CO 602: Fundamentals of Optimization
Fall 2010
Practice Final Exam
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Handed out: 2010-Dec-1, web only.

This exam had eight questions for a total of 160 points. However, 150 points were considered a full mark, so the remaining 10 were bonus points. The exam lasted 150 minutes. Students were permitted a single 8.5×11 inch paper with notes written on both sides during the exam. No calculators or other electronic devices were permitted.

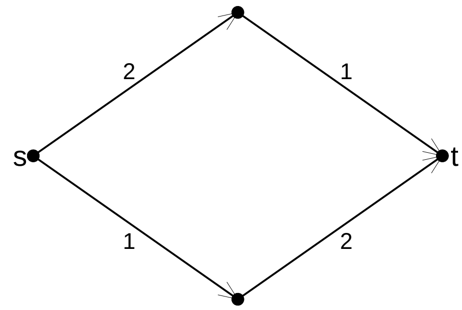
1. [**20 points**] Consider a pivot step for the simplex method applied to an instance of s.e.f. linear programming. Suppose $(\mathbf{x}, B)$ the current BFS and $\mathbf{d}$ is the search direction associated with a particular nonbasic variable j with a negative reduced cost. In the case that $d(i) \geq 0$ for all $i \in B$, the LP instance is unbounded. Explain why.

2. [**20 points**] Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the columns of the $n \times n$ identity matrix. Let $A = \text{conv}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ and let $B = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x} \geq \mathbf{0};\ x_1 + \cdots + x_n = 1\}$. Prove that $A = B$. [Hint: Prove separately that $A \subset B$ and $B \subset A$.]

3. [**20 points**] Consider the *undirected* shortest path problem, that is, given an undirected graph $G = (N, E)$, a distinguished $s \in N$, and nonnegative weights on the edges, find the shortest path from s to every other node of N. Propose an algorithm for this problem. If you use an algorithm presented in lecture, then you do not need to provide any detail about it. [Hint: Reduce the problem to the shortest path problem covered in lecture.]

4. [**20 points**] Consider the following generalization of the assignment problem. There are n workers and p tasks. There are positive integers $w_1, \ldots, w_p$, such that task j, for $j = 1, \ldots, p$ requires exactly w_j workers assigned to it. Furthermore, $w_1 + \cdots + w_p = n$. There are costs c_{ij} specified for $i = 1, \ldots, n$ and $j = 1, \ldots, p$ for assigning worker i to task j. Find the minimum-cost assignment of workers to tasks such that each worker is assigned to one task and task j is assigned exactly w_j workers. Propose an algorithm for this problem. If you use an algorithm presented in lecture, then you do not need to provide any detail about it. [Hint: Reduce the problem to the usual assignment problem.]

5. [**20 points**] Show by hand the steps of the Ford-Fulkerson algorithm for finding the maximum (s, t) flow in the following directed graph. The arc labels are capacities. On each iteration, exhibit the flow found so far and the residual network. Upon termination, exhibit also the minimum cut.



6. **[20 points]** The *disjoint path* problem asks, given a directed graph $G = (N, A)$ and two pairs of nodes (s_1, t_1) and (s_2, t_2) with $s_1, s_2, t_1, t_2 \in N$, find a directed path P_1 from s_1 to t_1 and a directed path P_2 from s_2 to t_2 such that P_1 and P_2 are disjoint (i.e., no common arcs or nodes between P_1 and P_2). Formulate this problem as an integer linear programming feasibility problem. There should be two variables per arc.

7. **[20 points]** Consider applying the Armijo line search with parameters (β, σ) in a descent algorithm to minimize the univariate function $f(x) = x^2$. Assume $x^0 = 1$. As σ gets close to 1, the allowable step gets shorter. Explain why.

8. **[20 points]** In Problem Set 8, Question 4, we considered the objective function $f(\mathbf{x}) = \sum_{i=1}^{m} \exp(\mathbf{a}_i^T \mathbf{x} - b_i)$, where $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$ are the rows of an $m \times n$ given matrix A, and $\mathbf{b} \in \mathbf{R}^m$ is a given vector. Prove that this objective function is convex.