Strong Duality, Complementarity, and Duality Gaps, in Conic Convex Optimization

Henry Wolkowicz
Dept. Comb. & Opt, Univ. Waterloo

Joint work with: Simon Schurr and Levent Tunçel

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Cone Optimization, (e.g. $K = \mathbb{S}^n_+$, SDP, $K = \mathbb{R}^n_+$, SDP)

Primal-Dual Pair of Optimization Problems in Conic Form

(assumed finite) $\quad v_P = \sup_y \{\langle b, y\rangle \;:\; \mathcal{A}^* y \preceq_K c\},\qquad (\mathbb{P})$

$(v_P \leq) \quad v_D = \inf_x \{\langle c, x\rangle \;:\; \mathcal{A}x = b,\; x \succeq_{K^*} 0\}.\qquad (\mathbb{D})$

where

- $\mathcal{A}$ - an onto linear transformation; adjoint is $\mathcal{A}^*$
- K - a proper convex cone with dual/polar cone $K^* = \{x : \langle s, x\rangle \geq 0,\ \forall s \in K\}$.
- $s' \preceq_K s''(s' \prec_K s'')$ - partial order, $s'' - s' \in K (\in \mathrm{int} K)$

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Motivation/Outline

Strong Duality Failure (NO Constraint Qualification, CQ)

- Instances: SDP relaxations for hard combinatorial problems (e.g. QAP, GP, strengthened MC)
- Fresh look at known Characterizations of Optimality using Subspace Formulation

Regularization, EFficient Solutions

- theme: use MINIMAL REPRESENTATIONS

Connections Complementarity/Duality

- Surprising Connections Complementarity of Homog. Probl. and duality/Numerical implications

Faces of Cones

Face

A convex cone F is a face of K, denoted $F \trianglelefteq K$, if

$$x, y \in K \text{ and } x + y \in F \implies x, y \in F.$$

If $F \trianglelefteq K$ and $F \neq K$, write $F \triangleleft K$.

Conjugate Face

If $F \trianglelefteq K$, the conjugate face (or complementary face) of F is

$$F^c := F^\perp \cap K^* \trianglelefteq K^*.$$

If $x \in \mathrm{ri}(F)$, then $F^c = \{x\}^\perp \cap K^*$.

Minimal Face (Minimal Cone)

Feasible sets

$$\begin{array}{lcll} \mathcal{F}_P^y & := & \{y : c - \mathcal{A}^* y \succeq_K 0\} & \text{primal} \\ \mathcal{F}_P^s & := & \{s : s = c - \mathcal{A}^* y \succeq_K 0, \text{ for some } y\} & \text{primal slacks} \\ \mathcal{F}_D^x & := & \{x : \mathcal{A}x = b, x \succeq_K^* 0\} & \text{dual} \end{array}$$

Minimal Faces

$$f_P := \mathrm{face}\mathcal{F}_P^s \trianglelefteq K \qquad f_D := \mathrm{face}\mathcal{F}_D^x \trianglelefteq K^*$$

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(Modified) SDP Example from Ramana, 1995

Primal SDP

$$0 = v_P = \sup_y \left\{ y_2 : \begin{pmatrix} 0 & 0 & y_2 \\ 0 & y_2 & 0 \\ y_2 & 0 & y_1 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$y^* = \begin{pmatrix} y_1^* & 0 \end{pmatrix}^T, \quad y_1^* \leq 0, \quad s^* = c - \mathcal{A}^* y^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -y_1^* \end{pmatrix}$$

Slater's CQ fails for primal and dual; $v_D = 1 > v_P = 0$

Minimal Face for Ramana Example

Feasible Set/Minimal Face

$$\mathcal{F}_P^y = \{y \in \mathbb{R}^2 : y_1 \leq 0,\ y_2 = 0\}$$

$$\begin{array}{rcl} f_P & = & \bigcap \{F \trianglelefteq K : \mathcal{F}_P^s = c - \mathcal{A}^*(\mathcal{F}_P^y) \subset F\} \\ & = & \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{S}^2_+ \end{pmatrix} \\ & \lhd & \mathbb{S}^3_+ \end{array}$$

Slater CQ and Minimal Face

If $(\mathbb{P})$ is feasible, then

$$c - \mathcal{A}^* y \not\succ_K 0 \ \forall y \text{ (Slater's CQ fails for } (\mathbb{P}) \text{) } \iff f_P \lhd K$$

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If $(\mathbb{P})$ is feasible, then

$c - \mathcal{A}^* y \not\succ_K 0\ \forall y$ (Slater's CQ fails for $(\mathbb{P})$) $\iff$ $f_P \lhd K$

