CO350 Linear Programming Chapter 8: Degeneracy and Finite Termination

27th June 2005

Recap

The perturbation method

$$(P) \quad \begin{array}{rrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Assumption: B is a feasible basis with $A_B = I$.

(P'): perturb b to $b' = b + [\varepsilon, \varepsilon^2, \ldots, \varepsilon^m]^T$.

Theorem 8.3 (pg 111)

(a) (P') is nondegenerate.

(b) Feasible bases of (P') are feasible bases of (P).

(c) Optimal bases of (P') are optimal bases of (P).

(d) Valid pivots for (P') are valid for (P).

(e) Bases detecting (P') unbounded also detects (P) unbounded.

Corollary 8.3 (pg 112)

The simplex method applied to (P') starting from B will terminate after a finite number of iterations.

Moreover, B' optimal for $(P') \implies B'$ optimal for (P),

and (P') unbounded $\implies (P)$ unbounded.

Recap

The lexicographical simplex method

Simplex method on (P) with a special choice rule for leaving variables that mimics the choice of leaving variables for (P').

This special rule is called the lexicographical rule.

The lexicographical rule:
pick the lexicographical minimum of

$$\frac{(\bar{b}_i, \beta_{i1}, \beta_{i2}, \ldots, \beta_{im})}{\bar{a}_{ik}} \qquad \text{over} \qquad \{i \in B : \bar{a}_{ik} > 0\}$$

where $[\beta_{i1}, \beta_{i2}, \ldots, \beta_{im}]$ is the h-th row of the matrix A_B^{-1} and i is the h-th index in the basis B.

A shortcut: A_B^{-1} appears in the tableau corresponding to B as columns indexed by the initial basis B'.

Smallest-subscript rules (§8.5)

Recall: **smallest subscript rule for entering variable**
"Pick the nonbasic variable with the least subscript among those with positive reduced cost".

Similar rule for leaving variable:
"Pick the basic variable with the least subscript among those that tie for minimum ratio".

Combining them gives the smallest-subscript rules:

1. Among the nonbasic x_j with $\bar{c}_j > 0$, pick the one for which j is the least.
2. Suppose x_k is entering. Among the basic x_i with
$$\bar{a}_{ik} > 0 \quad \text{and} \quad \bar{b}_i/\bar{a}_{ik} = \text{min. ratio},$$
pick the one for which i is the least.

Amazingly, using these rules also prevent cycling!
(Robert Bland)

These are also known as Bland's rules.

Lexicographical vs Smallest-subscript (Not in notes)

- Choice rule for entering variables.

  Lexicographical rule: may use ANY choice rule for entering variables.

  Smallest-subscript rules: can only use the smallest subscript rule for entering variables.

- Simplicity of implementation.

  Lexicographical rule: requires lexicographical comparisons.

  Smallest-subscript rules: simplest rule to implement.

Example (Not in notes)

Solve using simplex method with smallest-subscript rules.

Initial tableau:

$$
\begin{array}{rrrrrrrrrrrrrrr}
z & - & x_1 & - & 2x_2 & & & & & & & & & = & 0 \\
 & & x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & = & 4 \\
 & & 2x_1 & + & 4x_2 & + & 6x_3 & & & + & x_5 & & & = & 6 \\
 & & x_1 & + & 3x_2 & + & 3x_3 & & & & & + & x_6 & = & 3
\end{array}
$$

$\bar{c}_1$ and $\bar{c}_2$ are only positive reduced costs, so x_1 enters. $\min\{4/1, 6/2, 3/1\} = 3$ with x_5 and x_6 tied for min. ratio, so x_5 leaves. Pivot on $(5, 1)$:

$$
\begin{array}{rrrrrrrrrrrrrr}
z & & & & + & 3x_3 & & & + & \frac{1}{2}x_4 & & & = & 3 \\
 & & - & x_2 & - & 2x_3 & + & x_4 & - & \frac{1}{2}x_5 & & & = & 1 \\
 & x_1 & + & 2x_2 & + & 3x_3 & & & + & \frac{1}{2}x_5 & & & = & 3 \\
 & & & x_2 & & & & & - & \frac{1}{2}x_5 & + & x_6 & = & 0
\end{array}
$$

This tableau is optimal.

The Fundamental Theorem of Linear Programming (§8.6)

There are three versions of the fundamental theorem.

Theorem 8.5. Suppose (P) is an LP problem in SEF, where A is $m \times n$ with rank m. Then

(a) (P) has a feasible solution $\implies$ it has a b.f.s.

(b) (P) has an optimal solution
$\implies$ it has an optimal basic solution.

(c) (P) has feasible solution but no optimal solution
$\implies$ (P) unbounded.

Note: part (b) is Theorem 5.5.
part (a) can be proved similarly.

Theorem 8.6. Suppose (P) is an LP problem in SIF. Then

(a) (P) has a feasible solution $\implies$ it has a b.f.s.

(b) (P) has an optimal solution
$\implies$ it has an optimal basic solution.

(c) (P) has feasible solution but no optimal solution
$\implies$ (P) unbounded.

