CO 602/CM 740: Fundamentals of Optimization
Problem Set 1

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Contents

1 Chebychev Center

1. (Exercise 1.12, Pg 36 in the text) Consider a set P described by linear inequality constraints, that is $P = \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$. A ball with center y and radius r is defined as the set of all points within (Euclidean) distance r from y. We are interested in finding a ball with the largest possible radius, which is entirely contained within the set P. (The center of such a ball is called the *Chebychev center* of P.) Provide a linear programming formulation of this problem. (You can assume that $a_i \neq 0, \forall i$.)

2. The following MATLAB program can be used to generate a random (feasible) polytope P.

```
%generate random feasible problem
m=3;n=2;Amax=20;
A=randi(Amax,m,n)+1;
A=A*diag(sign(randn(n,1)));
A=diag(sign(randn(m,1)))*A;
x=randi(Amax,n,1);
b1=A*x+4*ones(m,1);   % ensures feasibility
b2=2*A*x-8*ones(m,1);   % ensures boundedness
A=[A;-2*A];
```

```
b=[b1; -b2];
m=2*m;
```

The MATLAB program was used to generate the following data given by the rows in A, b.

```
A =[ 6    19
     9    10
   -19    -8
   -12   -38
   -18   -20
    38    16];

b =[ 283
   256
  -410
  -550
  -496
   836];
```

Find the optimal radius and center for this data and plot the resulting circle and lines representing the polytope.
(Please use MATLAB; e.g., help linprog. Suggestion: Install and use CVX. See URL: http://cvxr.com/ Also, A MATLAB program to plot a circle, *circle.m*, is available online at
www.mathworks.com/matlabcentral/fileexchange/2876-draw-a-circle)

2 Basis for Nullspace of A

1. Suppose that A is an $m \times n$ matrix and suppose there exists a matrix P such that $PA = \begin{bmatrix} I_m & E \end{bmatrix}$, where I_m denotes the $m \times m$ identity matrix.
   (a) What are the ranks of A, P.
   (b) What is the dimension of the matrix E.
   (c) Use the matrix E and find a matrix whose columns form a basis for the nullspace of A.
2. Let
$$A = \begin{bmatrix} 5 & 4 & 18 & 14 & 2 \\ -10 & -11 & -36 & -28 & -4 \end{bmatrix}$$
Find the matrices P, E defined above and the matrix whose column space is the nullspace of A.

3 Piecewise Linear LP

Iron is the main material in three different products produced by Company A. The company has access to 250 units of iron per day. The cost is $2 per unit of iron. Additional iron can be obtained at $5 per unit.
Electricity usage costs $30 per unit for the first 1000 units used per day; $45 per unit for the next 500 units per day and $75 per unit for anything beyond 1500 units.
Water usage costs $6 per unit for the first 800 units used per day; $7 per unit for any amount over 800 units.
Fuel usage costs $4 per unit with a maximum of 3000 units available per day.
The labour force provides 640 man hours of labour per-day. But if more hours are needed, the cost is $12 dollars per hour for a maximum of 160 more man-hours per day.

The company wants a strategy to decide how much of each of the three products to produce in order to maximize the net profit.

	Units Input Needed for One Unit Product					
product	iron	Electricity	Water	Fuel	Labour	Selling Price/day
1	1/2	3	1	1	2	$300/unit for first 50; $250/unit beyond first 50;
2	1	2	1/4	1	1	$350/unit with limit 100/day
3	3/2	5	2	3	1	$450/unit

1. Model the problem using a piecewise linear objective net profit function and show that the cost function is concave.
2. Transform the model into an LP.