Taking Advantage of Degeneracy in Cone Optimization: with Applications to Sensor Network Localization

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Motivation: Loss of Slater CQ/Facial reduction

- optimization algorithms rely on the KKT system;
  and require that some constraint qualification (CQ) holds
  (Slater's CQ/strict feasibility for convex conic optimization)

- However, surprisingly many conic opt, SDP relaxations,
  instances arising from applications (QAP, GP, strengthened MC, SNL, POP, Molecular Conformation)
  do not satisfy Slater's CQ/are degenerate (ref. INFORMS meeting)

- lack of Slater's CQ results in: unbounded dual solutions;
  theoretical and numerical difficulties,
  in particular for *primal-dual interior-point methods*.

- solution:
  - theoretical *facial reduction* *(Borwein, W.'81)*
  - preprocess for regularized smaller problem (Cheung, Schurr, W.'11)
  - take advantage of degeneracy (for SNL)
    (Krislock, W.'10; Krislock, Cheung, Drusvyatskiy, W.'14)

Outline: Regularization/Facial Reduction

1. Preprocessing/Regularization
   - Abstract convex program
     - LP case
     - CP case
   - Cone optimization/SDP case
2. Appl.: QAP, GP, Polyn Opt., SNL, Molecular conformation ...

Background/Abstract convex program

$$\text{(ACP)} \quad \inf_x f(x) \text{ s.t. } g(x) \preceq_K 0, x \in \Omega$$

where:

- $f : \mathbb{R}^n \to \mathbb{R}$ convex; $g : \mathbb{R}^n \to \mathbb{R}^m$ is K-convex
  - $K \subset \mathbb{R}^m$ closed convex cone; $\Omega \subseteq \mathbb{R}^n$ convex set
  - $a \preceq_K b \iff b - a \in K$, $a \prec_K b \iff b - a \in \operatorname{int} K$
  - $g(\alpha x + (1-\alpha y)) \preceq_K \alpha g(x) + (1-\alpha) g(y)$, $\forall x, y \in \mathbb{R}^n, \forall \alpha \in [0,1]$

Slater's CQ: $\exists \hat{x} \in \Omega$ s.t. $g(\hat{x}) \in -\operatorname{int} K$ $\quad (g(x) \prec_K 0)$

- guarantees strong duality
- essential for efficiency/stability in p-d i-p methods
- ((near) loss of strict feasibility, nearness to infeasibility correlates with number of iterations & loss of accuracy)

Case of Linear Programming, LP

Primal-Dual Pair: $A, m \times n;\ \mathcal{P} = \{1, \ldots, n\}$

$$\text{(LP-P)} \quad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & A^\top y \le c \end{array} \qquad \text{(LP-D)} \quad \begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax = b,\ x \ge 0. \end{array}$$

Slater's CQ for (LP-P); Theorem of alternative

$\exists \hat{y}$ s.t. $c - A^\top \hat{y} > 0$, $\quad \left((c - A^\top \hat{y})_i > 0, \forall i \in \mathcal{P} =: \mathcal{P}^< \right)$

iff

$$Ad = 0,\ c^\top d = 0,\ d \ge 0 \implies d = 0 \qquad (*)$$

implicit equality constraints: $i \in \mathcal{P}^=$

expose minimal face containing feasible slacks; finding $0 \neq d^*$ to $(*)$ with max number of non-zeros determines

$$d_i^* > 0 \quad \implies (c - A^\top y)_i = 0, \forall y \in \mathcal{F}^y \quad (i \in \mathcal{P}^=)$$

($\mathcal{F}^y$ is primal feasible set)

Rewrite implicit-equalities to equalities/ Regularize LP

Facial Reduction: $A^\top y \leq_f c$; minimal face $f \trianglelefteq \mathbb{R}^n_+$

$$(\text{LP}_{reg}\text{-P}) \quad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & (A^<)^\top y \leq c^< \\ & (A^=)^\top y = c^= \end{array} \qquad (\text{LP}_{reg}\text{-D}) \quad \begin{array}{ll} \min & (c^<)^\top x^< + (c^=)^\top x^= \\ \text{s.t.} & \begin{bmatrix} A^< & A^= \end{bmatrix} \begin{pmatrix} x^< \\ x^= \end{pmatrix} = b \\ & x^< \geq 0, x^= \text{ free} \end{array}$$

Mangasarian-Fromovitz CQ (MFCQ) holds

(after deleting redundant equality constraints!)

$$\left(\exists \hat{y} : \quad \underset{\underline{i \in \mathcal{P}^<}}{(A^<)^\top \hat{y} < c^<} \quad \underset{\underline{i \in \mathcal{P}^=}}{(A^=)^\top \hat{y} = c^=} \right) \qquad (A^=)^\top \text{ is onto}$$

MFCQ holds iff dual optimal set is compact

Numerical difficulties if MFCQ fails; in particular for interior point methods! Modelling issue?

