A Characterization of Stability in Linear Programming

STEPHEN M. ROBINSON

University of Wisconsin, Madison, Wisconsin

(Received original July 1975; final, August 1976)

We prove that a necessary and sufficient condition for the primal and dual solution sets of a solvable, finite-dimensional linear programming problem to be stable under small but arbitrary perturbations in the data of the problem is that both of these sets be bounded. The distance from any pair of solutions of the perturbed problem to the solution sets of the original problem is then bounded by a constant multiple of the norm of the perturbations. These results extend earlier work of Williams.

IN THIS PAPER we shall be concerned with the pair of dual linear programming problems given by

$$\min \langle c, x \rangle$$

$$Ax - b \in Q^*, \quad x \in P,$$  \hspace{1cm} (P)

and

$$\max \langle u, b \rangle$$

$$c - uA \in P^*, \quad u \in Q,$$  \hspace{1cm} (D)

where $P$ and $Q$ are nonempty polyhedral convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and where the asterisk denotes the dual cone $P^* = \{ z \in \mathbb{R}^n \mid \langle z, x \rangle \geq 0 \text{ for each } x \in P \}$. It is clear that any pair of dual linear programming problems can be represented in the form of (P) and (D). The purpose of the paper is to establish what conditions must be placed on (P) and on (D) in order that for all small but otherwise arbitrary perturbations in the data $A, b, c$, the problems (P) and (D) will remain solvable, and the solution sets will be stable in the sense that for any pair of primal and dual solutions $x'$ and $u'$ of the perturbed problems, the distance of $x'$ from the solution set of (P) and that of $u'$ from the solution set of (D) will be bounded by some constant multiple of the size of the perturbations.

In 1963 Williams [22] proved that a solvable pair of dual linear programming problems would remain solvable under all small perturbations of a certain form if and only if the primal and dual optimal sets were both bounded. In terms of our notation for (P) and (D), the types of
perturbations that he considered were those obtainable by choosing a fixed matrix $A$, and fixed vectors $b_0$, $c_0$, of the same dimensions as $A$, $b$, and $c$, respectively, and considering the perturbed systems $(P_\alpha)$ and $(D_\alpha)$ constructed by replacing $A$, $b$, and $c$ by $A + \alpha A_0$, $b + \alpha b_0$, and $c + \alpha c_0$ for small positive $\alpha$. He proved that for each fixed $A_0$, $b_0$ and $c_0$ there exists a positive $\alpha_0$ (depending on $A_0$, $b_0$, and $c_0$) such that for all $\alpha \in [0, \alpha_0]$ the problems $(P_\alpha)$ and $(D_\alpha)$ are solvable and their common optimal objective value is differentiable from the right at $\alpha = 0$, if and only if the primal and dual optimal sets in $(P)$ and $(D)$ are both bounded. He also verified an expression given by Mills [16] for the right derivative of the objective value.

Our results here differ from Williams' in two principal ways. First, we shall show that $\alpha_0$ depends only on the magnitudes of the perturbations, and not on their directions; thus we need not restrict the perturbations to the special form just cited but can simply consider perturbed problems $(P')$ and $(D')$ in which $A$, $b$, and $c$ are replaced by $A'$, $b'$, and $c'$, respectively, as long as the latter are sufficiently close (in norm) to $A$, $b$, and $c$. Second, we shall provide error bounds for the distance from the optimal sets of $(P')$ and $(D')$ to those of $(P)$ and $(D)$, respectively, by using the solvability of $(P')$ and $(D')$ together with the technique used in [17] for bounding solution error by using Hoffman's theorem. For simplicity, we shall use the Euclidean norm on both $\Re^m$ and $\Re^n$ (and the corresponding induced norm for linear operators) since any other norms would be equivalent.

The results to be established here are special cases of more general stability properties, applicable to complementarity problems and to nonlinear variational inequalities, which we plan to present elsewhere. However, because of the fundamental importance of linear programming in applications and because of the possible implications of the present results for computational practice, it seemed appropriate to deal with them in a separate paper, using only elementary methods and emphasizing the application and interpretation of the theory.

