CO 602/CM 740: Fundamentals of Optimization
Problem Set 2

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1 Extreme Points and Linear Transformations

1. Let S_n, S_m be convex sets in $\mathbb{R}^n, \mathbb{R}^m$, respectively. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation that establishes a one-one correspondence between elements in the set S_n and elements in the set S_m. Show that there is a one-one correspondence between extreme points in S_n and extreme points in S_m.

2. Let $P = \{x \in \Re^n : Ax \geq b, x \geq 0\}$, where A is a $k \times n$ matrix. Let $Q = \{(x, z) \in \Re^{n+k} : Ax - z = b, x \geq 0, z \geq 0\}$. Show that P, Q are isomorphic, i.e. that there exist affine mappings f, g from P to Q and from Q to P such that $g(f(x)) = x, \forall x \in P$ and $f(g(y)) = y, forally \in Q$.

2 Polyhedrons

For each of the following (feasible) sets, determine whether it is a polyhedron.

1. The of all $(x, y) \in \mathbb{R}^2$ satisfying the constraints

$$\begin{array}{rcl} x\cos\theta + y\sin\theta & \leq & 1, \quad \forall \theta \in [0, \pi/2] \\ x & \geq & 0 \\ y & \geq & 0 \end{array} \tag{1}$$

2. The set of all $x \in \mathbb{R}$ satisfying the constraint $x^2 - 8x + 15 \leq 0$.
3. The empty set.

3 Basic Solutions with Interval Constraints

First consider the constraints $Ax = b$ and $x \geq 0$ and assume that the $m \times n$ matrix A has linearly independent rows. Recall that vector $\bar{x} \in \mathbb{R}^n$ is a basic solution if

1. all equality constraints are active at $\bar{x}$;
2. out of the constraints that are active at $\bar{x}$, there are n of them that are linearly independent.

Recall the procedure for finding basic solutions:

1. Choose m linearly independent columns $A[:, B]$, where $B \subseteq \{1, \ldots, n\}$.[1]
2. Set $x_i = 0, \forall i \notin B$. (called *nonbasic* variables)
3. Solve the system of m equations $Ax = b$.

Now consider the constraints $Ax = b$ as above but with the upper and lower bounds $\ell \leq x \leq u, \ell < u$.

1. Provide a procedure to the one above for finding basic solutions.
2. Prove that all basic feasible solutions, BFSs, can be found this way.
3. Now suppose that $\ell = 0, u = 1$, i.e., the lower bounds are all zero and the upper bounds are all ones. Write a MATLAB code that outputs all the basic solutions and counts the number of basic and number of basic feasible solutions. Test it on the instance with A, b given by

```
A=[
     0     1     0     1    -1     0     0    -1
     0     0     0     0    -2     1     0    -2
     0     0    -1     2     1     0     0    -1
     1     0     0    -1     1     0     3    -1
];

b =[-1
    -3
     2
     2];
```

[1] The are called *basic columns*; they exist because rank $A = m$.

4 Basic Solution

Consider the standard form polyhedron $P = \{x \in \Re^n : Ax = b, x \geq 0\}$, where the rows of A are linearly independent. Let x be a basic solution and let $J = \{i | x_i \neq 0\}$. Show that a basis is associated with the basic solution x if, and only if, every column $A_i, i \in J$ is in the basis.