Efficient Solutions for the Large Scale Trust Region Subproblem, TRS

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Outline

Background

- Generalized Trust Region Subproblem and Characterizations of Optimality
- MoSo (1983) algorithm for TRS; within a primal-dual framework; handling the hard case

Duality/Exploiting Sparsity & Degeneracy/Hard Case

- Various primal-dual pairs with strong duality
- Parametric eigenvalue problem and sparsity
- Hard case is actually easiest case, with an explicit solution
- Underdetermined least squares, Numerics

Trust Region Subproblem, (TRS)

“Nonconvex” quadratic minimization

$$\text{(TRS)} \qquad \begin{array}{rll} q^* = & \min & q(x) := x^T A x - 2a^T x \\ & \text{s.t.} & \|x\| \leq s, \\ & & x \in \mathbb{R}^n \end{array}$$

- $A \in \mathcal{S}^n$ - $n \times n$ symmetric (possibly indefinite) matrix
- $a \in \mathbb{R}^n$; $s > 0$ (TR radius);

Generalized TRS, (GTRS)

$$\text{(GTRS)} \qquad \begin{array}{rll} q^* = & \min & q(x) := x^T A x - 2a^T x \\ & \text{s.t.} & \ell \leq q_1(x) := x^T B x - 2b^T x \leq u \end{array}$$

Applications

Many important applications:

- forming subproblems for constrained optimization
- regularization of ill-posed problems
- theoretical applications
- **trust region (TR) methods for unconstr. min.** uses TRS formed from second order Taylor series approximation

(dis)Advantages for TR approach

- obtains second order optimality conditions
- q-quadratic convergence
- **BUT:** popularity?:
  (i) difficulties exploiting sparsity?
  (ii) hard case (degenerate case)?
  (iii) evaluating Hessian?

Characterization of Optimality for TRS

Surprising; TRS is on *watershed* of convex/nonconvex

- a characterization of optimality of a (possibly) nonconvex problem
- second order positive semidefinite necessary conditions hold on all of $\mathbb{R}^n$

Characterization of x^* optimal for TRS (iff)

(Gay-81 [2], More-Sorensen-83 [6])

$$\left.\begin{array}{c} (A-\lambda^* I)x^* = a, \\ A-\lambda^* I \succeq 0, \lambda^* \leq 0 \end{array}\right\} \qquad \text{dual feasibility}$$

$$\|x^*\|^2 \leq s^2 \qquad \text{primal feasibility}$$

$$\lambda^*(s^2 - \|x^*\|) = 0 \qquad \text{complementary slackness}$$

Characterization of Optimality for (homog) GTRS

(Mild) Slater Type Constraint Qualification

$$\text{(GTRS)} \qquad \begin{array}{rll} q^* = & \min & q(x) := x^T A x - 2a^T x \\ & \text{s.t.} & \ell \leq q_1(x) := x^T B x \leq u \end{array}$$

$$(CQ) \qquad \exists \hat{x} \text{ s.t. } l < q_1(\hat{x}) := \hat{x}^T B \hat{x} < u$$

Characterization of x^* optimal for GTRS (iff)

(More-93 [5], Stern-W-93 [10])

$$\left.\begin{array}{c} (A - \lambda^* B) x^* = a, \\ A - \lambda^* B \succeq 0 \end{array}\right\} \qquad \text{dual feasibility}$$

$$\ell \leq {x^*}^T B x \leq u \qquad \text{primal feasibility}$$

$$\lambda^*(u - {x^*}^T B x^*) \geq 0 \geq \lambda^*({x^*}^T B x^* - \ell) \qquad \text{compl. slack.}$$

S-Lemma Yakubovich-71,73

(survey in Polik-Terlaky:07 [7])

$f, g : \mathbb{R}^n \to \mathbb{R}$ quadratic functions
$\exists \bar{x}$ s.t. $g(\bar{x}) < 0$
then TFAE:

1. $\nexists x \in \mathbb{R}^n : f(x) < 0, g(x) \leq 0$
2. $\exists \lambda \geq 0$ s.t. $f(x) + \lambda g(x) \geq 0, \forall x \in \mathbb{R}^n$

Finsler:37 , Hestenes-McShane:40
$\{x^T Bx = 0, x \neq 0 \implies x^T Ax > 0\}$
implies
$\{\exists \lambda \geq 0, A + \lambda B \succ 0\}$

Mo-So Algorithm Framework

Recall: Opt. conditions of TRS

$$\left.\begin{array}{c}(A-\lambda^* I)x^* = a, \\ A-\lambda^* I \succeq 0, \lambda^* \leq 0\end{array}\right\} \qquad \text{dual feasibility}$$

$$\|x^*\|^2 \leq s^2 \qquad \text{primal feasibility}$$

$$\lambda^*(s^2 - \|x^*\|) = 0 \qquad \text{complementary slackness}$$

Original view: if $A - \lambda^* I \succ 0, \lambda^* < 0$

- define: $x(\lambda) = (A - \lambda I)^{-1} a$
- <u>SOLVE</u>: $\psi(\lambda) := \|x(\lambda)\| - s = \|(A - \lambda I)^{-1} a\| - s = 0$
- maintain: $A - \lambda I \succ 0, \ \lambda \leq 0$

Less Nonlinear $\phi(\lambda)$ for Newton's Method

Newton for $\psi(\lambda) = 0$?

