C&O 463/663 Convex Optimization and Analysis (Fall 2009)

Assignment 4 - Additional Problems

Instructor: Dr. Henry Wolkowicz
Due Date: Thursday, Dec. 3, 2009

1 Conjugate Duality

1. (text page 57) Suppose that $f, g : \mathbb{E} \to (-\infty, +\infty]$ are convex and $(f \odot g)(y) := \inf_x \{f(x) + g(y - x)\}$ is the *infimal convolution*.

   (a) Prove that $f \odot g$ is convex. (On the other hand, if g is concave prove that so is $f \odot g$.)

   (b) Prove $(f \odot g)^* = f^* + g^*$.

2. Generalize and prove the above two problems 1a and 1b to the infimal convolution $h(y) = \inf \left\{ \sum_{i=1}^k f_i(x_i) : \sum_{i=1}^k x_i = y \right\}$. (Thus the operations $+$ and $\odot$ are dual to each other with respect to taking conjugates.)

2 Entropy Minimization

Define the (Boltzmann-Shannon) entropy function on $\mathbb{R}$ and $\mathbb{R}^n$, by

$$p(t) := \begin{cases} t \ln t - t & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ +\infty & \text{if } t < 0, \end{cases}$$

and $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x) := \sum_{i=1}^n p(x_i),$$

respectively. Define the entropy minimization problem

$$(PE) \qquad \begin{array}{cc} \min & f(x) + \langle c, x \rangle \\ \text{subject to} & Ax = b, \end{array}$$

where f is as above, $c \in \mathbb{R}^n$, A is a $m \times n$ matrix, and $b \in \mathbb{R}^m$. Show that:

1. for any $c \in \mathbb{R}^n$, the function $f(x) + \langle c, x \rangle$ is strictly convex on $\mathbb{R}^n_+$ and has compact sublevel (level) sets;

2. for any $\bar{x} \in \operatorname{int}(\mathbb{R}^n_+)$ and $x \in \operatorname{bd}(\mathbb{R}^n_+)$ (the boundary), the directional derivative $f'(x; \bar{x} - x) = -\infty$.

3. if there exists a $x \in \operatorname{int}(\mathbb{R}^n_+)$ such that $Ax = b$, then problem (PE) has a unique solution $\bar{x} \in \mathbb{R}^n$ determined by the conditions on the components of $\bar{x}$,

$$\bar{x}_j = \exp(A^T \bar{\phi} - c)_j, \quad j = 1, \dots, n,$$

where $\bar{\phi}$ is a solution to the dual problem

$$\max_{\phi \in \mathbb{R}^m} \left\{ \langle \phi, b \rangle - (f + c)^*(A^T \phi) \right\}.$$

(Is $\bar{\phi}$ a Lagrange multiplier for problem (PE)? Why or why not?)

3 BONUS Questions

1. (*DAD* Problem) The matrix A is doubly stochastic if it is nonnegative, $A \geq 0$, and both its row and columns sums are 1, $Ae = e, A^T e = e$. The matrix $A \geq 0$ has a *doubly stochastic pattern* if there is a doubly stochastic matrix with exactly the same zero entries as A.
Show that $A \geq 0$ $n \times n$ has a doubly stochastic pattern if and only if there exist diagonal matrices $D_i, i = 1, 2$ with strictly positive diagonal entries such that $D_1 A D_2$ is doubly stochastic.
Hint: For necessity, use the following program to find the desired doubly stochastic matrix $B = D_1 A D_2$:

$$\begin{array}{ll} \min & \sum_{ij: A_{ij} > 0} \left(p(X_{ij}) - X_{ij} \log A_{ij} \right) \\ \text{subject to} & \sum_{i: A_{ij} > 0} X_{ij} = 1, \quad \text{for } j = 1, \dots n \\ & \sum_{j: A_{ij} > 0} X_{ij} = 1, \quad \text{for } i = 1, \dots m, \end{array}$$

where p is the (Boltzmann-Shannon) entropy function defined above.

2. (Fenchel vs Lagrangian Duality) Consider the abstract convex program defined in class

$$(CP) \qquad \begin{array}{ll} \min & f(x) \\ \text{subject to} & g(x) \preceq_K 0 \\ & x \in \Omega, \end{array}$$

where K is a ccc, Ω is a convex set, f is a finite valued convex function, and g is $K-$convex. Can you find an example of (CP) where strong duality holds for the Lagrangian dual and not the Fenchel dual? What about the converse, i.e. it holds for the Fenchel dual and not the Lagrangian dual? Why and/or why not? Do the dual optimal values make sense in these examples?
(For the Fenchel dual, use the indicator function of the feasible set, i.e. $f_1 \cong f$ and $f_2 \cong \mathbf{I}_S$, where S is the feasible set, and the use the primal problem $\inf\{f_1(x) + f_2(x)\}$.)