We shall show that the solvability of the slightly perturbed problems $(P')$ and $(D')$ is equivalent to two other conditions: the boundedness of the primal and dual solution sets, mentioned above, and a regularity condition imposed on the constraints of the problems. Following the terminology of [18] we shall say that the constraints of $(P)$ are regular if $b \in \text{int}\{A(P) - Q^p\}$ and that those of $(D)$ are regular if $c \in \text{int}\{(Q)A + P^*\}$. If constraints are not regular, we call them singular. Here, of course, $(Q)A + P^* = \bigcup \{qA + p^* : q \in Q, p^* \in P^*\}$, and similarly for $A(P) - Q^p$; int denotes interior. If, for example, we take $P = \Re^m$ and $Q = \Re^n$, so that the system $Ax - b \in Q^p$, $x \in P$ is simply the system of linear equation given by $Ax = b$, then the definition of regularity says
that \( b \) belongs to the interior of the range of \( A \), which will be true if and only if \( A \) has full row rank. Thus, the concept of regularity may be thought of as a natural generalization of the idea of full row rank (or of nonsingularity for square systems) to more general systems involving inequalities and constrained variables [18, 19].

It is worth remarking that for \( A(P) - Q^* \) to have an interior it is not necessary that either \( A(P) \) or \( Q^* \) have one. For example, consider the system

\[
\begin{align*}
x_1 & \leq 1 \\
x_2 & \leq 1 \\
x_1 - x_2 & = 0 \\
x_1, x_2 & \geq 0.
\end{align*}
\]

We can write this as \( Ax - b \in Q^*, x \in P \) by taking

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},
\]

\( Q = \mathbb{R}_+^2 \times \mathbb{R}, P = \mathbb{R}_+^2 \), where \( \mathbb{R}_+^2 \) denotes the non-negative (non-positive) orthant in \( \mathbb{R}^2 \). Here neither \( A(P) \) nor \( Q^* \) has an interior, but \( A(P) - Q^* \) does and \( b \) belongs to it.

This example also illustrates why we have used the cones \( P \) and \( Q \), instead of writing everything in terms of inequalities. It is not hard to verify that there is no way of writing this system as a system of pure inequalities in the non-negative variables \( x_1 \) and \( x_2 \), i.e., as a \( k \times 2 \) system of the form \( Dx \leq d, x \geq 0 \), while satisfying the requirement \( d \in \text{int} \{ D(\mathbb{R}_+^2) + \mathbb{R}_+^k \} \). It is thus essential to retain the capability to handle equations (as well as unconstrained variables) if the regularity conditions considered here are to be satisfied; this was observed by Williams [22]. The use of \( P \) and \( Q \) eliminates the necessity for distinguishing between inequalities and equations, and between non-negative and unconstrained variables, and thereby greatly simplifies the notation.

Although the results we obtain here certainly bear on the question of the behavior of linear programming problems when roundoff errors occur during computation, we do not deal with the propagation of such errors in specific solution techniques. That subject is analyzed for the simplex method in the interesting paper of Wolfe [23]. The present work is rather an attempt to determine what attributes of a given linear programming problem cause it to behave well or badly under small changes in the data, one source of which could very well be roundoff errors.
1. Principal Results

Before proving the main results, we state three lemmas. The first is a form of the fundamental theorem of Hoffman [13] on approximate solutions of linear systems. Since it may be derived easily from the theorem in Hoffman's paper and the properties of polyhedral convex cones, we shall not prove it here.

**Lemma 1 (Hoffman).** Let $K$ and $L$ be nonempty polyhedral convex cones in $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively, let $M$ be a $k \times l$ matrix, and let $m \in \mathbb{R}^k$. Suppose that the set $F \subset \mathbb{R}^l$ consisting of all points $x$ such that

$$Mx - m \in K, \quad x \in L$$  \hspace{1cm} (1)

is nonempty. Then there exists some $\theta \geq 0$ such that for each $x \in L$, $d(x, F) \leq \theta d(Mx - m, K)$, where for a point $y$ and a set $A$, $d(y, A) \equiv \inf \{\|y - a\| : a \in A\}$.

