

CO 602/CM 740: Fundamentals of Optimization

Problem Set 1

H. Wolkowicz

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1 MATLAB and Linear Equations

Consider the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \dots 0 & 1 \\ -1 & 1 & 0 & 0 \dots 0 & 1 \\ -1 & -1 & 1 & 0 \dots 0 & 1 \\ \dots & \dots & \dots & & \\ \dots & \dots & \dots & -1 & 1 \end{bmatrix}$$

i.e., 1 on the diagonal and the last column and -1 below the diagonal. Look at the file `Aoneseps.m` (link at this webpage). For $n = 10 : 10 : 200$, i.e., starting at 10 with steps 10 up to 200,

1. generate a random $x_0 \in \mathbb{R}^n$, $x_0 = \text{randn}(n)$, and set $b = A * x_0$;
2. solve the equation $Ax = b$, $x_e = A \setminus b$, and evaluate the residual

$$r_e = \text{norm}(b - A * x_e);$$

3. perturb $A_{eps} := A + \epsilon E$ with a random E of unit norm and $\epsilon > 0$ small; solve the equation $A_{eps}x_{eps} = b$, $x_{eps} = A_{eps} \setminus b$, and evaluate the residuals

$$r_{eps} = norm(b - A_{eps} * x_{eps}); r_{Ax_{eps}} = norm(b - A * x_{eps});$$

4. compare the three residual values and try and explain the results.

Please use MATLAB version 2015a if possible. If not, please state which version of MATLAB you are using. Also, if possible use several different versions of MATLAB and see if results change.

Please capture the output using e.g., diary command. You can hand in the output or email it in.

2 Linear Programming Standard Form

Consider the linear program, LP, with three constraints and six variables

$$(LP) \quad \begin{aligned} p^* &= \min && (76 \ 58 \ 100 \ 38 \ 28 \ 36) x \\ &\text{s.t.} && \begin{bmatrix} 7 & 25 & 14 & 26 & 70 & 98 \\ 97 & 81 & 15 & 85 & 91 & 90 \\ 43 & 20 & 60 & 3 & 25 & 25 \end{bmatrix} x \begin{pmatrix} = \\ \leq \\ \geq \end{pmatrix} \begin{pmatrix} 77 \\ 98 \\ 27 \end{pmatrix} \\ &&& x_1 \text{ free}, x_2 \leq 0, x_i \geq 0, i = 3, 4, 5, 6. \end{aligned} \quad (1)$$

Put (LP) into standard form. Solve it using *linprog* in MATLAB, i.e., solve the problem first in the original form and then in the standard form. Verify that the slacks/surplus variables make sense.

3 Linear Fractional Programming

Suppose we have the linear fractional program

$$(LFP) \quad \begin{aligned} p^* &= \min && \frac{c^T x + d}{f^T x + g} \\ &\text{s.t.} && Ax \leq b \\ &&& x \in \mathbb{R}^n \\ &&& |x_i| \leq M, \forall i \end{aligned} \quad (2)$$

Moreover, suppose we know

$$x \in F := \{x \in \mathbb{R}^n : Ax \leq b, |x_i| \leq M, \forall i\} \implies f^T x + g > 0,$$

and that the optimal value satisfies the bounds

$$p^* \in [K, L].$$

1. Develop an algorithm that uses linear programming as a subroutine and finds the optimal value p^* to any given tolerance $\epsilon > 0$.¹

¹HINT: Consider the problem of deciding whether the optimal value is less than or equal to a certain number. Then change $\frac{c^T x + d}{f^T x + g} \leq p$ to a linear objective function. Use this to decrease the length of the interval $[K, L]$.

2. Code the algorithm in MATLAB. Solve the following instance with tolerance $\epsilon = .00001$ for the optimal value. Provide the best approximations x, p that you obtain.

```

c = -8
    -4
     1
     2
    -2

d = -2

f = 0
     5
    13
     6
    -3

g = 4187

A = -6   -7    8    1   -1
     -7   -6   24   -6  -18
        0    6    3   -2  -3

b = -170
    -17
     33

M = 155

```

Choose K very small and L very large in order to find the optimal value in $[K, L]$.

4 Linear Assignment

Consider a school district with I neighbourhoods, J schools, and G grades at each school. Each school j has a capacity of C_{jg} for grade g . In each neighbourhood i , the student population of grade g is S_{ig} . Finally, the distance of school j from neighbourhood i is d_{ij} . Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance traveled by all students. (You may ignore the fact that the number of students must be integer.)

5 Absolute Values

Consider the problem

$$\begin{array}{ll} \min & 2x_1 + 3|x_2 - 10| \\ \text{s.t.} & |x_1 + 2| + |x_2| \leq 5 \end{array}$$

and reformulate it as a linear programming problem (without absolute values). Find an optimal solution.

6 Chebychev center

Consider a set P described by linear inequality constraints

$$P = \{x \in \mathbb{R}^n : a_i^T x \leq b_i, i = 1, \dots, m\}.$$

A ball with center y and radius r is defined as the set of all points within (Euclidean) distance r from y . We are interested in finding a ball with the largest possible radius, which is entirely contained within the set P . (The center of such a ball is called the *Chebychev center of P* .) Provide a linear programming formulation of this problem.