Theorem 8.7. Suppose (P) is an LP problem in general form. Then either

- (P) has an optimal solution,
- (P) is infeasible, or
- (P) is unbounded.

Proof:

Suppose (P) has feasible solution but no optimal solution.

Convert (P) to SIF (P').

So (P') has feasible solution but no optimal solution.

Theorem 8.6(c) $\implies$ (P') is unbounded.

So (P) is unbounded. ■

Theorem 8.5. Suppose (P) is an LP problem in SEF, where A is $m \times n$ with rank m. Then

(a) (P) has a feasible solution $\implies$ it has a b.f.s.

(b) (P) has an optimal solution
$\implies$ it has an optimal basic solution.

(c) (P) has feasible solution but no optimal solution
$\implies$ (P) unbounded.

Proof:

(a) Construct auxiliary problem (A) of (P).

Apply lexicographical simplex method to get an optimal basis of (A).

(P) has feasible solution $\implies$ (A) has optimal value 0.

From optimal basis of (A), build feasible basis of (P).

(b) From feasible basis of (P) in part (a), start lexicographical simplex method to get an optimal basis of (P).

(c) From feasible basis of (P) in part (a), start lexicographical simplex method to conclude that (P) is unbounded. ■

Theorem 8.6. Suppose (P) is an LP problem in SIF. Then

(a) (P) has a feasible solution $\implies$ it has a b.f.s.

(b) (P) has an optimal solution
$\implies$ it has an optimal basic solution.

(c) (P) has feasible solution but no optimal solution
$\implies$ (P) unbounded.

Proof: Convert (P) to SEF (P').

(a) (P) has feasible solution
$\implies$ (P') has feasible solution
$\implies$ (P') has b.f.s. (by Thm 8.5(a))
$\implies$ (P) has b.f.s. (by definition)

(b) (P) has optimal solution with value v^*
$\implies$ (P') has optimal solution with value v^*
$\implies$ (P') has b.f.s. with value v^* (by Thm 8.5(b))
$\implies$ (P) has b.f.s. with value v^* (by definition)
$\implies$ (P) has optimal basic solution

(c) (P) has feasible solution but no optimal solution
$\implies$ (P') has feasible solution but no optimal solution
$\implies$ (P') is unbounded (by Thm 8.5(c))
$\implies$ (P) is unbounded ∎

Proof of Duality Theorem (§8.7)

Theorem 8.8

(Duality Theorem of Linear Programming)

(P) in SEF has an optimal solution $\hat{x}$
$\Longrightarrow$ its dual (D) has an optimal solution $\hat{y}$ with $c^T\hat{x} = b^T\hat{y}$.

Proof: Case 1, A has rank m.

Two-phase method on (P) with lexicographical rule gives optimal basis B with optimal basic solution x^*.

Let $\hat{y}$ be solution of $A_B^T y = c_B$.

We learned in Chapter 6 that $\hat{y}$ is feasible for (D) and satisfies C.S. condition with x^*.

So $\hat{y}$ is optimal for (D) and $c^T x^* = b^T\hat{y}$.

Finally $c^T\hat{x} = c^T x^* = b^T\hat{y}$.

Proof of Duality Theorem (§8.7)

Theorem 8.8

(Duality Theorem of Linear Programming)

(P) in SEF has an optimal solution $\hat{x}$

$\Longrightarrow$ its dual (D) has an optimal solution $\hat{y}$ with $c^T\hat{x} = b^T\hat{y}$.

Proof: <u>Case 2, A has rank $< m$.</u>

Let R be the set of indices of redundant rows.

Let $\hat{A}x = \hat{b}$ be the derived from $Ax = b$ by removing the rows indexed by R.

Then $rank(\hat{A}) = \#$ rows in $\hat{A}$.

Consider

$$(P') \quad \begin{array}{lrcl} \text{max} & c^T x & & \\ \text{s.t.} & \hat{A}x & = & \hat{b} \\ & x & \geq & 0 \end{array}$$

and its dual LP (D').

By Case 1, $\exists y'$ optimal for (D') with $b^T y' = c^T\hat{x}$.

Construct a feasible solution $\hat{y}$ for (D) by

$$\hat{y}_i = \begin{cases} 0 & \text{if } i \in R, \\ y'_k & \text{if } i\text{-th row of } A = k\text{-th row of } \hat{A}. \end{cases}$$

Then $b^T\hat{y} = c^T\hat{x}$

$\Longrightarrow$ $\hat{y}$ optimal for (D). ■

Review of Part IV: Fundamental Theorem of LP

Upon successful completion of the two-phase method, we will conclude that an LP in SEF

- is infeasible,
- has an optimal solution, or
- is unbounded.

To ensure successful completion of simplex method, we need to prevent cycling.

Two sets of rules that prevents cycling:

- Lexicographical rule (for leaving variables)
- Smallest-subscript rules (for both entering and leaving variables)

Upon successful completion of two-phase method with optimal x^*, we have optimal solution $\hat{y}$ for the dual LP that satisfies

$$c^T x^* = b^T \hat{y}$$

This proves the Duality Theorem.