The next lemma examines the behavior of the solution set of (1) when $M$ and $m$ are slightly perturbed. We have shown elsewhere [18] that if (1) is regular, its solution set has certain stability properties. However, we shall want to consider later a system that is not regular. The following lemma shows that if the solution set is bounded, it will exhibit a property similar to, but stronger than, upper semicontinuity (for which see, e.g., Berge [2], p. 109). This property might be called “upper Lipschitz continuity.” We use the symbol $B$ to denote the unit ball in $\mathbb{R}^t$, and we note that, for a closed set $A \subset \mathbb{R}^t$ and $\epsilon \geq 0$, $A + \epsilon B = \{x \in \mathbb{R}^t : d(x, A) \leq \epsilon\}$.

**Lemma 2.** Let $K$, $L$, $M$, $m$, $\theta$ and $F$ be as in Lemma 1. Suppose that for some $\mu \geq 0$, $F \subset \mu B$. Then for each $\epsilon > 0$ and each $M'$, $m'$ with $\delta' = \max \{|M' - M|, |m' - m|\} < \theta^{-1} \epsilon/(1 + \epsilon)$, the solution set $F'$ of the system $M'x - m' \in K$, $x \in L$ satisfies $F' \subset F + \lambda \delta' B$, with $\lambda = (1 + \epsilon)(1 + \mu)\theta$. (Here $\theta^{-1}$ is interpreted as $\pm \infty$ if $\theta = 0$.)

**Proof.** Choose $\epsilon > 0$ and select $M'$ and $m'$ as described. If $F' = \phi$, the result is surely true; if $F' \neq \phi$, let $x'$ be any point in $F'$, and let $x_0$ be the closest point in $F$ to $x'$. By Lemma 1, $\|x' - x_0\| = d(x', F) \leq \theta d(Mx' - m, K)$. But $Mx' - m' \in K$; hence $d(Mx' - m, K) \leq \|Mx' - m - (Mx' - m')\| \leq \|M' - M\| \|x'\| + \|m' - m\| \leq \delta'(1 + \|x'\|)$. We also have $\|x'\| \leq \|x_0\| + \|x' - x_0\| \leq \mu + \|x' - x_0\|$, so $d(Mx' - m, K) \leq \delta'(1 + \mu) + \delta' \|x' - x_0\|$. Therefore, $\|x' - x_0\| \leq \theta d(Mx' - m, K) \leq \theta \delta'(1 + \mu) + \theta \delta' \|x' - x_0\|$. As $1 - \theta \delta' > 1 - \theta \delta'(1 + \mu) \leq (1 + \epsilon)^{-1}$, we have $\|x' - x_0\| \leq (1 - \theta \delta'(1 + \mu) \leq (1 + \epsilon)^{-1} \theta \delta' (1 + \mu) \|x' - x_0\|$. But since $x'$ was arbitrary in $F'$, the result follows.

The third lemma shows that regular linear systems are precisely those that remain solvable under all sufficiently small perturbations in the
data. It is a very special case of Theorem 1 in [18]; that theorem deals with nonpolyhedral ordering cones in infinite-dimensional spaces, and it includes error bounds for changes in the solution set as well as solvability results. As we are dealing here with polyhedral cones and as we do not need the error bounds, we give a simple self-contained proof that requires nothing more complicated than the Farkas lemma. The idea of the proof is the same as that used by Williams [22] in his Lemma 2, but we permit arbitrary small variations in the data of the problem rather than the particular variations used by Williams.

**Lemma 3.** The linear system \( Ax - b \in Q^*, \ x \in P \) is regular (i.e., \( b \in \text{int}\{A(P) - Q^*\} \)) if and only if there exists some \( \eta > 0 \) such that for any \( A', b' \) with \( \max\{\|A' - A\|, \|b' - b\|\} < \eta \), the system

\[
A'x - b' \in Q^*, \quad x \in P
\]  

(2)

is solvable.