Facial Reduction on Dual/Preprocessing

Slater's CQ for (LP-P) / Theorem of alternative

$\exists \hat{x}$ s.t. $A\hat{x} = b, \hat{x} > 0$

iff

$z = A^\top y \geq 0,\ b^\top y = 0,\ \implies z = 0 \qquad (**)$

Linear Programming Example, $x \in \mathbb{R}^5$

$$\begin{array}{ll} \min & \begin{pmatrix} 2 & 6 & -1 & -2 & 7 \end{pmatrix} x \\ \text{s.t.} & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \end{bmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, x \geq 0 \end{array}$$

Sum the two constraints ($y^T = (1\ 1)$):

$2x_1 + x_4 + x_5 = 0 \implies x_1 = x_4 = x_5 = 0$

equiv. simplified problem/smaller face/less constr.

$$\min 6x_2 - x_3 \text{ s.t. } x_2 + x_3 = 1, x_2, x_3 \geq 0, \quad (x_1 = x_4 = x_5 = 0)$$

Case of ordinary convex programming, CP

$$\text{(CP)} \qquad \sup_y \ b^\top y \text{ s.t. } g(y) \leq 0,$$

where

- $b \in \mathbb{R}^m$; $g(y) = \big(g_i(y)\big) \in \mathbb{R}^n$, $g_i : \mathbb{R}^m \to \mathbb{R}$ convex, $\forall i \in \mathbb{P}$
- Slater's CQ: $\exists \hat{y}$ s.t. $g_i(\hat{y}) < 0, \forall i$ (implies MFCQ)
- Slater's CQ fails implies implicit equality constraints exist, i.e.:
  $\mathcal{P}^= := \{i \in \mathcal{P} : g(y) \leq 0 \implies g_i(y) = 0\} \neq \emptyset$
  Let $\mathcal{P}^< := \mathcal{P} \backslash \mathcal{P}^=$ and

$$g^< := (g_i)_{i \in \mathcal{P}^<}, \qquad g^= := (g_i)_{i \in \mathcal{P}^=}$$

Rewrite implicit equalities to *equalities*/ Regularize CP

(CP) is equivalent to $g(y) \preceq_f 0$, f is minimal face

$$(\mathrm{CP}_{\mathrm{reg}}) \qquad \begin{array}{cl} \sup & b^\top y \\ \text{s.t.} & g^<(y) \leq 0 \\ & y \in \mathcal{F}^= \end{array} \qquad \text{or } (g^=(y) = 0)$$

where $\mathcal{F}^= := \{y : g^=(y) = 0\}$. Then

$\mathcal{F}^= = \{y : g^=(y) \leq 0\}$, so is a convex set!

Slater's CQ holds for (CP_{reg}) $\quad \exists \hat{y} \in \mathcal{F}^= : g^<(\hat{y}) < 0$

modelling issue again?

Faces of Cones - Useful for Charact. of Opt.

Face

A convex cone F is a face of K, denoted $F \trianglelefteq K$, if

$$x, y \in K \text{ and } x + y \in F \implies x, y \in F$$

Conjugate Face

If $F \trianglelefteq K$, the conjugate face of F is

$$F^c := F^\perp \cap K^* \trianglelefteq K^*$$

If $x \in \mathrm{ri}(F)$, then $F^c = \{x\}^\perp \cap K^*$.

where polar cone: $K^* = \{\phi : \langle \phi, y \rangle \geq 0, \forall y \in K\}$

Recall: (ACP) $\inf_x f(x)$ s.t. $g(x) \preceq_K 0, x \in \Omega$

- $K^f = \text{face}(F)$ minimal face containing feasible set F.

