C&O 463/663 Convex Optimization and Analysis (Fall 2009)

Assignment 3 - Additional Problems

Instructor: Dr. Henry Wolkowicz
Due Date: Thursday, Nov. 12, 2009

1 Convex Functions, Convex Sets, Fenchel Conjugates

Suppose that f is a proper convex function from a Euclidean space $\mathbb{E}$ to $\mathbb{R} \cup +\infty$.

1. Show that the following conditions are equivalent.

   (a) f is lower semi-continuous on $\mathbb{E}$.

   (b) The level set $L_\alpha = \{x \in \mathbb{E} : f(x) \leq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.

   (c) The epigraph of f is a closed set.

2. Let $f(x) = |x|$ be the absolute value function on $\mathbb{R}$. Find the conjugate f^*.

3. Let $\mathbb{E}$ and $\mathbb{Y}$ be Euclidean spaces, let $f : \mathbb{E} \to (-\infty, +\infty]$ and $g : \mathbb{Y} \to (-\infty, +\infty]$, and let $A : \mathbb{E} \to \mathbb{Y}$ be a linear map. Consider the *constraint qualification* type condition
$$0 \in \operatorname{int}(\operatorname{dom} g - A\operatorname{dom} f). \tag{1}$$
   Show that
$$\partial(f + g \circ A)(x) \supset \partial f(x) + A^* \partial g(Ax), \forall x \in \mathbb{E},$$
   with equality if (1) holds.

4. Suppose that the set $S \subseteq \mathbb{R}^n$ is open and convex, and consider a function $f : S \to \mathbb{R}$. For points $x \notin S$ define $f(x) = +\infty$.

   (a) Prove $\partial f(x)$ is nonempty for all $x \in S$ if and only if f is convex.

   (b) Prove that a continuous function $h : \operatorname{cl} S \to \mathbb{R}$ is convex if and only if its restriction to S is convex. What about strictly convex functions?

5. If the function $f : \mathbb{R}^2 \to (-\infty, +\infty]$ is defined by
$$f(x_1, x_2) = \begin{cases} \max\{1 - \sqrt{x_1}, |x_2|\} & \text{if } x_1 \geq 0 \\ +\infty & \text{otherwise,} \end{cases}$$
   prove that f is convex but that $\operatorname{dom} \partial f$ is not convex.

6. Suppose that $f : \mathbb{E} \to \mathbb{R}$ is differentiable, not necessarily convex, and bounded below. Show that for each positive integer n, there exists $x_n \in \mathbb{E}$ such that $\|\nabla f(x_n)\| \leq \frac{1}{n}$. Can you get a similar result if f is convex but not necessarily differentiable?

7. (a) Suppose that $K = \{x \in \mathbb{R}^n : x_1 \geq \sqrt{x_2^2 + \ldots + x_n^2}\}$, the second order cone. Show that $N_K(0) = -K$.

   (b) Suppose that $K = \mathcal{S}^n_+$, the cone of positive semidefinite matrices. Show that $N_K(0) = -K$.

2 Convex Optimization Problems

1. Let A and B be non-empty compact convex subsets of $\mathbb{E}$. Use Fenchel duality to show that
$$\min_{x\in A}\max_{y\in B}\langle x, y\rangle = \max_{y\in B}\min_{x\in A}\langle x, y\rangle$$