**Proof.** The "if" part is obvious, since if \( b \notin \text{int}\{A(P) - Q^*\} \), then there exist points \( b' \) arbitrarily close to \( b \) such that \( b' \notin A(P) - Q^* \); thus with \( A' = A \) the system (2) is not solvable. To prove the "only if" part, suppose that there is no \( \eta \) with the cited properties; we shall show that \( b \notin \text{int}\{A(P) - Q^*\} \). By assumption, we can find sequences \( \{A_n\} \) and \( \{b_n\} \) converging to \( A \) and \( b \), respectively, such that for each \( n \) the system

\[
A_nx - b_n \in Q^*, \quad x \in P
\]  

(3)

has no solution. By a variant of the Farkas lemma (easily derivable from, e.g., Theorem 3.5 of Ben-Israel [1]), (3) is not solvable if and only if there exists some \( w_n \in \sigma^m \) such that

\[
w_nA_n \in -P^*, \quad w_n \in Q, \quad \langle w_n, b_n \rangle > 0.
\]  

(4)

Evidently \( w_n \) is not zero, so we lose no generality by supposing that \( \|w_n\| = 1 \). Thus the sequence \( \{w_n\} \) is contained in the (compact) unit sphere of \( \sigma^m \), and so it has a convergent subsequence. Again without loss of generality, we may suppose that \( \{w_n\} \) itself converges to some \( w \) with \( \|w\| = 1 \). Taking the limit in (4) and using the fact that \( P^* \) and \( Q \) are closed, we have

\[
wA \in -P^*, \quad w \in Q, \quad \langle w, b \rangle \geq 0.
\]  

(5)

Now consider any point \( y \) in \( A(P) - Q^* \). We have \( y = Ap - q^* \) for some \( p \in P, q^* \in Q^* \). Thus \( \langle w, y \rangle = \langle w, Ap - q^* \rangle = \langle wA, p \rangle - \langle w, q^* \rangle \leq 0 \leq \langle w, b \rangle \), where we have used (5) and the properties of polar cones. It follows, since \( w \neq 0 \), that the hyperplane \( \{v \mid \langle w, v \rangle = 0\} \) separates \( b \) from the convex cone \( A(P) - Q^* \); therefore, \( b \) cannot lie in the interior of the cone. This completes the proof.
Observe that \( \langle w_n, b_n \rangle \geq 0 \), with \( w_n \neq 0 \), would suffice in (4); hence the full strength of the Farkas lemma is not required. The lemma could therefore be established also for nonpolyhedral cones by using another separation theorem.

We can now state our principal result.

**Theorem 1.** The following are equivalent:

(a) The constraints of (P) and of (D) are regular.

(b) The sets of optimal solutions of (P) and of (D) are nonempty and bounded.

(c) There exists an \( \varepsilon_0 > 0 \) such that for any \( A', b' \) and \( c' \) with

\[
\varepsilon' = \max \{ \| A' - A \|, \| b' - b \|, \| c' - c \| \} < \varepsilon_0,
\]

the two dual problems

\[
\begin{align*}
(P') & \quad \min \langle c', x \rangle \\
A'x - b' & \in Q^b, \quad x \in P
\end{align*}
\]

\[
\begin{align*}
(D') & \quad \max \langle u, b' \rangle \\
c' - uA' & \in P^*, \quad u \in Q
\end{align*}
\]

are solvable.

If these conditions are satisfied, then there exist constants \( \varepsilon_1 \in (0, \varepsilon_0) \) and \( \gamma \) such that for any \( A', b' \), and \( c' \) with \( \varepsilon' < \varepsilon_1 \), any \( x' \) solving (P'), and any \( u' \) solving (D'), one has \( d([x', u'], S_P \times S_D) \leq \gamma \varepsilon' \), where \( S_P \) and \( S_D \) are the sets of optimal solutions for (P) and (D), respectively.

**Proof.** If (P) and (D) are not both solvable, then by the duality theorem of linear programming ([1], Theorem 4.6) at least one must be infeasible; thus (a) and (b) are both false. To prove their equivalence for the case in which (P) and (D) are both solvable, it suffices to prove that the set of solutions of (D) is bounded if and only if \( b \in \text{int} \{ A(P) - Q^b \} \). The corresponding result for the solution set of (P) then follows by symmetry.