$A = Q\Lambda Q^T$ orthogonal diagonalization; $\gamma = Q^T a$;
$\psi(\lambda) = \|x(\lambda)\|^2 - s^2 = \sum_{j=1}^{n} \frac{\gamma_j^2}{(\lambda_j(A)-\lambda)^2} - s^2$ *highly* nonlinear in λ, in particular near $\lambda_1(A)$.

less nonlinear $\phi(\lambda)$ (Reinsch:67 [8, 9], Hebden:73 [3])

SOLVE: $\phi(\lambda) := \frac{1}{s} - \frac{1}{\|x(\lambda)\|} = 0$

Newton iterates $\lambda^{(k)}$ for $\phi(\lambda) = 0$; exploit Cholesky

$$\begin{aligned}
\lambda^{(k+1)} &= \lambda^{(k)} - \frac{\phi(\lambda^{(k)})}{\phi'(\lambda^{(k)})} \\
&= \lambda^{(k)} - \left(\frac{\|x\|}{\|y\|}\right)^2 \left(\frac{\|x\|-s}{s}\right),
\end{aligned}$$

where: $A - \lambda^{(k)} I = R^T R$ (<u>Cholesky</u>); $R^T R x = a$; $R^T y = x$

Computing the Newton direction/step Gay:81, MoSo:83

Assume $\lambda^{(k)} \leq 0$ and $A - \lambda^{(k)} I \succ 0$ (i.e. $\lambda^{(k)} < \lambda_1(A)$)

- Factor $A - \lambda^{(k)} I = R^T R$ (Cholesky factorization).
- Solve for x, $R^T R x = a$ $\quad$ ($x = x(\lambda^{(k)})$).
- Solve for y, $R^T y = x$.
- Let $\lambda^{(k+1)} = \lambda^{(k)} - \left[\frac{\|x\|}{\|y\|}\right]^2 \left[\frac{(\|x\| - s)}{s}\right]$ $\quad$ (Newton step); but, *safeguard/maintain* positive definiteness

Nearly linear

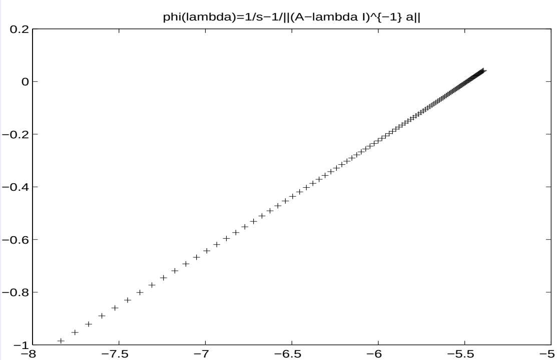


Figure: “Easy” case: $\phi(\lambda) = \frac{1}{s} - \frac{1}{||(A-\lambda I)^{\dagger} a||}$

“Easy” Case; Newton Quadr. Cvgnce from One Side

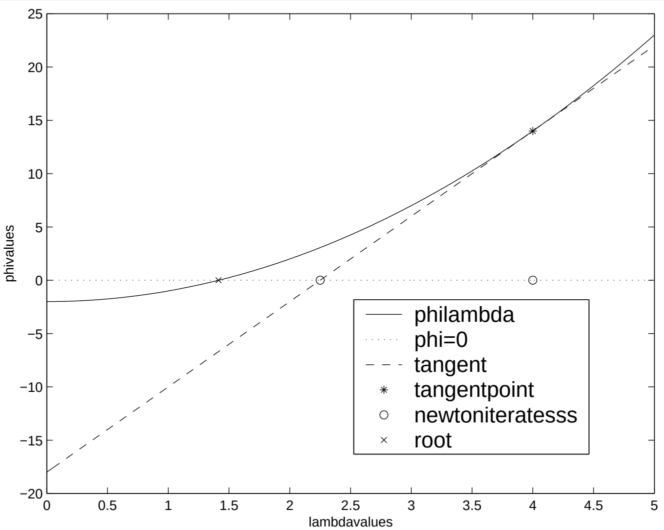


Figure: Newton’s method with the secular function, $\phi(\lambda)$.

Easy/Hard Cases for TRS

$\lambda_1(A) = \lambda_{\min}(A) > \lambda^* \implies A - \lambda^* I \succ 0$

1. Easy case	2.(a) Hard case (case 1)	2.(b) Hard case (case 2)
$a \notin \mathcal{R}(A - \lambda_1(A)I)$ (stationarity implies $\lambda^* \neq \lambda_1(A)$, so that $\lambda^* < \lambda_1(A)$)	$a \perp \mathcal{N}(A - \lambda_1(A)I)$ but $\lambda^* < \lambda_1(A)$	$a \perp \mathcal{N}(A - \lambda_1(A)I)$ and $\lambda^* = \lambda_1(A)$ (i) $\|(A - \lambda^* I)^\dagger a\| = s$ or $\lambda^* = 0$ (ii) $\|(A - \lambda^* I)^\dagger a\| < s, \lambda^* < 0$

Table: The three different cases for the trust region subproblem. We include two subcases (i) and (ii) for the hard case (case 2), where $A - \lambda^* I$ is positive semidefinite, singular. ($\cdot^\dagger$ denotes Moore-Penrose gen. inverse.)

Hard Case is really easiest?

We take advantage of Lanczos algorithm used in large sparse case.
We will see: hard case becomes easiest case.

$\phi(\lambda)$ in the (near) Hard Case

Newton steps (linearizations) are *"useless"*

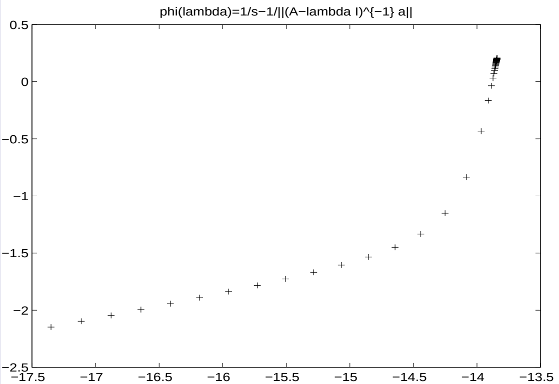


Figure: (near) hard case case: $\phi(\lambda) = \frac{1}{s} - \frac{1}{||(A-\lambda I)^{\dagger} a||}$

Nonlinear Primal-Dual Pair of SDPs; MoSo Alg.