Lemma (Facial Reduction)

Suppose $\bar{x}$ is feasible. Then the LHS system

$$\left\{ \begin{array}{ll} (\Omega - \bar{x})^+ \cap \partial\langle\phi, g(\bar{x})\rangle \neq \emptyset & \\ \phi \in K^+, & \langle\phi, g(\bar{x})\rangle = 0 \end{array} \right\} \quad \textit{implies} \quad K^f \subseteq \phi^\perp \cap K.$$

Proof

line 1 of system implies $\bar{x}$ global min for convex function $\langle\phi, g(\cdot)\rangle$ on Ω; i.e., $0 = \langle\phi, g(\bar{x})\rangle \leq \langle\phi, g(x)\rangle \leq 0, \forall x \in F$; implies $-g(F) \subseteq \phi^\perp \cap K$. □

Semidefinite Programming, SDP

$K = \mathcal{S}^n_+ = K^*$ nonpolyhedral cone!

$$\text{(SDP-P)} \quad v_P = \sup_{y \in \mathbb{R}^m} b^\top y \text{ s.t. } g(y) := \mathcal{A}^* y - c \preceq_{\mathcal{S}^n_+} 0$$

$$\text{(SDP-D)} \quad v_D = \inf_{x \in \mathcal{S}^n} \langle c, x \rangle \text{ s.t. } \mathcal{A}x = b, \; x \succeq_{\mathcal{S}^n_+} 0$$

where:

- PSD cone $\mathcal{S}^n_+ \subset \mathcal{S}^n$ symm. matrices
- $c \in \mathcal{S}^n$, $b \in \mathbb{R}^m$
- $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ is a linear map, with adjoint $\mathcal{A}^*$
  $\mathcal{A}x = (\operatorname{trace} A_i x) = (\langle A_i, x \rangle) \in \mathbb{R}^m, \quad A_i \in \mathcal{S}^n$
  $\mathcal{A}^* y = \sum_{i=1}^m A_i y_i \in \mathcal{S}^n$

Slater's CQ/Theorem of Alternative

Assume feasibility: $\exists \tilde{y}$ s.t. $c - \mathcal{A}^* \tilde{y} \succeq 0$.

Exactly one of the following statements hold:

$$\exists \hat{y} \text{ s.t. } s = c - \mathcal{A}^* \hat{y} \succ 0 \qquad (Slater)$$

or

$$\mathcal{A}\Phi = 0,\ \langle c, \Phi \rangle = 0,\ 0 \neq \Phi \succeq 0 \qquad (*)$$

Φ exposes minimal face of slacks

If Φ is a non-zero max-rank solution to ($*$-SDP(P)), *and* $\Phi = PD_+P^T$ is compact spectral decomp., $[P\ Q]$ is orthogonal matrix, then

$$Z = C - \mathcal{A}^* y \succeq_{\mathcal{S}^n_+} 0 \quad \implies \quad Z = QD_ZQ^T$$

so Φ determines/exposes a smaller face of $\mathcal{S}^n_+$ for feasible region/slacks.

Regularization Using Minimal Face

Borwein-W.'81 , $f_P = \text{face}\,\mathcal{F}_P^s$

(SDP-P) is equivalent to the regularized

$$\text{(SDP}_{reg}\text{-P)} \qquad v_{RP} := \sup_y \{\langle b, y\rangle : \mathcal{A}^* y \preceq_{f_P} c\}$$

f_p is mininimal face of primal feasible slacks
slacks: $s = c - \mathcal{A}^* y \in f_p$

Lagrangian Dual DRP Satisfies Strong Duality:

$$\text{(SDP}_{reg}\text{-D)} \quad v_{DRP} := \inf_x \{\langle c, x\rangle : \mathcal{A}x = b,\ x \succeq_{f_P^*} 0\}$$
$$= v_P = v_{RP}$$

and v_{DRP} is attained.

SDP Regularization process

Alternative to Slater CQ

$$\mathcal{A}d = 0,\ \langle c, d\rangle = 0,\ 0 \neq d \succeq_{\mathcal{S}^n_+} 0 \qquad (*)$$

Determine a proper face $f_p \trianglelefteq f = Q\mathcal{S}^{\bar{n}}_+ Q^T \lhd \mathcal{S}^n_+$

Let d solve $(*)$ with compact spectral decomosition
$d = Pd_+P^\top$, $d_+ \succ 0$, and $[P\ Q] \in \mathbb{R}^{n\times n}$ orthogonal. Then

$$c - \mathcal{A}^*y \succeq_{\mathcal{S}^n_+} 0 \implies \langle c - \mathcal{A}^*y, d^*\rangle = 0$$
$$\implies \mathcal{F}^s_P \subseteq \mathcal{S}^n_+ \cap \{d^*\}^\perp = Q\mathcal{S}^{\bar{n}}_+ Q^\top \lhd \mathcal{S}^n_+$$