We note first that the (translated) perturbation function \( f(y) = \inf_x \{ \langle c, x \rangle | Ax - y \in Q^b, x \in P \} \) associated with (P) is a proper convex function since (P) and (D) are solvable, and that the effective domain of \( f \) is \( A(P) - Q^b \). By Theorem 23.4 of Rockafellar [20] the set of subgradients \( \partial f(b) \) is nonempty and bounded if and only if \( b \in \text{int} \{ A(P) - Q^b \} \). However, it is well known (and not hard to prove) that the set of solutions of (D) is precisely \( \partial f(b) \), and this establishes the equivalence of (a) and (b). The equivalence of (a) and (c) is shown by applying Lemma 3 to the constraints of (P) and, mutatis mutandis, to those of (D) and then noting that the feasibility of both (P') and (D') is equivalent, by the duality theorem, to their solvability. This proves the first part of the theorem.

Now suppose that the three equivalent conditions in the statement of the theorem hold. To prove the final assertion, we note that the set of
points \((x, u)\) solving \((P)\) and \((D)\) is the solution set of the linear system
\[
\begin{bmatrix}
0 & -A^T \\
A & 0 \\
c & -b^T
\end{bmatrix}
\begin{bmatrix}
x \\
0
\end{bmatrix}
- \begin{bmatrix}
-c^T \\
b \\
0
\end{bmatrix}
\in \begin{bmatrix}
P^* \\
Q^* \\
\{0\}
\end{bmatrix},
\]
and that this set is bounded by hypothesis. As the set of points \((x', u')\)
solving \((P')\) and \((D')\) is the solution set of the system obtained from \((6)\)
by replacing \(A, b,\) and \(c\) by \(A', b',\) and \(c',\) respectively, we can invoke
Lemma 2 to complete the proof.

We point out that the equivalence of (a) and (b) in Theorem 1 is well
known; for the linear programming problems considered by Williams it
coincides with his Theorem 3, for which he cites p. 49 of Goldman [11].
It is easy to show by using a separation argument that his regularity
conditions \((R1)\) and \((R2)\) are equivalent, for the special \(P\) and \(Q\) which
he employed, to the interior conditions used here. Also, results of this kind
are known in the more general context of convex programming; see Corol-
lary 29.1.5 of Rockafellar [20]. Nevertheless, we have included this equiva-
lence in the statement of the theorem for the sake of completeness and
because (b) provides a very convenient test for regularity.

Theorem 1 shows that linear programming problems that satisfy the
regularity conditions given earlier are quite well behaved under small
perturbations in the data. The solution sets are not necessarily continuous,
but they are upper Lipschitz continuous in the sense defined earlier. For
previous results about upper semicontinuity, see Dantzig, Folkman, and
Shapiro [5] and Evers [8, 9]. The results of Evans and Gould [7], of Green-
berg and Pierskalla [12], and of Stern and Topkis [21] are also related to
the present work, but require, e.g., the stronger assumption that the
primal feasible set be bounded. Another related work is the recent paper
of Böhm [3], which establishes the continuity of the primal optimal set
under certain restricted perturbations of the right-hand side. Finally,
Martin [15] has studied the continuity of the optimal value (not the
optimal set) under boundedness conditions similar to those used here.

Since linear programming problems satisfying the regularity conditions
are well behaved, one might reasonably ask whether problems that do not
satisfy those conditions will be badly behaved. This question is answered
in part by the regularity conditions themselves, as it is clear that if, say,
the constraints of \((P)\) are singular then arbitrarily small changes in \(b\)
alone can render \((P)\) infeasible, so that the problem will have no solution.
A similar conclusion holds for \((D)\) with respect to changes in \(c\). However,
we shall now show that another kind of instability is also present when-
ever the regularity conditions are not satisfied: If the constraints of (P) are singular, then the common optimal value of the problems (P) and (D) can be made to "jump" up to any objective value attained on the primal feasible set, by arbitrarily small changes in one row of $A$ and in the corresponding element of $b$. We state this result formally in the next theorem. The technique used in the proof is an adaptation of that used in Theorem 4 of [18].

**Theorem 2.** Suppose that (P) and (D) are solvable but that the constraints of (P) are singular. Let $x_1$ be any feasible point of (P) and $\epsilon$ be any positive number. It is then possible to modify a single element of $b$ and the entries in the corresponding row of $A$ by amounts less than $\epsilon$ in absolute value, leaving $c$ unaltered, to produce a new dual pair of problems whose common optimal value is $\langle c, x_1 \rangle$.

