CO 602/CM 740: Fundamentals of Optimization
Problem Set 8

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Contents

1 Convex Functions

Let $x \in \mathbb{R}^n$ and let $x_{[i]}$ denote the i-th largest component of x, i.e. $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$. Let $1 < r < n$. Show that the function $g(x) := \sum_{i=1}^{r} x_{[i]}$ is convex. (HINT: Recall that the sup or max of convex functions is convex.)

2 Steepest Descent

1. Consider the function $f(x) = x_1^2 + 2x_2^2 + 4x_1 + 4x_2$. Prove by induction that the method of steepest descent (with exact line search) with the initial point at the origin, $x^0 = 0$, produces the sequence of iterates

$$x^k = \begin{pmatrix} \frac{2}{3^k} - 2 \\ (-\frac{1}{3})^k - 1 \end{pmatrix}.$$

   What is the minimizer of f?

2. Apply the steepest descent algorithm (using MATLAB) with constant stepsize α to the function

$$f(x) = \begin{cases} x^2\left(\sqrt{2} - \sin\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the gradient ∇f satisfies the Lipschitz condition

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathbb{R},$$

for some constant L. Write a MATLAB program to verify that the method is a descent method for $\alpha \in (0, 2/L)$. Do you expect to converge to the global minimum $x^* = 0$?

3. Consider the gradient method $x^{k+1} = x^k + \alpha^k d^k$ for the case where f is a positive definite quadratic, and let $\bar{\alpha}^k$ be the stepsize corresponding to the line minimization rule. Show that a stepsize α^k satisfies the inequalities of the Goldstein rule if and only if

$$2\sigma\bar{\alpha}^k \leq \alpha^k \leq 2(1 - \sigma)\bar{\alpha}^k.$$

3 Trust Region Subproblem, TRS

Let $f(x) = 10(x_2 - x_1^2)^2 + (1 - x_1)^2$. At $x = (0, -1)^T$ draw the contour lines of the quadratic model

$$m(p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T \nabla^2 f(x) p.$$

Draw the family of solutions of

$$\min m(p) \text{ s.t. } \|p\| \leq \delta$$

as the trust region radius varies from $\delta = 0$ to $\delta = 2$. Repeat this at the point $(0, 0.5)^T$. (Explain the details on how you found the points to do the drawing.)