Regularization of $(\mathbb{P})$ Using Minimal Face

Borwein-W (1981), $f_P = \mathrm{face}\mathcal{F}_P^S$

$(\mathbb{P})$ is equivalent to regularized $(\mathbb{P})$

$$v_{RP} := \sup_y \{\langle b, y\rangle \ : \ \mathcal{A}^* y \preceq_{f_P} c\}. \qquad (\mathbb{RP})$$

Lagrangian Dual DRP Satisfies Strong Duality:

$$v_P = v_{RP} = v_{DRP} := \inf_x \{\langle c, x\rangle \ : \ \mathcal{A}x = b,\ x \succeq_{f_P^*} 0\} \qquad (\mathbb{DRP})$$

and v_{DRP} is attained

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(SYMMETRIC) Subspace Form for ($\mathbb{P}$) and ($\mathbb{D}$)

Assume Linear Feasibility for $\tilde{s}, \tilde{y}, \tilde{x}$; with data A, b, c, K

$$\mathcal{A}^*\tilde{y} + \tilde{s} = c \qquad \mathcal{A}\tilde{x} = b$$

$$\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*) \text{ (range)} \qquad \mathcal{L} = \mathcal{N}(\mathcal{A}) \text{ (nullspace)}$$

Equivalent Primal-Dual Pair in Subspace Form, (e.g. N&N '94)

Particular solution $\hat{+}$ solution of homogeneous equation

$$v_P = c\tilde{x} - \inf_s \left\{ s\tilde{x} : s \in (\tilde{s} + \mathcal{L}^\perp) \cap K \right\}. \qquad (\mathbb{P})$$

$$v_D = \tilde{y}b + \inf_x \{ \tilde{s}x : x \in (\tilde{x} + \mathcal{L}) \cap K^* \}. \qquad (\mathbb{D})$$

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Numerical/Stability Advantages of Subspace Form

Single Bilinear Equation For Interior Point Methods

Current p-d i-p methods introduce ill-conditioning (ill-posedness) from symmetrization & block elimination. Instead solve:

$$0 = F_\mu(y, v) = (c - \mathcal{A}^* y)(\tilde{x} + \mathcal{V}^* v) - \mu I$$

$\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*)$ (range) $\quad \mathcal{L} = \mathcal{N}(\mathcal{A}) = \mathcal{R}(\mathcal{V}^*)$ (nullspace)

Some Applications

- Gauss-Newton method for SDP (Kruk et al 2001)
- GN and CG method for SDP relaxation of Max-Cut (W. 2004)
- Large Scale LP (Wei and W. 2004)

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For $(\mathbb{P})$ and $(\mathbb{D})$

Faces of Recession Directions (feasible case)

$$f_P^0 := \operatorname{face}\left(\mathcal{L}^\perp \cap K\right) (\subset f_P) \qquad f_D^0 := \operatorname{face}\left(\mathcal{L} \cap K^*\right) (\subset f_D)$$

Recall

minimal faces $\quad f_P = \operatorname{face}\mathcal{F}_P^s \quad f_D = \operatorname{face}\mathcal{F}_D^x$

Minimal Subspaces/Linear Transformations

min. subsp. $\quad \mathcal{L}_{PM}^\perp := \mathcal{L}^\perp \cap (f_P - f_P) \quad \mathcal{L}_{DM} := \mathcal{L} \cap (f_D - f_D)$

min. Lin. Tr. $\quad \mathcal{A}_{PM}^* \quad \mathcal{A}_{DM}$

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Regularization of ($\mathbb{P}$) Using Minimal Subspace

Assume K Facially Dual Complete, FDC (Pataki/07, 'nice')

i.e. $F \lhd K \implies K^* + F^\perp$ is closed. (e.g. $\mathbb{S}^n_+, \mathbb{R}^n_+$,SOC).

$\mathcal{L}^\perp_{PM} = \mathcal{L}^\perp \cap (f_P - f_P)$

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Nice and Devious Cones

Alternate Form of BW Characterization

$v_P \geq \sup_{y \in g^{-1}(f_P - f_P)} \langle b, y \rangle + \langle (c - \mathcal{A}^* y), \bar{x} \rangle$, for some $\bar{x} \in f_P^* = K^* + f_P^\perp$; WOLG $\bar{x} \in K^*$

Lemma for SDP Case (Ramana,Tuncel,W./97)

Let $0 \neq F \lhd \mathbb{S}^n_+$. Then

$\mathbb{S}^n_+ + F^\perp$ is closed (nice)

$\mathbb{S}^n_+ + \mathrm{span} F^c$ is not closed (devious)

$\mathbb{S}^n_+ + F^\perp = \overline{\mathbb{S}^n_+ + \mathrm{span} F^c}$

Let $\mathcal{L} = \mathrm{span} F^c$; choose $c = \tilde{s} = 0$ and $\tilde{x} \in (\mathbb{S}^n_+ + F^\perp) \backslash (\mathbb{S}^n_+ + \mathrm{span} F^c)$; (subspace repr. (P),(D): (1)). then $0 = v_P < v_D = \infty$.