**Proof.** By hypothesis, $b$ must lie on the boundary of the convex cone $A(P) - Q^e$. By Corollary 11.6.1 of Rockafellar [20], there is a nonzero (outward) normal $w$ to $A(P) - Q^e$ at $b$, and it is easily checked that since $\langle w, y \rangle \leq \langle w, b \rangle$ for each $y \in A(P) - Q^e$, we have $w \in Q$, $wA \in -P^*$ and $\langle w, b \rangle = 0$ (compare the proof of Lemma 3; the set of all such vectors is in fact just the recession cone of $S_D$). Choose any $i$ with $w_i \neq 0$; we can normalize $w$ so that $|w_i| = 1$. Let $x$ and $\epsilon$ be chosen, and choose some dual feasible point $u_1$. Let $\gamma$ be a real number having the same sign as $w$, and so small in absolute value that

$$|\gamma| \max \{|\langle c, x_1 \rangle|, |c_1|, \ldots, |c_n|, \epsilon |(u_1)|, |\| < \epsilon.$$ 

Let $e_i$ be a vector in $\alpha_i^{m}$ with one in its $i$th component and zero elsewhere; define $A^* = A + \gamma e_i$, $b^* = b + \gamma(c, x_1)e_i$, and $c^* = c$. Thus only the $i$th element of $b$ and the entries in the $i$th row of $A$ have been changed, and the absolute values of the perturbations are all less than $\epsilon$. Since $1 + \gamma(u_1), > 0$, the point $u_2 = (1 + \gamma(u_1))^{-1}u_1$ belongs to $Q$, and we have

$$c^* - u_2A^* = (1 + \gamma(u_1))^{-1}[c^* + \gamma(u_1)c^* - u_1A^*]$$

$$= (1 + \gamma(u_1))^{-1}[c + \gamma(u_1)c - u_1A - \gamma(u_1)c]$$

$$= (1 + \gamma(u_1))^{-1}(c - u_1A) \in P^*.$$ 

Hence $u_2$ is feasible for (D'). Also,

$$A'x_1 - b' = A x_1 + \gamma(c, x_1)e_i - b - \gamma(c, x_1)e_i = Ax_1 - b \in Q^e;$$

thus $x_1$ is feasible for (P'). Hence by the duality theorem (P') and (D') are both solvable. Now let $x$ be any feasible point for (P'). Then $A'x - b' \in Q^e$ and $x \in P$, so $Ax = (A'x - b') \in A(P) - Q^e$, and therefore $\langle w, b \rangle \geq \langle w, Ax - (A'x - b') \rangle = \langle w, b \rangle + \gamma(w, e_i)[(c, x_1) - \langle c, x \rangle]$. As $\gamma(w, e_i) > 0$, we have $\langle c, x \rangle \geq \langle c, x_1 \rangle$, which completes the proof.
It is clear that an analogous result could be established if the constraints of \((D)\) were singular. In that case we could have made the optimal objective value jump downward by modifying one column of \(A\) and the corresponding element of \(c\) by arbitrarily small amounts. In a way, this kind of instability seems even worse than the infeasibility mentioned earlier since the perturbed system remains solvable, even though very inaccurate results may be forthcoming.

A simple example of an unstable problem is provided by

\[
\begin{align*}
\min \ [3 & \ 1 & \ 1 & \ 3] x \\
[1 & \ 4/3 & \ 2 & \ 0] x = \begin{bmatrix} 3/2 \\ 3/2 \\ 1 \end{bmatrix}, \quad x \geq 0.
\end{align*}
\]

A pair of primal and dual optimal solutions, corresponding to the optimal objective value of 1, is \([0 \ 3/4 \ 1/4 \ 0]^T\) and \([1/2 \ -1/6 \ 1/2]\). However, if the number \(4/3\) in the coefficient matrix is decreased to \(4/3 - \epsilon\), for some \(\epsilon > 0\), then the optimal objective value jumps up to 2 and an optimal pair is \([1/2 \ 0 \ 1/2 \ 0]^T\) and \([0 \ -2/3 \ 3] + \epsilon^{-1}[4/3 \ -4/9 \ -4/3]\). For example, rounding \(4/3\) to 1.3333333 for entry into an eight-digit computer corresponds to taking \(\epsilon = \frac{1}{3} \times 10^{-5}\). Here the difficulty is that the primal constraints are not regular (although the dual constraints are, since the primal optimal set is bounded).