Lagrangian: $L(x,\lambda) := x^tAx - 2a^tx - \lambda(||x||^2 - s^2)$

(Stern-W.:93 [10])

- Strong duality holds for (TRS)
  $$\mu^* = \min_x \max_\lambda L(x,\lambda) = \max_\lambda \min_x L(x,\lambda).$$
  Moreover, attainment holds for x and uniquely for λ.
- A (Wolfe) dual problem for (TRS), without a duality gap:
  $$(D) \qquad \mu^* = \max_{A-\lambda I \succeq 0} h(\lambda), \quad \text{where}$$
  $$h(\lambda) = \lambda s^2 - a^t(A-\lambda I)^\dagger a$$

MoSo is a dual algorithm

solves a modified:
$$h'(\lambda) = s^2 - a^t\left((A-\lambda I)^\dagger\right)^2 a = s^2 - \|x(\lambda)\|^2 = 0$$
a duality gap is: $q(x_{feas}) - h(\lambda_{feas})$

Easy case: $h(\lambda) = \lambda s^2 - a^t(A - \lambda I)^\dagger a$

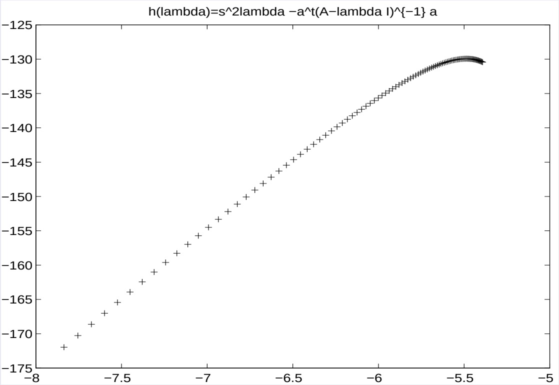


Figure: Easy case: $h(\lambda) = \lambda s^2 - a^t(A - \lambda I)^\dagger a$

Easy case, solve: $h'(\lambda) = s^2 - a^t((A - \lambda I)^\dagger)^2 a = 0$

Can be very nonlinear

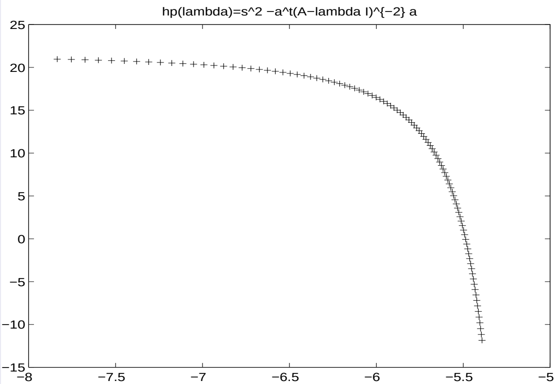


Figure: Easy case: $h'(\lambda) = s^2 - a^t((A - \lambda I)^\dagger)^2 a = s^2 - \|x(\lambda)\|^2$

(near) Hard case: $h(\lambda) = \lambda s^2 - a^t(A - \lambda I)^\dagger a$

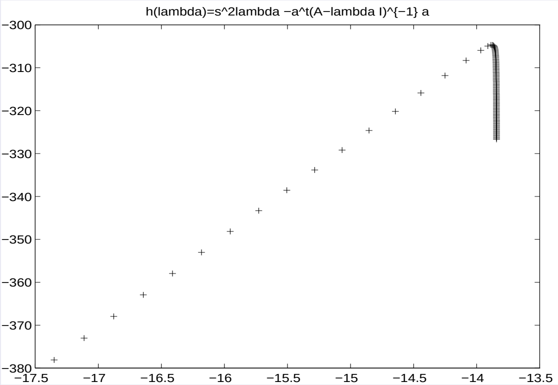


Figure: (near) Hard case: $h(\lambda) = \lambda s^2 - a^t(A - \lambda I)^\dagger a$

Dual to SDP

Slater's CQ holds for dual program (D)

Strong duality holds with the (Wolfe) dual (DD) of the dual (D):

$$
(DD) \qquad \begin{array}{rcl} \mu^* = & \min & h(\lambda) + \mathrm{trace}\, X(A-\lambda I) \\ & \text{s.t.} & s^2 - \|(A-\lambda I)^\dagger a\|^2 - \mathrm{trace}\, X = 0 \\ & & X \succeq 0. \end{array}
$$

Duality gap view to handling hard case in MoSo

$h(\lambda) + \mathrm{trace}\, X(A-\lambda I) - h(\lambda) = \mathrm{trace}\, X(A-\lambda I)$ (compl. slack.)

if $\|x_\lambda\| < s$ and $\|x_\lambda + z\|^2 = s^2$, then:

$$
\begin{array}{rcl} q(x_\lambda + z) & = & (\lambda s^2 - x_\lambda^t (A-\lambda I) x_\lambda) + z^t (A-\lambda I) z \\ & = & h(\lambda) + z^t (A-\lambda I) z. \end{array}
$$

find z (a direction of negative curvature) to make $z^t(A-\lambda I)z$ small. Set $X = zz^t$ to make duality gap term, or complementary slackness term, small. (Error estimate now comes from duality gap.)

Towards a Parametrization for Large Sparse Case

Motivation: try to avoid Cholesky in large sparse case

Homogenization, $y_0^2 = 1 \quad \mu^* =$

$$
\begin{array}{rl}
= & \min\limits_{||x||=s,\ y_0^2=1} x^tAx - 2y_0a^tx \\
= & \max\limits_{t} \min\limits_{||x||=s,\ y_0^2=1} x^tAx - 2y_0a^tx + ty_0^2 - t \\
\geq & \max\limits_{t} \min\limits_{||x||^2+y_0^2=s^2+1} x^tAx - 2y_0a^tx + ty_0^2 - t \quad \text{**eig prob**} \\
\geq & \max\limits_{t,\lambda} \min_{x,y_0} x^tAx - 2y_0a^tx + ty_0^2 - t + \lambda(||x||^2 + y_0^2 - s^2 - 1) \\
= & \max\limits_{r,\lambda} \min_{x,y_0} x^tAx - 2y_0a^tx + ry_0^2 - r + \lambda(||x||^2 - s^2) \\
= & \max\limits_{\lambda} \left(\max\limits_{r} \min_{x,y_0} x^tAx - 2y_0a^tx + ry_0^2 - r + \lambda(||x||^2 - s^2) \right) \\
= & \max\limits_{\lambda} \min_{x,y_0^2=1} x^tAx - 2y_0a^tx + \lambda(||x||^2 - s^2) \\
= & \mu^*,
\end{array}
$$

equated r with $t + \lambda$.

used strong duality for TRS for last two equalities.

