CO 602/CM 740: Fundamentals of Optimization
Problem Set 2

H. Wolkowicz

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Contents

1 Basic and Basic Feasible Solutions

1. Suppose that $P = \{x \in \mathbb{R}^n : Ax = a, Bx \geq b\}$ is a given polyhedron, where A, B are $m_A \times n$ and $m_B \times n$ matrices, respectively.
   (a) What conditions on the data guarantees that P has a nonempty set of extreme points?
   (b) If P has a nonempty set of extreme points, prove that the number of extreme points is finite.
   (c) What is the maximum number of extreme points that P can have? (An upper bound, not necessarily the best upper bound, is fine.)
2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 10 & 2 & 2 & 1 \\ 1 & 1 & -1 & 0 & 4 & 2 & 1 \\ 2 & 3 & -2 & 3 & 10 & 5 & 3 \end{bmatrix}; b = \begin{bmatrix} 10 \\ 6 \\ 17 \end{bmatrix}; c = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T.$$

   Consider the standard form polyhedron $P = \{x \in \mathbb{R}^7 : Ax = b, x \geq 0\}$. Write a MATLAB program that takes A, b, with A full row rank, as input, and then it outputs the following:
   (a) verification that A is full row rank (or not);
   (b) the maximum number of possible basic solutions for a polyhedron of type P dependent only on the size of A, call it nCm; (see Item (1c) above)

(c) the minimum number of 0 elements/components in a basic solution, call it ζ.

(d) an $nCm \times m$ matrix, call it RowsnCk, where each row of the matrix RowsnCk corresponds to the possible indices of columns of A (of nonzero components of basic solution x) corresponding to a basic solution; (Hint: use the MATLAB command: nchoosek)

(e) an $n \times k$ matrix X where each column corresponds to a basic solution;

(f) a vector with the indices of the columns of X that correspond to basic feasible solutions and a corresponding vector with the values of $c^T x$ and the basic feasible solutions for the maximum value and the minimum value for $c^T x$;

(g) a submatrix formed from the rows of the matrix RowsnCk where each row corresponds to a basic feasible solution, i.e. this new matrix provides the indices of columns that form feasible bases.

2 Isomorphic Polyhedra

The mapping of type $f(x) = Ax + b$, where A is a matrix and b is a vector, is called an *affine map*. Let P, Q be polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The polyhedra P, Q are called *isomorphic* if there exist affine maps $f : P \to Q$ and $g : Q \to P$ such that $g(f(x)) = x, \forall x \in P$ and $f(g(y)) = y, \forall y \in Q$. Show the following:

1. If P, Q are isomorphic polyhedra, show that there is one-one correspondence between their extreme points. In particular, if f, g are as above, then show that x is an extreme point of P if and only if $y = f(x)$ is an extreme point of Q.

2. (slack/surplus variables) Let $P = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}$, where A is $m \times k$. Let $Q = \{(x, z) \in \mathbb{R}^{n+k} : Ax - z = b, x \geq 0, z \geq 0\}$. Show that P and Q are isomorphic.