CO 602/CM 740: Fundamentals of Optimization
Problem Set 3

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Fall 2016.
Handed out: Friday 2016-Oct-7.
Due: Thursday 2016-Oct-20 in class before lecture starts.
(Please use LEARN dropbox to hand in assignments.)

Contents

1 Adjacent Vertices

Consider the polyhedron

$$P = \{x \in \Re^n : a_i^T x \geq b_i,\ i = 1, \ldots, m\}.$$

Suppose that u, v are two distinct basic feasible solutions (BFSs) that satisfy

$$a_i^T u = a_i^T v = b_i,\ i = 1, \ldots, n-1,$$

and that the set of vectors $\{a_i\}_{i=1}^{n-1}$ is linearly independent. (In particular, u, v are adjacent BFSs.) Let

$$L = \{\lambda u + (1-\lambda) v : 0 \leq \lambda \leq 1\}$$

be the line segment, $[u, v]$, joining u, v. Prove that

$$L = \{z \in P : a_i^T z = b_i,\ i = 1, \ldots, n-1\}.$$

2 Polyhedra and Dimension

Let $P \subseteq \mathbb{R}^n$ be a nonempty polyhedron in *standard form* whose definition involves m linear independent equality constraints $Ax = b$. Its dimension is defined as the smallest integer k such that P is contained in some k-dimensional affine subspace of $\mathbb{R}^n$.

1. Explain why the dimension of P is at most $n - m$.
2. Suppose that P has a nondegenerate BFS. Show that the dimension of P is exactly $n - m$.
3. Suppose that x is a degenerate BFS. Show that x is degenerate under every standard form representation of the same polyhedron (in the same space $\mathbb{R}^n$).[1]

3 Klee-Minty Example

Consider the following linear program

$$\begin{array}{ll} \max & \sum_{j=1}^{n} 10^{n-j} x_j \\ \text{s.t.} & \left(2 \sum_{j=1}^{i-1} 10^{i-j} x_j\right) + x_i \leq 100^{i-1}, \qquad \forall i = 1, \dots, n \\ & x_j \geq 0, \quad \forall j = 1, \dots, n \end{array} \tag{1}$$

1. Solve the problem by hand for $n = 3$ using the largest coefficient rule (Dantzig's rule). Show your work.
2. Solve the program in MATLAB using both the simplex method and the interior-point method. Start with $n = 8$ and increase n. For how large a value of n can you solve the problem (in a reasonable amount of time)? Which method gives a better solution in terms of accuracy? In terms of cpu seconds?
3. How many iterations do you expect the simplex method to take for general n?

4 Infeasibility and Boundedness

1. Construct an example of an LP where both the primal and dual problem are infeasibile.
2. (a) Construct an example of an LP where the *optimal set* is bounded.
   (b) Construct an example of an LP where the *optimal set* is unbounded.
3. **BONUS** Can you find an example of an LP where both the primal and dual *feasible sets* are *bounded*?

[1] Hint: Using parts 1 and 2, compare the number of equality constraints in two representations of P under which x is degenerate and nondegenerate, respectively. Then, count active constraints.

5 Diet Problem

Polly would like to find a nutritious diet but minimize the cost. She plans on having the diet yield a minimum of:

energy 2,000 Kcal; protein 55g; calcium 800 mg.

(She will get the remaining nutrients from vitamin pills and nutrient drinks.) She chooses the six foods with their nutritive values given in Table 1.

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (cents)
Oatmeal	28 g	110	4	2	12
Chicken	100 g	205	32	12	164
Eggs	2 large	116	13	54	30
Whole milk	237 cc	160	8	285	54
Cherry pie	170 g	420	4	22	120
Pork with beans	260 g	260	14	80	116

Table 1: Nutritive values per serving

1. Find the optimal diet and show each iteration with an economic interpretation.[2]
2. What is the economic interpretation of the dual player?

[2]Fractional servings are allowed.