CO 602/CM 740: Fundamentals of Optimization
Problem Set 4

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Contents

1 Two Phase Simplex Method

1. Use the Phase I and Phase II simplex method and solve the following LP.

$$\begin{array}{llllllllllll} \min & 2x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & - & 2x_5 & & \\ \text{s.t.} & x_1 & + & 3x_2 & & & + & 4x_4 & + & x_5 & = & 2 \\ & x_1 & + & 2x_2 & & & - & 3x_4 & + & x_5 & = & 2 \\ & -x_1 & - & 4x_2 & + & 3x_3 & & & & & = & 1 \\ & x \geq 0 & & & & & & & & & & \end{array} \quad (1)$$

2. Use the Phase I and Phase II simplex method and solve the following LP. But use only ONE artificial variable!

$$\begin{array}{llllllllllllll} \min & -x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & + & 3x_5 & - & x_6 & & \\ \text{s.t.} & x_1 & & & & & - & 2x_4 & + & x_5 & + & x_6 & = & 3 \\ & & & x_2 & & & - & 2x_4 & + & -2x_5 & + & x_6 & = & -6 \\ & & & & & x_3 & + & x_4 & + & -2x_5 & - & 2x_6 & = & -12 \\ & x \geq 0 & & & & & & & & & & & & \end{array} \quad (2)$$

Hint: Start with a basic (infeasible) solution. Then add one artificial variable with a coefficient -1 in each row, e.g. $-y$ in each row. Then let this artificial variable enter the basis but choose the leaving variable by picking the row with the most negative b_i component. Show/prove that this yields a BFS. You can now proceed as in the usual Phase I to eliminate the artificial variable.

3. Use the MATLAB LP solver and verify the two solutions in Parts 1 and 2, above.

2 Duality

1. Consider the general LP:

$$\begin{array}{lccccccc} \max & c^1x^1 & + & c^2x^2 & + & c^3x^3 & & \\ \text{s.t.} & A_{11}x^1 & + & A_{12}x^2 & + & A_{13}x^3 & = & b^1 \\ & A_{21}x^1 & + & A_{22}x^2 & + & A_{23}x^3 & \leq & b^2 \\ & A_{31}x^1 & + & A_{32}x^2 & + & A_{33}x^3 & \geq & b^3 \\ & x^1 \geq 0 & & x^2 \text{ free} & & x^3 \leq 0 & & \end{array} \tag{3}$$

for appropriate **size vectors** c^i, x^i, b^i, and matrices A_{ij}. Use the *game theory/min-max* approach and derive a dual LP problem.

2. Give an example of a pair (primal and dual) of linear programming problems, both of which have multiple optimal solutions.

3. Let A be a symmetric square matrix. Consider the LP

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq c \\ & x \geq 0. \end{array} \tag{4}$$

Prove that if x^* satisfies $Ax^* = c$, and $x^* \geq 0$, the x^* is an optimal solution.

3 Optimality and Perturbations

Consider the LP

$$\begin{array}{ll} \min & (c + \lambda \bar{c})^T x \\ \text{s.t.} & Ax = a \\ & Bx \geq b \end{array} \tag{5}$$

Suppose that $\bar{x}$ is an optimal solution if $\lambda = 0$ and also is optimal if $\lambda = \alpha$, for a given fixed $\alpha > 0$. Show that $\bar{x}$ remains an optimal solution for the LP for all $0 \leq \lambda \leq \alpha$.