CO 602: Fundamentals of Optimization
Fall 2010
Second Practice Final Exam
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Handed out: 2010-Dec-5, web only.

1. **[20 points]** Consider a linear programming problem in standard equality form. Suppose the simplex algorithm is at a BFS given by $(\mathbf{x}, B)$. Recall the following definitions of a pivot step. The reduced costs are defined as $\bar{c}_j = c_j - \mathbf{c}(B)^T A(:, B)^{-1} A(:, j)$ for $j \in N$. Let $j \in N$ be a nonbasic variable with negative reduced cost. Direction $\mathbf{d}$ is defined so that $d(j) = 1$, $d(i) = 0$ for $i \in N - \{j\}$, and $\mathbf{d}(B)$ is chosen (uniquely) to satisfy $A\mathbf{d} = \mathbf{0}$. Parameter θ is defined as $\min\{-x(i)/d(i) : i \in B, d(i) < 0\}$. Finally $\mathbf{x}^{\mathrm{NEW}} = \mathbf{x} + \theta\mathbf{d}$.

   Describe in 2–3 words each what is the interpretation of the following conditions that might hold during the pivot step.

   (a) No $j \in N$ with negative reduced cost exists.

   (b) No $i \in B$ exists such that $d(i) < 0$.

   (c) $\theta = 0$.

2. **[20 points]** Let $P = \{\mathbf{x} \in \mathbf{R}^n : |x_1| + \cdots + |x_n| \leq 1\}$.

   (a) Draw a figure of this set in the case $n = 2$. In the drawing, identify all four extreme points.

   (b) This set P can be defined with 2^n linear inequalities. Explain.

   (c) If one is allowed to introduce n auxiliary variables, then one can describe P with many fewer inequalities. Provide such a system of inequalities, which should have at most const$\cdot n$ inequalities. (In other words, come up with a system of linear inequalities involving $x_1, \ldots, x_n, t_1, \ldots, t_n$ such that the set of $\mathbf{x}$'s for which there exists a feasible $(\mathbf{x}, \mathbf{t})$ is exactly P.)

3. **[20 points]** Suppose $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are positive real numbers. Consider the linear programming problem of minimizing $a_1x_1 + \cdots + a_nx_n$ subject to the constraints that $b_1x_1 + \cdots + b_nx_n = 1$, $x_1, \ldots, x_n \geq 0$.

   Figure out the optimal solution, and demonstrate that it is truly optimal using duality or complementarity. Hint: The optimal solution has a single nonzero entry.

4. **[20 points]** Consider an instance of the max-flow problem in which the network G consists entirely of k disjoint directed paths from s to t. For consistency, use the following notation. For $i = 1, \ldots, k$, let the vertices on the ith such path be denoted $v_{i,1}, \ldots, v_{i,l_i-1}$ and let the arcs on the ith such path be denoted $a_{i,1}, \ldots, a_{i,l_i}$. Here, l_i denotes the length (number of arcs) of the ith path. Let the capacities be denoted $u(a_{i,j})$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l_i$.

(a) Describe an efficient algorithm to find the maximum flow in such a network.

(b) Describe an efficient algorithm to find the minimum cut in such a network.

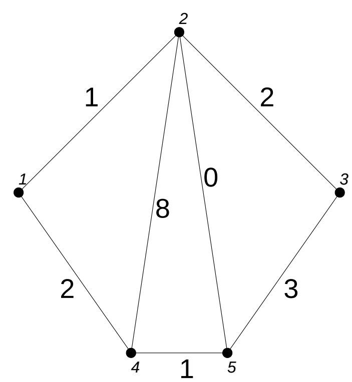
Both algorithms should involve a single pass over the nodes and arcs of G.

5. **[20 points]** Consider an assignment problem with costs $c_{i,j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$, where there are n tasks and n workers. Write the dual as in lecture, namely, maximize

$$\sum_{i=1}^{n} r_i + \sum_{j=1}^{n} p_j$$

subject to $r_i + p_j \leq c_{i,j}$ for all i, j. Suppose one is given an optimal (integer) solution $x^*_{i,j}$ to the assignment problem and is also given the optimal values for r^*_i, $i = 1, \ldots, n$. Explain how to deduce the optimal values for p^*_j, $j = 1, \ldots, n$, and how to confirm that all these variable settings are indeed optimal.

6. For the following graph, describe the steps of Prim's algorithm to find a minimum cost spanning tree, assuming that the tree is initialized with node set $\{1\}$. In particular, indicate on each step which edge and node is inserted into the tree built by Prim. Note: the node numbers are in smaller typesize, and the edge weights are in a larger typesize.



7. (a) **[20 points]** Consider applying the Armijo line-search to the univariate quadratic objective function $f(x) = x^2$ at current iterate $x^k > 0$. Let d^k be a descent direction. Show that if $\sigma \in (1/2, 1)$, then x^{k+1} will have the same sign as x^k.

(b) **[10 points]** Suppose d^k is the Newton step for this problem. Show that the Newton step with $\alpha^k = 1$ (i.e., $m = 0$ if $\alpha^k = \beta^m$) will be accepted by the Armijo inequality if and only if $\sigma \leq 1/2$.

8. **[20 points]** Let $\mathbf{a}_1$ be a nonzero vector in $\mathbf{R}^n$, $n > 1$. Consider the function $f(\mathbf{x}) = (\mathbf{a}_1^T \mathbf{x} - b_1)^2$.

(a) Rewrite f in the standard form for a quadratic function (i.e., using a symmetric matrix). [Hint: the square of a scalar may be written as the inner product of the scalar with itself.]

(b) Argue that f is convex.

(c) Argue that f has an infinite number of minimizers.