headrest covers on each of N consecutive days. The company can buy new covers at a price of p dollars each, or launder the used ones. Laundering can be done at a fast rate that makes the covers unavailable for the next n days and costs f dollars per cover, or at a slower rate that makes covers unavailable for the next m days (with $m > n$) at a cost of g dollars per cover ($g < f$). The laundry company's problem is to decide how to meet the airline's demand at minimum cost, starting with no headrest covers and under the assumption that any leftover covers have no value and no clean covers remain at the end.

1. Show that the problem can be formulated as a network flow problem.[2]

2. Show explicitly the form of the network if $N = 7, n = 1, m = 2$.

3. Solve the problem using MATLAB for the data:

$$N = 7, r = (r_i) = \begin{pmatrix} 100 \\ 100 \\ 100 \\ 100 \\ 100 \\ 125 \\ 125 \end{pmatrix}, p = \$10, f = \$2, g = \$1, n = 1, m = 2$$

   Obtain the dual solution and provide an economic interpretation for the dual variables.

4 Minimization of the mean cost of a cycle using linear programming

Consider a directed graph in which each arc (edge) is associated with a cost c_{ij}. For any directed cycle, we define its mean cost as the sum of the costs of its arcs, divided by the number of the arcs. We are interested in a directed cycle whose mean cost is minimal. We assume that there exists at least one directed cycle.

Consider the linear programming problem

$$\begin{array}{ll} \max & \lambda \\ \text{s.t.} & p_i + \lambda \leq p_j + c_{ij}, \quad \text{for all arcs } (i, j). \end{array}$$

1. Show that this maximization problem is feasible.

2. Show that if (λ, p) is a feasible solution to the maximization problem, then the mean cost of every directed cycle is at least λ.

3. Show that the maximization problem has an optimal solution.

4. Show how an optimal solution to the maximization problem can be used to construct a directed cycle with minimal mean cost.

[2] Hint: Use a node corresponding to clean covers and a node corresponding to dirty covers for each day; more nodes may also be needed.