Parametric Eigenvalue Problem

$$\max_t \boxed{\min_{||x||^2+y_0^2=s^2+1} x^tAx - 2y_0a^tx + ty_0^2} - t$$

$$k(t) := (s^2+1)\lambda_1(D(t)) - t \qquad D(t) = \begin{bmatrix} t & -a^t \\ -a & A \end{bmatrix}$$

An Unconstrained dual problem to (TRS)

$$\max_t k(t) \qquad \text{(concave max)}$$

$k'(t) = (s^2+1)y_0(t)^2 - 1$

If $\lambda_1(D(t))$ is simple

- $y(t)$ normalized eigenvector for $\lambda_1(D(t))$
- $y(t) = \begin{pmatrix} y_0(t) \\ x(t) \end{pmatrix}$
- $$\frac{1}{y_0(t)}\|x(t)\| = s \text{ if and only if } k'(t) = 0$$

Linear Primal-Dual SDP Pair

From parametric eigenvalue problem

$$(DSDP) \qquad \begin{array}{rll} \mu^* = & \max & (s^2+1)\lambda - t \\ & \text{s.t.} & D(t) \succeq \lambda I \end{array}$$

(with $\lambda \leq 0$ for inequality constraint TRS)

$$(PSDP) \qquad \begin{array}{rll} \mu^* = & \min & \operatorname{trace} D(0)X \\ & \text{s.t.} & \operatorname{trace} X = s^2+1 \\ & & X_{11} = 1 \\ & & X \succeq 0 \end{array}$$

$y_0(t)$ in the easy case

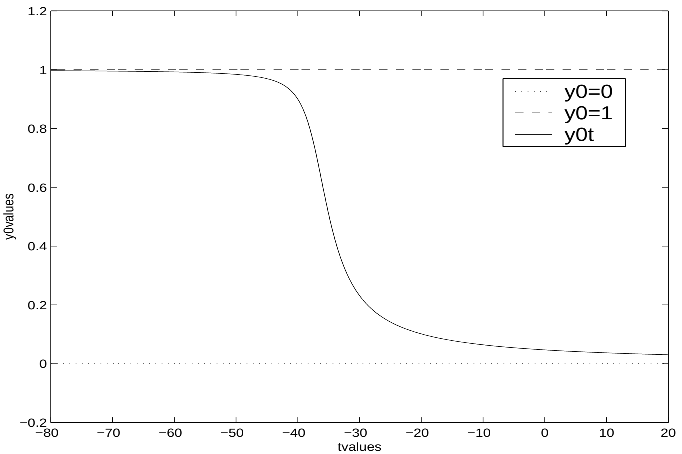


Figure: $y_0(t)$ in the easy case

$y_0(t)$ in the hard case

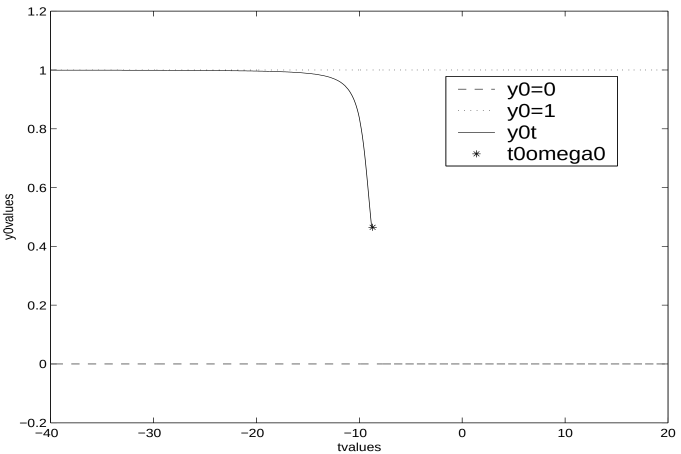


Figure: $y_0(t)$ in the hard case

$k'(t)$ in the easy case

Can be highly nonlinear

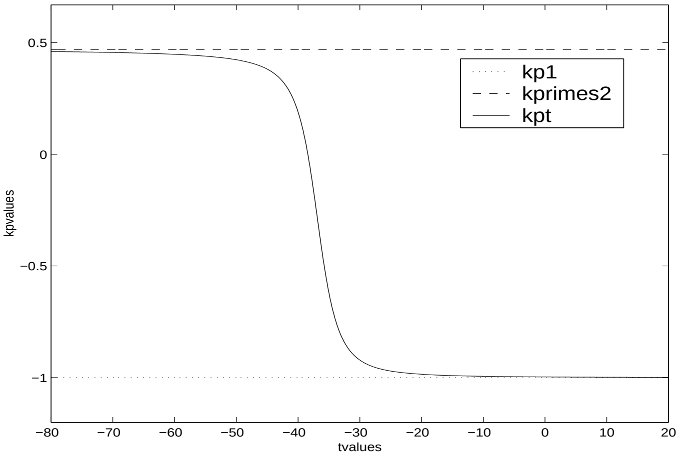


Figure: $k'(t)$ in the easy case

$k'(t)$ in the hard case (*case 1*)

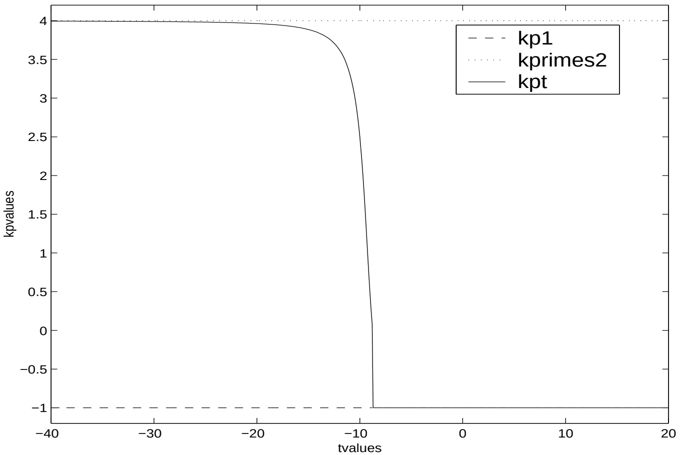


Figure: $k'(t)$ in the hard case (*case 1*)

Solving $\psi(t) = 0$ or $k'(t) = 0$ is equivalent; but less nonlinear.

