A Strengthened Barvinok-Pataki Bound on $\mathbb{SDP}$ Rank

(how to take advantage of facial reduction AGAIN)

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Sunday, Dec. 5, 2021, 14:00-14:30, ET

CMS W21

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Outline

- The Barvinok-Pataki bound provides an upper bound on the rank of extreme points of a spectrahedron (intersection of SDP cone $\mathbb{S}^n_+$ and a linear manifold)
- bound depends solely on algebra of problem: triangular number of the rank r,

  $t(r) \leq m$, the number of affine constraints.
- We provide a strengthened upper bound on rank using the singularity degree of the spectrahedron.
- Thus we bring in the geometry and stability of the spectrahedron, i.e., paradox?:

  increased instability, as seen by higher singularity degree, yields a lower, strengthened rank bound.

Background/Notation

- Semidefinite programming, SDP, over symmetric matrices

$$p^* = \begin{array}{lll} \min_{X \in \mathbb{S}^n} & f(X) & (f : \mathcal{S}^n \to \mathbb{R}) \\ \text{s.t.} & \mathcal{A}(X) = b & (\in \mathbb{R}^m) \\ & X \succeq 0 & (X \in \mathbb{S}^n_+) \end{array} \tag{1}$$

- the spectrahedron (feasible set/intersection of an affine set and the positive semidefinite cone) is:

$$\mathcal{F} = \{X \succeq 0 : \mathcal{A}(X) = b\}.$$

- onto linear map $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$; $\quad \mathcal{A}(X) = (\langle A_i, X \rangle)_i \in \mathbb{R}^m$; where $A_i \in \mathcal{S}^n$, $i \in \{1, \ldots, m\}$, $\langle A_i, X \rangle = \text{trace}(A_i X)$.

Applications

Having an upper bound, $\mathrm{rank}(X) \leq r, \forall X \in \mathcal{F}\}$, is useful in many applications, e.g.,

- splitting methods where a projections onto the psd cone $\mathbb{S}^n_+$ is one of the subproblems;
  one can cheat on projection onto $\mathbb{S}^n_+$ and apply the Eckart-Young Theorem and obtain the nearest psd matrix rank $\leq r$.
- low rank SDP algorithms, e.g., [3, Burer-Monteiro '05], where the variable X with $X \succeq 0$ is replaced by VV^T where $V \in \mathbb{R}^{n \times r}$, thus reducing the number of unknowns triangular number $t(n) = n(n+1)/2 \leftarrow nr$.

Recall

Definition (F is a face of C, $F \trianglelefteq C$)

A nonempty convex subset F of a convex set C is a face of C, if
$x, y \in C, \lambda \in (0,1), \lambda x + (1-\lambda) y \in F \implies x, y \in F$.

Properties

- an intersection of faces is a face (for minimal face)
- a face of a face is a face (for FR algorithm)

Known Bounds I

Theorem ([4, Pataki, '98, Theorem 2.1])

Suppose that $X \in F$, where F is a face of the feasible set $\mathcal{F}$. Let $d = \dim F$, $r = \operatorname{rank} X$. Then

$$t(r) \leq m + d. \tag{2}$$

Application: Extreme points; (Barvinok-Pataki bound)

Given the number of constraints is m, (2), gives an upper bound on the rank of a solution.
E.g., extreme points X: $\text{dim}(\text{face}(\{X\})) = 0 \implies$

$$t(\operatorname{rank}(X)) \leq m, \quad \text{for all extreme points } X \in \mathcal{F}. \tag{3}$$

Known Bounds II

Theorem ([1, Barvinok, 2001, Theorem 1.1])

Let $\mathcal{L} \subset \mathcal{S}^n$ be an affine manifold such that the intersection $\mathcal{F} = \mathbb{S}^n_+ \cap \mathcal{L} \neq \emptyset$ and $\operatorname{codim} \mathcal{L} \leq t(r+1) - 1$ for some nonnegative integer r. Then there exists $X \in \mathcal{F}$ such that $\operatorname{rank} X \leq r$.

