C&O 463/663 Convex Optimization and Analysis (Fall 2009)

Assignment 2 - Additional Problems

Instructor: Dr. Henry Wolkowicz
Due Date: Tuesday, Oct. 20, 2009

1 Convex Functions and Convex Sets

1. Show that the image and the inverse image of a convex cone under a linear transormation is a convex cone. Is this true for an affine transformation? Why or why not?

2. Let $\emptyset \neq C \subseteq \mathbb{E} = \mathbb{R}^n$ be a convex set. Suppose that f is a convex function on $\mathbb{E}$ with $C \subset \operatorname{dom} f$, and that $g : \mathbb{R} \to \mathbb{R}$ is a convex function that is monotonically nondecreasing over the convex hull, $\operatorname{conv}\{f(x) : x \in C\}$. Show that the composite function $h(x) := g(f(x))$ is convex over C. In addition, if g is monotonically increasing and f is strictly convex, then h is strictly convex.
   Recall: h is *strictly convex* on D if
   $$h(\lambda x + (1-\lambda)y) < \lambda h(x) + (1-\lambda)h(y), \quad \forall 0 < \lambda < 1, \forall x, y \in D.$$

3. (Characterizations of Convex Functions) Suppose that $f : \mathbb{E} \to \mathbb{R} \cup +\infty$ and C is an open set satisfying $C \subseteq \operatorname{dom} f$. Moreover, assume sufficient differentiability for f as needed. Show that the following are equivalent:

   (a) (zero order conditions) f is convex on C, i.e.
   $$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall 0 \leq \lambda \leq 1, \forall x, y \in C.$$

   (b) (first order condition)
   $$\nabla f(x)^T(y-x) \leq f(y) - f(x), \quad \forall x, y \in C$$

   (c) (first order condition)
   $$(\nabla f(y) - \nabla f(x))^T(y-x) \geq 0, \quad \forall x, y \in C$$

   (d) (second order condition)
   $$\nabla^2 f(x) \succeq 0, \quad \forall x \in C$$

4. Suppose that $K \subseteq \mathbb{E}$. Show that K is a closed convex cone if and only $K = (K^-)^-$.
   Hint: Use the hyperplane separation theorem for the difficult part.

2 Convexification

Let $\emptyset \neq X \subseteq \mathbb{E}$ and let f be defined on $\mathbb{E}$ and bounded below on X. Let $F := \mathrm{conv}\,(f)$. Show that

$$\inf_{\mathrm{conv}\,(X)} F(x) = \inf_X f(x),$$

and, moreover,

$$x^* \in \mathrm{argmin}\,_X f(x) \implies x^* \in \mathrm{argmin}\,_{\mathrm{conv}\,(X)} F(x).$$

3 Subgradients

1. (Subgradients of maximum eigenvalue) Prove

$$\partial \lambda_{\max}(0) = \{Y \in \mathcal{S}^n_+ : \mathrm{tr} Y = 1\}$$

2. * Define a function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, x_2, \ldots, x_n) = \max_j\{x_j\}$. Let $\bar{x} = 0$ and $d = (1, 1, \ldots, 1)^T$, and let $e_k = (1, 1, \ldots, 1, 0, 0, \ldots, 0)^T$ (ending in $k-1$ zeros). Calculate the functions p_k defined in the proof of Theorem 3.1.8 (Max formula), using Proposition 2.3.2 (directional derivatives of max functions). (The theorem and proposition are from the text.)