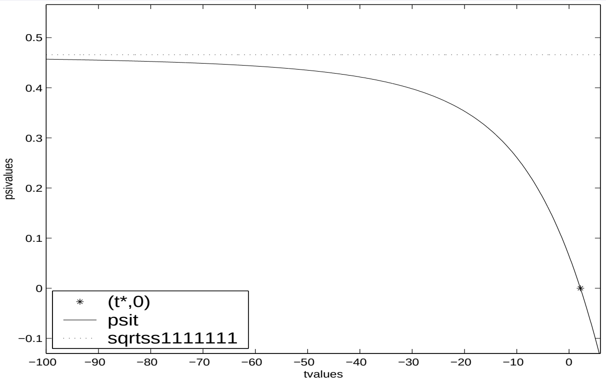


Figure: $\psi(t)$ in the easy case

$\psi(t)$ in the hard case (*case 1*)

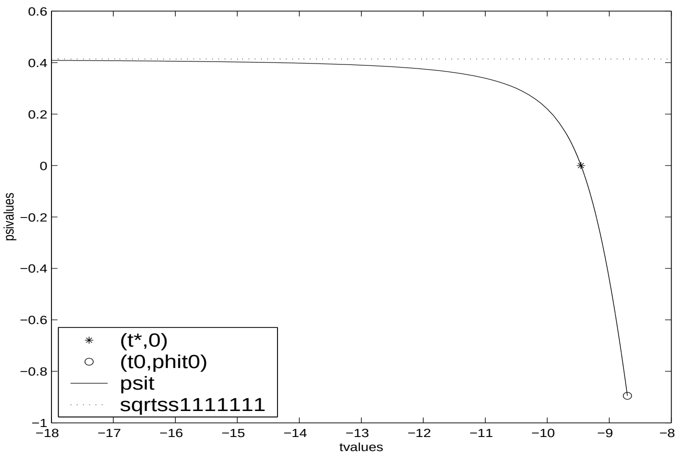


Figure: $\psi(t)$ in the hard case (*case 1*)

$\psi(t)$ in the hard case (*case 2*)

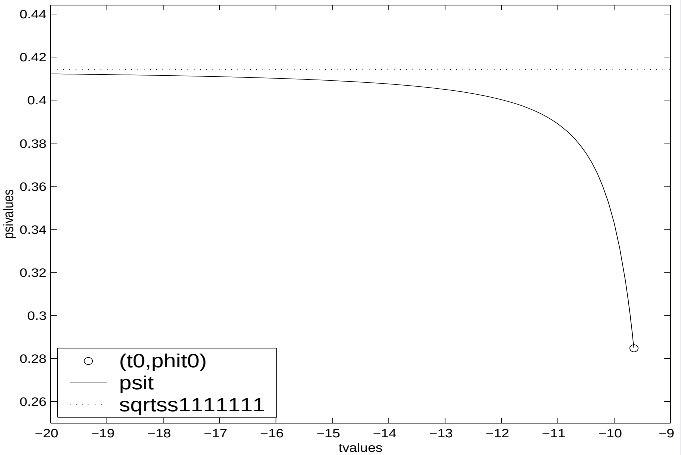


Figure: $\psi(t)$ in the hard case (*case 2*)

easy case: $k(t) = (s^2 + 1)\lambda_1(D(t)) - t$

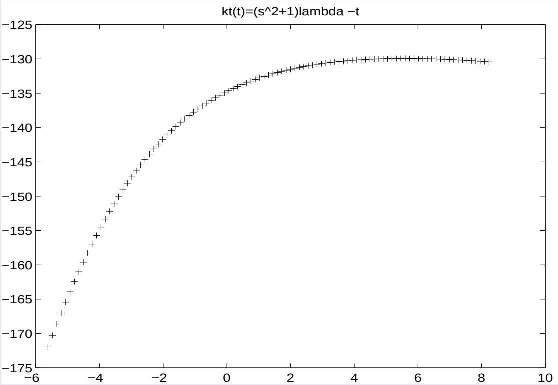


Figure: easy case: $k(t) = (s^2 + 1)\lambda_1(D(t)) - t$

$k(t)$ Hard Case (Case 1)

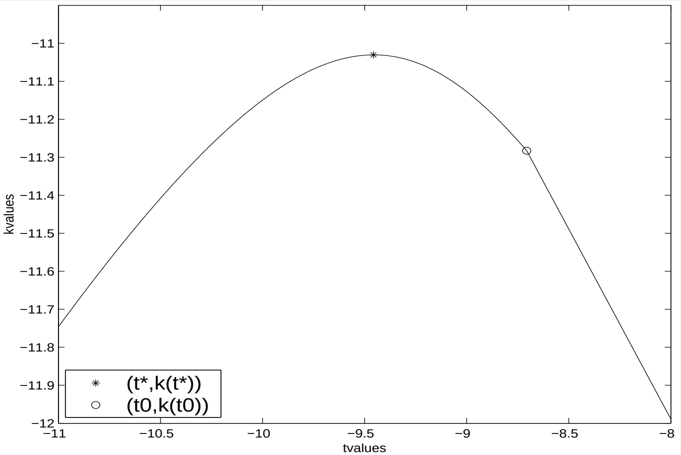


Figure: $k(t)$ in the hard case (case 1)

$k(t)$ in the hard case (case 2)