(implicit rank reduction, $\bar{n} < n$)

Conclusion

Part I

- Minimal representations of the data regularize (P); use min. face f_P (and/or implicit rank reduction)
- goals:
  - exploit structure to find efficient exact facial reduction
  - a backwards stable preprocessing algorithm to handle (feasible) conic problems for which Slater's CQ (almost) fails

For Part II

- Many instances of SDP that arise from relaxations of hard combinatorial problems fail the Slater CQ.
- The structure of the problem can be exploited to allow for an explicit facial reduction of the problem. We obtain a smaller (minimal) equivalent problem for which the Slater CQ holds.

Part II: Applications of SDP where Slater's CQ fails

Instances of NP-hard combinatorial optimization problems

- Quadratic Assignment (Zhao-Karish-Rendl-W.'96)
- Graph partitioning (W.-Zhao'99)
- Min cut (Hao-Pong-Wang-W. '13)

Low rank problems

- Sensor network localization (SNL) problem (Krislock-W.'10, Drusvyatskiy-Cheung-Krislock-W'14)
- Molecular conformation (Burkowski-Cheung-W.'11)
- general SDP relaxation of low-rank matrix completion problem
- solving polynomial equations

SNL (K-W'10)

Highly (implicit) degenerate/low-rank problem

- SNL is a graph realization problem that can be modelled exactly as an SDP with rank restriction
- cliques in the graph correspond to principal submatrices that are singular; high (implicit) degeneracy translates to low rank solutions
- fast, high accuracy solutions for exact data

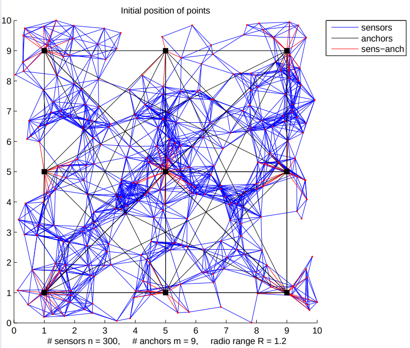
SNL - a Fundamental Problem of Distance Geometry

easy to describe - (dates back to Grasssmann 1886)

- r : embedding dimension
- n ad hoc wireless sensors $p_1, \ldots, p_n \in \mathbb{R}^r$ to locate in $\mathbb{R}^r$;
- m of the sensors $p_{n-m+1}, \ldots, p_n$ are anchors (positions known, using e.g. GPS)
- pairwise distances $D_{ij} = \|p_i - p_j\|^2, ij \in E$, are known within radio range $R > 0$
- $$P^\top = \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix} = \begin{bmatrix} X^\top & A^\top \end{bmatrix} \in \mathbb{R}^{r \times n}$$

Sensor Localization Problem/Partial EDM

Sensors ○ and Anchors ■



Underlying Graph Realization/Partial EDM NP-Hard

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- node set $\mathcal{V} = \{1, \ldots, n\}$
- edge set $(i,j) \in \mathcal{E}$; $\omega_{ij} = \|p_i - p_j\|^2$ known approximately
- The anchors form a clique (complete subgraph)
- Realization of $\mathcal{G}$ in $\mathbb{R}^r$: a mapping of nodes $v_i \mapsto p_i \in \mathbb{R}^r$ with squared distances given by ω.

Corresponding Partial Euclidean Distance Matrix, EDM

$$D_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise (unknown distance)}, \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors p_i, p_j; anchors correspond to a clique.

Connections to Semidefinite Programming (SDP)

$D = \mathcal{K}(B) \in \mathcal{E}^n$, $B = \mathcal{K}^\dagger(D) \in \mathcal{S}^n \cap \mathcal{S}_C$ (centered $Be = 0$)

$P^\top = \begin{bmatrix} p_1 & p_2 & \dots & p_n \end{bmatrix} \in \mathcal{M}^{r \times n}$;
$B := PP^\top \in \mathcal{S}^n_+$ (Gram matrix of inner products);
$\operatorname{rank} B = r$; let $D \in \mathcal{E}^n$ corresponding EDM; $e = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^\top$

$$
\begin{array}{rcl}
(\text{to } D \in \mathcal{E}^n) \quad D & = & \left(\|p_i - p_j\|_2^2\right)_{i,j=1}^n \\
& = & \left(p_i^T p_i + p_j^T p_j - 2p_i^T p_j\right)_{i,j=1}^n \\
& = & \boxed{\operatorname{diag}(B)\, e^\top + e \operatorname{diag}(B)^\top - 2B} \\
& =: & \mathcal{K}(B) \quad (\text{from } B \in \mathcal{S}^n_+).
\end{array}
$$

Popular Techniques; SDP Relax.; Highly Degen.