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Lemma for SDP Case (Ramana,Tuncel,W./97)

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Strong Duality for (P) ($v_P = v_D$ and v_D is attained)

Minimal Face and Minimal Subspace CQs for (P)

1. $f_P = K$ is a CQ
   (from BW: $f_P^* = K^*$)
2. $\mathcal{L}^\perp \cap (f_P - f_P) = \mathcal{L}_{PM}^\perp = \mathcal{L}^\perp$ is a CQ (if K is FDC (nice))
   ($\tilde{s} \in f_P - f_P : x^* = x_K^* + x_f^* \in f_P^* = K^* + f_P^\perp \implies x^*(\tilde{s} + \mathcal{L}^\perp) = x_K^*(\tilde{s} + \mathcal{L}^\perp)$)

Universal CQ, UCQ for (P) (i.e. independent of feasible data c, b)

$\mathcal{L}^\perp \subset f_P^0 - f_P^0$ is a UCQ (if K is FDC)
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Our Goals:

Goals: Derive an Algorithm that Satisfies

1. recognizes if Slater's CQ holds and if $(\mathbb{P})$–$(\mathbb{D})$ has a zero duality gap (improves on stability/efficiency of B-W algorithm)
2. size of any intermediate cone program solved does not exceed that of $(\mathbb{P})$ or $(\mathbb{D})$ (improves on size/efficiency of Ramana's dual)
3. intermediate cone programs to be solved are well behaved (in the Slater CQ sense)

Theorem of the Alternative for Slater's CQ

THEOREM

Suppose that $(\mathbb{P})$ is feasible. Then exactly one of the following two systems is consistent:

(1) $\mathcal{A}x = 0,\ \langle c, x\rangle = 0,$ and $0 \neq x \succeq_{K^*} 0$

(2) $\mathcal{A}^* y \prec_K c$ (Slater's CQ holds for $(\mathbb{P})$)

Difficult?

In theory, we can solve

$$(*) \quad \min\{0 : x \text{ satisfies } (1)\}$$

to determine if Slater's CQ fails for $(\mathbb{P})$.

But this problem $(*)$ need not satisfy the generalized Slater CQ!

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Stable Theorem of the Alternative

Stable Auxiliary Problem

Let $e \in \operatorname{int}(K) \cap \operatorname{int}(K^*)$; define $\mathcal{A}_c x := \begin{pmatrix} \mathcal{A}x \\ \langle c, x \rangle \end{pmatrix}$

$$\alpha^* := \left\{ \inf_{x,\alpha} \alpha : \mathcal{A}_c x = 0, x + \alpha e \succeq_{K^*} 0, \langle e, x \rangle \leq 1 \right\} \qquad (\mathcal{A})$$

Properties/Advantages

- size of $(\mathcal{A})$ essentially that of $(\mathbb{D})$
- A strictly feasible primal-dual point is easily found.
- Apply primal-dual IPM; assume a barrier for K^* such that the central path defined by it converges to a point in the relative interior of the optimal face; follow central path closely at end of algorithm.

Slater's Condition and the Auxiliary problem

Solution to $(\mathcal{A})$ yields info on $(\mathbb{P})$–$(\mathbb{D})$

Theorem: The x component of the central path for $(\mathcal{A})$ converges to a point in $\mathrm{ri}(\mathrm{face}(G_P))$, where

$$G_P := \{x \ : \ Ax = 0, \ \langle c, x \rangle = 0, \ x \succeq_{K^*} 0\}.$$

Moreover, since $f_P \subset \{x^*\}^\perp \cap K = [\mathrm{face}(G_P)]^c \trianglelefteq K$, one of the following holds:

1. $\alpha^* = 0$ and $x^* = 0$, so Slater's CQ holds for $(\mathbb{P})$, or
2. $\alpha^* = 0$ and $0 \neq x^* \succeq_{K^*} 0$, so $f_P \subset \{x^*\}^\perp \cap K \lhd K$, or
3. $\alpha^* < 0$ and $x^* \succ_{K^*} 0$, so the generalized Slater CQ holds for $(\mathbb{D})$.