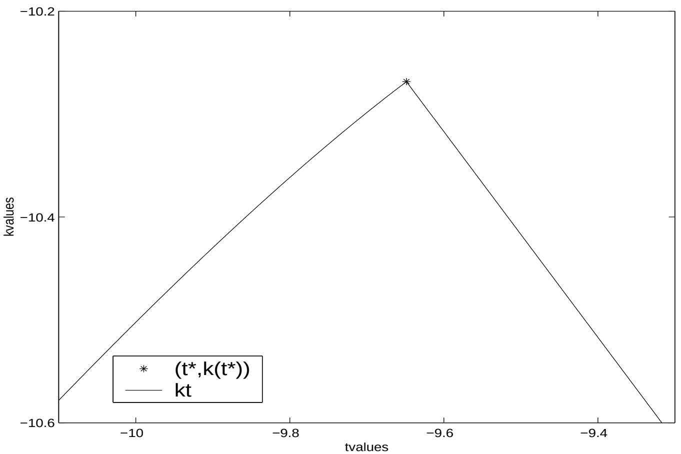


Figure: $k(t)$ in the hard case (case 2)

(Near) Hard Case $k(t) = (s^2 + 1)\lambda_1(D(t)) - t$

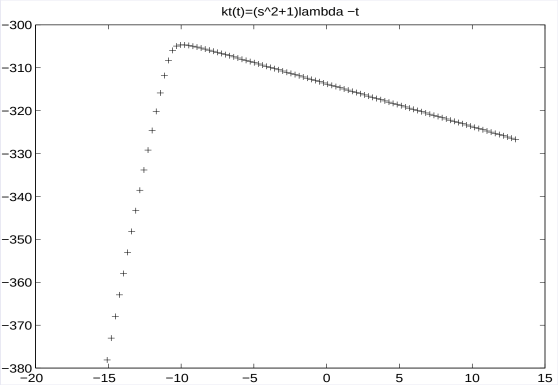


Figure: (near) hard case: $k(t) = (s^2 + 1)\lambda_1(D(t)) - t$

Hard Case: $k'(t) = (s^2+1)y_0^2(t) - 1$

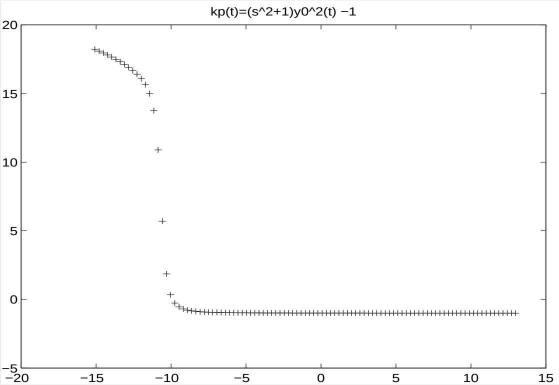


Figure: Hard Case: $k'(t) = (s^2+1)y_0^2(t) - 1$

Hard Case is Easiest Case: Shift and Deflation

If $\lambda_{\min} < 0$ use $A \leftarrow A - \lambda_{\min}(A)$

- hard case (case 2) happens only when $\lambda^* = \lambda_{\min}(A) < 0$
- $A \succeq 0$ after the shift (objective changes by a constant)

Lemma Fortin-W:03 [1]

$A = \sum_{i=1}^n \lambda_i(A) v_i v_i^T = Q \Lambda Q^T$ orthogonal spectral decomposition of A; $\gamma_i = (Q^T a)_i$

$$\begin{aligned}
S_1 &= \{i : \gamma_i \neq 0, \lambda_i(A) > \lambda_{\min}(A)\} \\
S_2 &= \{i : \gamma_i = 0, \lambda_i(A) > \lambda_{\min}(A)\} \\
S_3 &= \{i : \gamma_i \neq 0, \lambda_i(A) = \lambda_{\min}(A)\} \\
S_4 &= \{i : \gamma_i = 0, \lambda_i(A) = \lambda_{\min}(A)\}
\end{aligned}$$

For $k = 1, 2, 3, 4,$: $A_k = \sum_{i \in S_k} \lambda_i(A) v_i v_i^T$.

Then:

Lemma Conclusions

1. If $S_3 \neq \emptyset$ (easy case), then (x^*, λ^*) solves TRS <u>iff</u> (x^*, λ^*) solves TRS when A is replaced by $A_1 + A_3$.
2. If $S_3 = \emptyset$ (hard case), let $i_0 = 1 \in S_4$, then (x^*, λ^*) solves TRS <u>iff</u> (x^*, λ^*) solves TRS when A is replaced by $A_1 + \lambda_{i_0}(A) v_{i_0} v_{i_0}^T$. (DEFLATE)
3. Let $x(\lambda^*) = (A - \lambda^* I)^\dagger a$, then (x^*, λ^*), where $x^* = x(\lambda^*) + z, z \in \mathcal{N}(A - \lambda^* I)$ and $\|x^*\| = s$ solves TRS <u>iff</u> $(x(\lambda^*), \lambda^* - \lambda_{\min}(A))$ solves TRS when A is replaced by $A - \lambda_{\min}(A) I$.
4. If $\lambda_{\min}(A) \geq 0$, then (x^*, λ^*) solves TRS iff (x^*, λ^*) solves TRS when A is replaced by $A + \sum_{i \in S_4} \alpha_i v_i v_i^T$, with $\alpha_i \geq 0$. (DEFLATE all in S_4)

Shift/Deflate: Solve Hard Case Explicitly

Summary - sometimes you get !lucky!

- eigenpair $\lambda_{\min}(A) = \lambda_1(A), v_1$ found; large sparse case; so, assume possible hardcase: $\boxed{\lambda_1(A) < 0 \text{ and } v_1^T a = 0}$
- !we have $\lambda_1(A)$! so shift: $A \leftarrow A - \lambda_{\min}(A)I \succeq 0$; !optimum unchanged, objective value changed by $\lambda_1 s^2$!.
- !we have v_1, $v_1^T a = 0$! so deflate: $A \leftarrow A + \alpha_1 vv^T$, $\|A\| > \alpha_1 >> 0$; (repeat deflation if needed, i.e as long as $\boxed{\lambda_{\min}(A) = 0 \text{ and } v^T a = 0}$) !objective value/optima unchanged!.
- if $v^T a \neq 0$, then continue with TRS algorithm, i.e. hard case (case 1); otherwise, $A \succ 0$, so (using prec. conj grad.) calculate $\bar{x} = A^{-1}a$; if $\|x(\lambda^*)\| > s$, then continue with TRS algorithm, i.e. hard case (case 1); otherwise $\|x(\lambda^*)\| \leq s$, then we have hard case (case 2) and !we have an explicit solution : $\|x^*\| = \|\bar{x} + \beta v_1\| = s$, !since $v_1 \in \mathcal{N}(A - \lambda^* I)$!.