Remark

There exists $X \in \mathcal{F}$ with $\operatorname{rank}(X) \leq \lfloor \frac{\sqrt{8m+1}-1}{2} \rfloor$. We may obtain an equivalent bound by defining the smallest $r \in \mathcal{N}$ satisfying $\binom{r+2}{2} > m$. Therefore if we have $\binom{r+2}{2} - 1 \geq m$, where m is the number of linearly independent constraints, we obtain the statement in the theorem.

Known Bounds III

Theorem ([1, Barvinok, 2001, Theorem 1.2])

Let $r > 0, n \geq r+2$. Let $\mathcal{L} \subset \mathcal{S}^n$ be an affine manifold such that the intersection $\mathcal{F} = \mathbb{S}^n_+ \cap \mathcal{L} \neq \emptyset$ and bounded, and $\operatorname{codim} \mathcal{L} = t(r+1)$, for some nonnegative integer r. Then there exists $X \in \mathcal{F}$ such that $\operatorname{rank} X \leq r$.

Remark; bounded spectrahedron case

Given triple (r, m, n), where r is upper bound on target rank; $m = \binom{r+2}{2}$ is the number of linearly independent constraints; and the embedding space $\mathcal{S}^n$ satisfies $n \geq r+2 \geq 3$. Then there exists a point $X \in \mathcal{F}$ such that $\operatorname{rank}(X) \leq r$.

Facial Reduction, FR, [2, Borwein-W 1981]

Minimal Face of $C \subseteq \mathbb{S}^n_+$, face$(C)$

face(C) is the intersection of all faces containing C.

- face F is exposed if it is the intersection of $\mathbb{S}^n_+$ and a hyperplane: $F = \mathbb{S}^n_+ \cap Z^\perp$, for some $Z \in \mathbb{S}^n_+$
- vector Z is called an exposing vector of F and it is maximal if it is of the highest rank over all exposing vectors.
- FR is a process of identifying the minimal face of $\mathbb{S}^n_+$ containing the affine set $\{X : \mathcal{A}(X) = b\}$. Since $\mathbb{S}^n_+$ is facially exposed, the process can be characterized as identifying an exposing vector.

Theorem of the Alternative

For the feasible constraint system for $\mathcal{F}$, exactly one of the following statements holds:

1. There exists $X \succ 0$ such that $\mathcal{A}(X) = b$,
2. There exists $y \in \mathbb{R}^m$ such that

$$(0 \neq Z =) \quad \mathcal{A}^*(y) \in \mathbb{S}^n_+ \setminus \{0\}, \ \langle b, y \rangle = 0. \tag{4}$$

Pseudo Code for Facial Reduction Algorithm

- REQUIRE: data $(\mathcal{A}, b)$ for affine set $\{X : \mathcal{A}(X) = b\}$
- WHILE: $\nexists X \succ 0$ satisfying $\mathcal{A}(X) = b$
-     find an exposing vector Z
-     compute V such that $\operatorname{Range}(V) = \operatorname{Null}(Z)$
-     $\mathcal{A} \leftarrow \mathcal{A}_V(\cdot) := \mathcal{A}(V(\cdot)V^T)$
- ENDWHILE
- OUTPUT: $\operatorname{face}(\mathcal{F}) = V\mathbb{S}^r_+V^T$, V a facial vector,
  substitute $X \succeq 0 \leftarrow VRV^T, R \succeq 0$

Dimension AND Constraint Reduction

- The dimension is reduced $n \leftarrow r$, $\text{face}(\mathcal{F}) = V\mathbb{S}^r_+ V^T$.
- And constraint reduction:

Lemma

At least one linear constraint of the SDP *becomes redundant after each step of* FR.

Proof.