Nearest, Weighted, SDP Approx. (relax/discard $\operatorname{rank} B$)

- $\min_{B \succeq 0} \|H \circ (\mathcal{K}(B) - D)\|$; $\operatorname{rank} B = r$;
  typical weights: $H_{ij} = 1/\sqrt{D_{ij}}$, if $ij \in E$, $H_{ij} = 0$ otherwise.
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex, BUT: expensive/low accuracy/implicitly highly degenerate (cliques restrict ranks of feasible Bs)

Instead: (Shall) Take Advantage of Degeneracy!

clique $\alpha, |\alpha| = k$ (corresp. $D[\alpha]$) with embed. dim. $= t \leq r < k$

$\implies \operatorname{rank} \mathcal{K}^\dagger(D[\alpha]) = t \leq r \implies \operatorname{rank} B[\alpha] \leq \operatorname{rank} \mathcal{K}^\dagger(D[\alpha]) + 1$

$\implies \operatorname{rank} B = \operatorname{rank} \mathcal{K}^\dagger(D) \leq n - \boxed{(k - t - 1)} \implies$

Slater's CQ (strict feasibility) fails

Facial Reduction for Cliques

- Using the basic theorem:
  each clique corresponds to a
  Gram matrix/corresponding subspace/corresponding face
  of SDP cone (implicit rank reduction)
- In the case where
  two cliques intersect, the union of the cliques correspond
  to the (efficiently computable)
  intersection of the corresponding faces/subspaces
- Finally, the
  positions are determined using a
  Procrustes problem

Results - Data for Random Noisless Problems

- 2.16 GHz Intel Core 2 Duo, 2 GB of RAM
- Dimension $r = 2$
- Square region: $[0,1] \times [0,1]$
- $m = 9$ anchors
- Using only Rigid Clique Union and Rigid Node Absorption
- Error measure: Root Mean Square Deviation

$$\text{RMSD} = \left(\frac{1}{n} \sum_{i=1}^{n} \|p_i - p_i^{\text{true}}\|^2 \right)^{1/2}$$

Results - Large n (SDP size $O(n^2)$)

n # of Sensors Located

n # sensors \ R	0.07	0.06	0.05	0.04
2000	2000	2000	1956	1374
6000	6000	6000	6000	6000
10000	10000	10000	10000	10000

CPU Seconds

# sensors \ R	0.07	0.06	0.05	0.04
2000	1	1	1	3
6000	5	5	4	4
10000	10	10	9	8

RMSD (over located sensors)

n # sensors \ R	0.07	0.06	0.05	0.04
2000	$4e-16$	$5e-16$	$6e-16$	$3e-16$
6000	$4e-16$	$4e-16$	$3e-16$	$3e-16$
10000	$3e-16$	$5e-16$	$4e-16$	$4e-16$

Results - N Huge SDPs Solved

Large-Scale Problems

# sensors	# anchors	radio range	RMSD	Time
20000	9	.025	$5e{-}16$	25s
40000	9	.02	$8e{-}16$	1m 23s
60000	9	.015	$5e{-}16$	3m 13s
100000	9	.01	$6e{-}16$	9m 8s

Size of SDPs Solved: $N = \binom{n}{2}$ (# vrbls)

$\mathcal{E}_n(\text{density of } \mathcal{G}) = \pi R^2$; $M = \mathcal{E}_n(|E|) = \pi R^2 N$ (# constraints)

Size of SDP Problems:

$M = \begin{bmatrix} 3,078,915 & 12,315,351 & 27,709,309 & 76,969,790 \end{bmatrix}$

$N = 10^9 \begin{bmatrix} 0.2000 & 0.8000 & 1.8000 & 5.0000 \end{bmatrix}$

Noisy SNL Case

200 Sensors; [-0.5,0.5] box; noise 0.05; radio range 0.1

- use sum of exposing vectors rather than intersection of faces obtained from cliques to do facial reduction
- use motivation: roundoff error cancels

show video

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Thanks for your attention!

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