Techniques Used for Maximizing $k(t)$

"good side" and "bad side" for t

Let $t_0 := \lambda_{\min}(A) + \sum_{k \in \{j | (Q^T a)_j \neq 0\}} \frac{(Q^T a)_k^2}{\lambda_k(A) - \lambda_{\min}(A)}$

- If $k'(t) < 0$ ($t > t^*$), then t is on good side
- If $k'(t) > 0$ ($t < t^*$), then t is on bad side

Note: if $\exists t, t^* < t < t_0$ (on good side), then the hard case does not occur.

Techniques:

- Bracketing Newton's Method
- Triangle Interpolation
- Vertical Cut
- Inverse Linear Interpolation

First order method to maximize $k(t)$

duality provides upper and lower bounds of $k^* = \max_t k(t)$: k_{up}, k_{low}, respectively; $M_j := \alpha k_{up} + (1-\alpha)k_{low}$, $\alpha \in (0,1)$
$t_{j+1} = t_j - \frac{k(t_j)-M_j}{k'(t_j)}$: if $k(t_{j+1}) > k(t_j)$, then $k_{low} \leftarrow k(t_{j+1})$ (else $k_{up} \leftarrow M_j$)

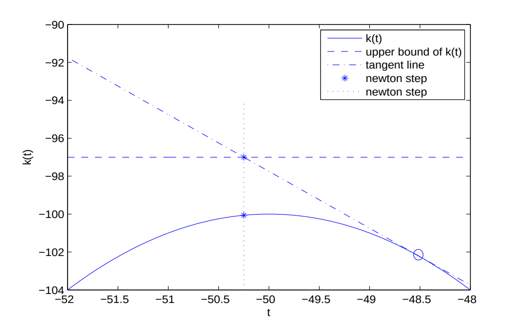


Figure: Bracketing Newton's Method step to solve $k(t) - M_j = 0$

Triangle Interpolation

Pick new t_+ if in current interval of uncertainty for t

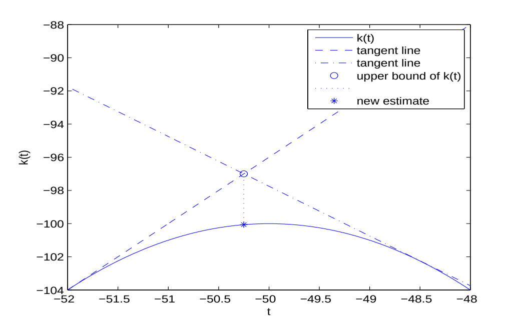


Figure: Triangle Interpolation

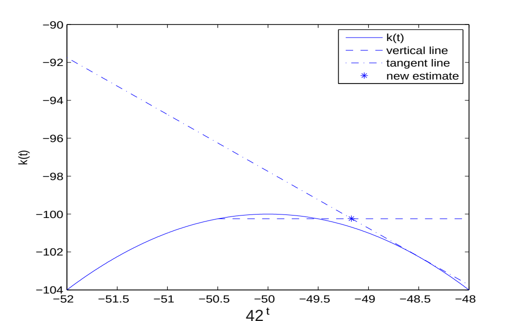
Vertical Cut

if $k'(t_g) < 0\ k'(t_b) > 0$, $k(t_g) < k(t_b)$, then:

use vertical cut to reduce the interval of the t values, i.e., solve

$$k(t_g) + k'(t_g)(t - t_g) = k(t_b)$$

the intersection of secant line on $(t_g, k(t_g))$ and the line $k = k(t_b)$ yields: the new t in (t_b, t_g) and on good side (Similarly if $k(t_b) < k(t_g)$)



Inverse Interpolation

$$\begin{bmatrix} \psi_1^2 & \psi_1 & 1 \\ \psi_2^2 & \psi_2 & 1 \\ \psi_3^2 & \psi_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ t_{\text{new}} \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

on $\psi(t)$

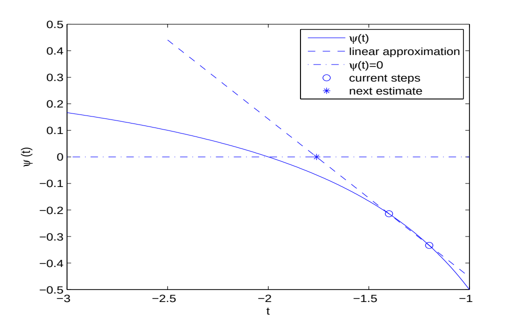


Figure: Linear Interpolation

Inverse interpolation; easy case & hard case (*case 1*)

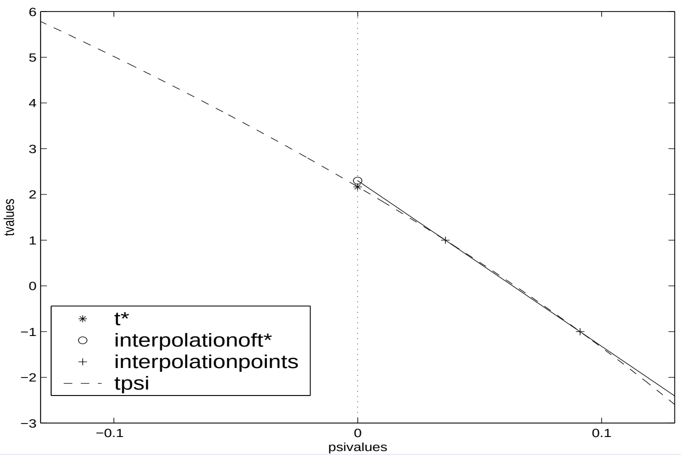


Figure: Inverse interpolation; easy case & hard case (*case 1*)