Let $Z = \mathcal{A}^*(y)$ be the exposing vector satisfying the system (4). Let V be a minimal facial vector satisfying $\text{Null}(\mathcal{A}^*(y)) = \text{Range}(V)$. Clearly, $V^T \mathcal{A}^*(y) V = \sum_{i=1}^m y_i V^T A_i V = 0$. After the reduction the constraints have the form $\text{trace}(V^T A_i V X) = b_i, \ \forall i$. Since $y \in \mathbb{R}^m$ is a nonzero vector, the matrices in $\{V^T A_i V\}_{i=1,\dots,m}$ are not linearly independent. □

Hölder Regularity; Singularty Degree, $\mathrm{sd}(\mathcal{F})$

Definition (Hölder regularity (projections))

A, B closed convex sets are γ-Hölder regular, if for any compact set U, $\exists c > 0$ such that:
$\operatorname{dist}(x, A \cap B) \leq c\left(\operatorname{dist}^{\gamma}(x, A) + \operatorname{dist}^{\gamma}(x, B)\right), \forall x \in U$
(and add displacement vector in)

Definition ([7, Sturm 2000] [6])

Given a spectrahehedron $\mathcal{F}$, the singularity degree of $\mathcal{F}$, denoted by $\mathrm{sd}(\mathcal{F})$, is the smallest number of facial reduction, FR, steps for finding $\operatorname{face}(\mathcal{F})$.

Theorem ([7, Sturm error bound 2000])

$\mathcal{F}$ is $(1/(2^{sd(\mathcal{F})}))$-Hölder regular with displacement.

$\mathrm{sd}(\mathcal{F}) \leq 1$ if $\mathcal{F} = \mathcal{L} \cap P$, P polyhedral cone (e.g. LP)

Two Lemmas

Lemma (Bound on singularity degree [5, 6])

Let $\mathcal{F}$ be a nonempty spectrahedron such that $\mathcal{F} \neq \{0\}$. Then the singularity degree of $\mathcal{F}$ satisfies the following bound:

$$\mathrm{sd}\,(\mathcal{F}) \leq \min\{n-1, m\}.$$

Lemma (rank of feasible points unchanged after FR)

Let $V \in \mathbb{R}^{n \times r}$ be a minimal facial vector containing the set $\mathcal{F} := \{X \succeq 0 : \mathcal{A}(X) = b\}$, i.e., $V\mathbb{S}^r_+ V^T \supseteq \mathcal{F}$. Then, for VRV^T feasible, we have $\mathrm{rank}\,(VRV^T) = \mathrm{rank}\,(R)$.

Main Result

Theorem

(A strengthened Barvinok-Pataki bound) Suppose that the singularity degree of the nonempty spectrahedron $\mathcal{F}$ satisfies $s = \mathrm{sd}\,(\mathcal{F}) > 0$. Then there exists a point $X \in \mathcal{F}$ with $r = \mathrm{rank}\,(X)$ that satisfies

$$t(r) \leq \min\{t(n-s), m-s\}. \tag{5}$$

Corollary

Let $s = \mathrm{sd}\,(\mathcal{F})$. Then there exists a solution $X \in \mathcal{F}$ such that

$$\mathrm{rank}\,(X) \leq \left\lfloor \frac{\sqrt{1 + 8\min\{t(n-s)\ ,\ m-s\}}}{2} - 1 \right\rfloor .$$

Conclusion

- given spectrahedron $\mathcal{F} = \{X \succeq 0 : \mathcal{A}(X) = b \in \mathbb{R}^m\}$
- Barvinok-Pataki bound: exists $X \in \mathcal{F}$ s.t. $\operatorname{rank}(X) = r$ and $t(r) \leq m$.
- our strengthened bound uses singularity degree $\operatorname{sd}(\mathcal{F})$

$$t(r) \leq \min\{t(n - \operatorname{sd}(\mathcal{F})),\ m - \operatorname{sd}(\mathcal{F})\} \leq m.$$

- important applications exist for existence of low rank solutions
- many open questions arise on understanding singularity degree and:
  complexity of feasible solutions; projections onto faces of cones; singularity degree and strength of SDP relaxations;
  ...

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Thanks for your attention!

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