We have now established that linear programming problems with regular constraints will behave well under small perturbations, and that those with singular constraints can behave very badly indeed. In the next section we shall interpret the regularity conditions economically in the context of an activity-analysis model, and we shall try to argue that, at least in this case, these conditions express economically sensible properties that one could reasonably expect a real model to have.

2. REGULARITY FOR AN ACTIVITY-ANALYSIS MODEL

We shall consider a firm producing \(m\) goods and consisting of \(n\) linear activities, each represented by a column of the \(m \times n\) matrix \(A\). Let \(c \in \mathbb{R}^n\) be a vector whose \(j\)th component is the cost of operating activity \(j\) at unit level, and suppose that the \(i\)th component of the vector \(b \in \mathbb{R}^m\) represents the amount of the \(i\)th good that is required to be produced. If we now take \(P = \partial^T\) and \(Q = \partial^{T*}\), then \((P)\) is the problem of selecting non-negative activity levels \(x_1, \ldots, x_n\) to fulfill the required production at least cost and \((D)\) is the problem of assigning non-negative prices \(u_1, \ldots, u_m\) to the goods \(b\), in such a way as to maximize the value of the required output while simultaneously ensuring that no activity makes a positive profit. Models such as this one, of course, are ubiquitous
in economic theory (see, e.g., Dorfman, Samuelson, and Solow [6], Gale [10], Koopmans [14]). Our purpose here is to examine the regularity conditions that we have given in the preceding section to see what economic interpretation they may have in terms of this model.

To begin with, we can interpret very easily the primal regularity condition \( b \in \text{int} \{ A(P) - Q^* \} \). It simply means that the firm must be able to produce, if required, a bill of goods strictly larger than \( b \); that is, we must be able to find some non-negative \( x \in \mathbb{R}^n \) with \( Ax > b \). This capability is evidently sufficient for regularity. That it is necessary follows since if \( b \in \text{int} \{ A(P) - Q^* \} \) then for any \( b' \) close enough to \( b \) we have \( b' \in A(P) - Q^* \); in particular, we may take \( b' > b \). This condition is extended to general ordering cones \( Q^* \) (rather than the cone \( Q^* = \alpha \mathbb{R}^n \) used here) in Theorem 2 of [18]. Put another way, regularity of the primal constraints means in this case that the requirement to produce \( b \) does not strain the firm’s productive capacity to its limit.

We could analyze the regularity of the dual constraints in a similar manner, but it may be more illuminating to interpret it in another way. We first observe that the condition required for this regularity, that \( c \) belong to the interior of \( (Q)A + P^* \), is equivalent to the statement that \( c \) cannot be separated from \( (Q)A + P^* \) by a hyperplane; that is, that there exists no nonzero \( x \in P \) with \( Ax \in Q^* \) and \( \langle c, x \rangle \leq 0 \). Put in the language of our problem, this says that for any semipositive (non-negative but not zero) vector \( x \) with \( Ax \geq 0 \), we must have \( \langle c, x \rangle > 0 \). This is the condition denoted by (R2) in [22]; translated into our notation. Now, the economic meaning of this is transparent: it is that for any production plan \( x \) that produces a non-negative bill of goods (i.e., that results in no net consumption), either \( x \) is the trivial (zero) plan or else \( x \) has a positive cost in terms of \( c \). Both this condition and that for primal regularity are thus seen to have economically meaningful interpretations, and it does not seem unreasonable (at least in this case) to expect that economic systems that one might wish to model by linear programming should satisfy these conditions.