Inverse interpolation in the hard case (*case 2*)

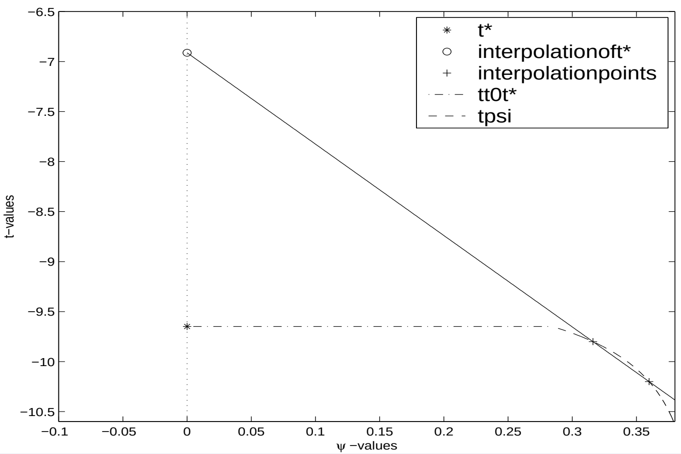


Figure: Inverse interpolation in the hard case (*case 2*)

$$\text{(TRS)} \qquad \begin{array}{rll} q^* = & \min & q(x) := \|Bx - b\|^2 \\ & \text{s.t.} & \|x\| = s \end{array}$$

- $A = B^T B \succeq 0, \operatorname{rank}(A) < n$; B is $m \times n, m > n$ and $\operatorname{rank}(B) < n$.
- $a = B^T b \in \mathcal{R}(A), a \perp \mathcal{N}(A - \lambda_{\min}(A)I)$
- so, hard case (case 2) if $\|A^\dagger a\| \leq s$.

Corollary: high accuracy explicit solution if s chosen large enough!

- deflate the eigenvectors corresponding to the 0 eigenvalues;
- use PCG to solve the system and use an eigenvector to move to the boundary.

cputime in the hard case (*case 2*)

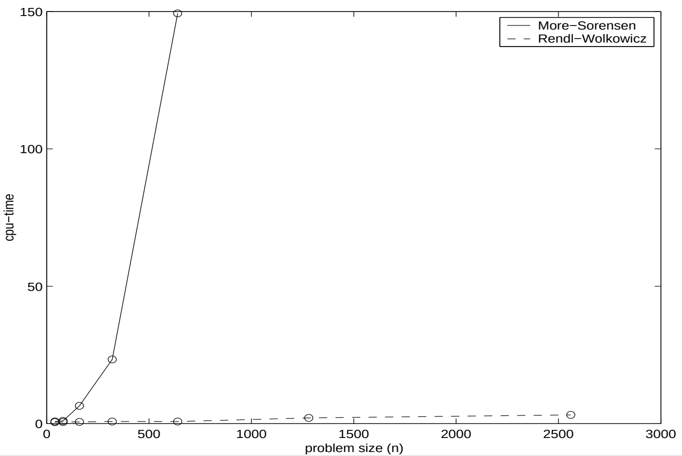


Figure: cputime in the hard case (*case 2*)

log of cputime in the hard case (*case 2*)

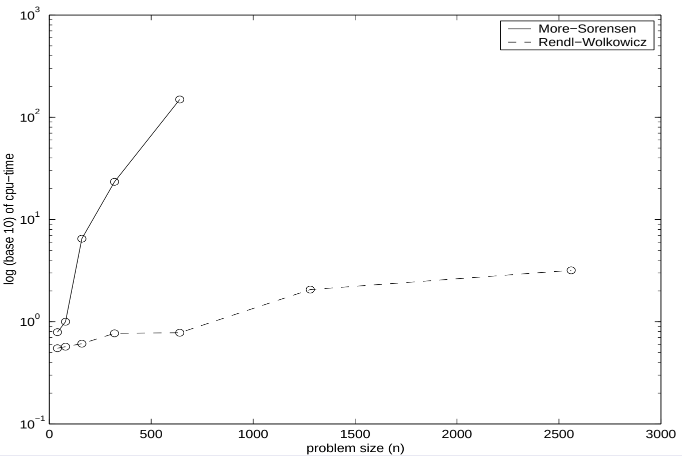


Figure: log of cputime in the hard case (*case 2*)

- C. Fortin and H. Wolkowicz, *The trust region subproblem and semidefinite programming*, Optim. Methods Softw. **19** (2004), no. 1, 41–67. MR MR2062235 (2005c:90065)

- D.M. Gay, *Computing optimal locally constrained steps*, SIAM J. Sci. Statist. Comput. **2** (1981), 186–197.

- M.D. Hebden, *An algorithm for minimization using exact second derivatives*, Tech. Report TP515, Atomic Energy Research Establishment, Harwell, England, 1973.

- Y. Levin and A. Ben-Israel, *The Newton bracketing method for convex minimization*, Computational Optimization and Applications (2001).

- J.J. Moré, *Generalizations of the trust region problem*, Optim. Methods Software **2** (1993), 189–209.

- J.J. Moré and D.C. Sorensen, *Computing a trust region step*, SIAM J. Sci. Statist. Comput. **4** (1983), 553–572.

- I. Pólik and T. Terlaky, *A survey of the S-lemma*, SIAM Rev. **49** (2007), no. 3, 371–418 (electronic). MR MR2353804

- C. Reinsch, *Smoothing by spline functions*, Numer. Math. **10** (1967), 177–183.

- ______, *Smoothing by spline functions II*, Numer. Math. **16** (1971), 451–454.

- R. Stern and H. Wolkowicz, *Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations*, SIAM J. Optim. **5** (1995), no. 2, 286–313. MR 96h:90077

Efficient Solutions for the Large Scale Trust Region Subproblem, TRS

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