One might also ask the following question: what conditions must one assume on \( A \) in order to ensure that the dual constraints will be regular for every cost vector \( c \in \alpha \mathbb{R}^n \)? It is easy to see that such regularity will hold if and only if the cone \( (\alpha \mathbb{R}^n)A + \alpha \mathbb{R}^n \) is the whole space \( \mathbb{R}^n \), since only then can each \( c \in \alpha \mathbb{R}^n \) lie in the interior of the cone. This condition, in turn, is precisely equivalent to the satisfaction of both Postulates \( A \) and \( B \) of [14], namely, the irreversibility of production (there is no semipositive \( x \) with \( Ax = 0 \)) and the impossibility of the land of Cockaigne (there is no non-negative \( x \) for which \( Ax \) is semipositive). The proof of this equivalence is easily carried out by separation arguments (theorems of the alternative), and we shall not give it here. We may also remark
that the dual regularity condition used in this paper (i.e., for a fixed \( c \)) can also be formulated in terms of Koopmans’ Postulates \( A \) and \( B \) by adjoining an additional good (money) to the model and considering the augmented matrix \( A_c = [A, -c]^T \). One verifies easily that dual regularity is equivalent to the satisfaction of Postulates \( A \) and \( B \) for the matrix \( A_c \).

One can analyze the regularity of the primal constraints in terms of similar postulates placed upon price systems for goods in the model. In this case the primal constraints will be regular for each right-hand side \( b \) if and only if there exists no semipositive price system \( p \) under which the total output of every activity of the model will have a non-positive value (i.e., \( pA \leq 0 \)). Regularity of the primal constraints for a fixed \( b \) can be expressed by adding another activity (the external world) to the model and requiring that the above condition on price systems be satisfied for the matrix \( A_b = [A, -b] \). This is Williams’ condition (R1). However, our earlier interpretation of primal regularity in terms of production capability seems at least as easily understandable as is this requirement. The former condition is closely related to Postulate \( C_1 \) of Koopmans [14], the difference being accounted for by the fact that in our simple model we have prescribed the levels of output of all final commodities, whereas Koopmans, in his more general analysis, has not.

3. CONCLUSION

We have presented necessary and sufficient conditions for stability of a linear programming problem. We have suggested that in our activity-analysis example, stability is “natural” in the sense that it corresponds to intuitively reasonable properties of the production system giving rise to the model. However, as we have previously shown [18], a regular system can be made singular by improper formulation, and this would destroy any stability that might otherwise be present. Examples of this “artificial singularity” can be found in the practice of writing an equation as two opposite inequalities (which must then be singular), or in replacing an unconstrained variable by the difference of two non-negative variables (equivalent to replacing an equation in the dual constraints by two opposite inequalities). Some computational packages for linear programming require such transformations in order to convert a problem into a prescribed form. It is not clear to what extent this has contributed to actual computational difficulties, as the numerical method of solution would then play a critical role in determining whether the potential for error, which we have shown to be present, would actually be realized. Nevertheless, as the computational modifications necessary to accommodate, e.g., unconstrained variables, are extremely slight, there would seem to be no reason not to allow for them.

One might also pose the following question: given a linear programming
problem that does not satisfy the regularity conditions given here, can it be "regularized" by rewriting it in an equivalent regular form? A moment’s reflection will make it clear that this cannot in general be done if we regard both primal and dual optimal sets as prescribed. Singularity implies that at least one of the optimal sets is unbounded, and we cannot regularize the problem without changing that set. However, if we concentrate on the primal problem and suppose that we are considering such a problem with a bounded optimal set but singular constraints, then a regularization can be carried out provided that the primal feasible set meets the relative interior of the cone $P$. In that case, we can apply Theorem 3 of [18] to show that we can rewrite the primal constraints in a regular way, without changing the primal feasible set, by changing certain inequalities to equations and then deleting redundant equations. As the primal feasible set is unchanged, the corresponding optimal set is still bounded. Hence the dual constraints are regular by Theorem 1, and as the primal constraints are now also regular, the linear programming problem will be stable in the sense described in this paper.

For previous work on regularization in linear programming, see Charnes, Cooper, and Thompson [4] who, however, emphasized boundedness of the feasible (rather than the optimal) sets.

ACKNOWLEDGMENT

The research reported here was sponsored in part by the United States Army under Contract DA-31-124-ARO-D-462 and in part by the National Science Foundation under Grant DCR74-20584.

Note Added in Proof: Another recent paper presenting results related to those derived here is: B. Berceanu, "The Continuity of the Optimum in Parametric Programming and Applications to Stochastic Programming,” J. Optimization Theory Appl. 18, 319–333 (1976), in which continuity of the optimal value is studied under regularity conditions which are specializations of those used in this paper.

REFERENCES