Algorithm Alternates to Obtain Minimal Representations

For Minimal Face

From auxiliary problem, find:
$0 \neq x \in K^*, \{x\}^\perp = H, \{x\}^\perp \cap K \supset f_P$

For Minimal Subspace

Find $\mathcal{A}_H$ so that $\mathcal{R}(A_H^*) = \mathcal{R}(A^*) \cap H$
to get reduced problem in H

Previous SDP with $K = \mathbb{S}^3_+$ and a Duality Gap of 1

SeDuMi 1.1 Results

$$y^* = \begin{pmatrix} -0.321 \times 10^6 & 0.372 \end{pmatrix}^T$$

$$s^* = \begin{pmatrix} 0 & 0 & -0.372 \\ - & 0.628 \times 10^5 & 0 \\ - & - & -0.321 \times 10^6 \end{pmatrix};$$

desired accuracy (10^{-6}) achieved but!!

$\langle c, x^* \rangle - \langle b, y^* \rangle \approx -0.12$! and s^* is not pos. semidef.

After One Step of the Reduction

Our code yields correct primal solution:

$$y^* = \begin{pmatrix} -1.50 \\ 0 \end{pmatrix}, \quad s^* = \begin{pmatrix} 0 & 0 & 0 \\ - & 1.00 & 0 \\ - & - & 1.50 \end{pmatrix}$$

Higher Dimensional Numerical Experiments

SDP with $m = n \geq 3$, $b = e_2$, $c = 0$

$$\mathcal{A}^* y = \begin{pmatrix} y_1 & y_2 & y_3 & & \cdots & y_{n-1} & y_n \\ y_2 & y_3 & & & & & \\ \vdots & & & & \ddots & & \\ y_{n-1} & & & & & y_n & \\ y_n & & & & & & 0 \end{pmatrix}$$

SeDuMi/Our Algorithm

SeDuMi gives incorrect primal/dual solution; duality gap of -1; our algorithm gives correct solution
$F_P = \{y \in \mathbb{R}^n : y_1 \leq 0,\ y_2 = \cdots = y_n = 0\}$
min. face $f_P = \{s \in \mathbb{S}^n_+ : s_{11} \geq 0,\ s_{ij} = 0\ \forall (i,j) \neq (1,1)\}$,
and $(\mathbb{D})$ is infeasible.

(Near) Loss of Slater Condition/Strict Feasibility

Theoretical/Numerical Difficulties

- Primal Slater condition implies strong duality, i.e. zero duality gap AND dual attainment.
- (Near) loss of strict feasibility is used as a measure in complexity theory. (e.g. Renegar/95, Freund/01, Lara and Tuncel/02)
- (Near) loss of strict feasibility correlates with number of iterations and loss of accuracy in interior-point methods (e.g. Freund/Ordonez/Toh 2006)

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Loss of Strict Complementarity, (SC)

Strict Complementary Optimal Primal-Dual Pair

- There exists an optimal primal-dual pair x, s such that
  $x + s \succ 0 \qquad (\in \operatorname{int}(K + K^*))$

Theoretical Difficulties/Convergence

- Convergence proofs for asymptotic quadratic superlinear convergence require SC.
- Proofs of convergence to the analytic center require SC

Numerical Difficulties/Relation to Duality Gaps???

increased number of iterations? loss of accuracy?

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Hard SDP Instances (Wei and W. 2006)

Maximal Complementary Solution Pair:

- A p-d pair of optimal solutions $(\bar{s}, \bar{x})$ is a *maximal complementary solution pair* if the pair maximizes the sum $\mathrm{rank}(s) + \mathrm{rank}(x)$ over all p-d optimal (s, x).

Strict Complementarity Nullity, g:

- $g = n - \mathrm{rank}(\bar{s}) - \mathrm{rank}(\bar{x})$, where $(\bar{s}, \bar{x})$ is a maximal complementary solution pair

Hard SDP Instances:

- problems where nullity is nonzero

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Algorithm for Given Nullity g

Algorithm

- Given: rank of optimum x is $r > 0$; number constraints is $m > 1$
- Let $Q = [Q_P|Q_N|Q_D]$ be orthogonal matrix; dimensions of Q_P, Q_N, Q_D are $n \times r, n \times g, n \times (n-r-g)$, resp.
  Construct: $x := Q_P D_x Q_P^T \quad s := Q_D D_s Q_D^T$,
  where $D_x \succ 0$ and $D_s \succ 0$.
- cont...

Algorithm cont...

Algorithm cont...

- Define

$$A_1 = [Q_P|Q_N|Q_D] \begin{bmatrix} 0 & 0 & Y_2^T \\ 0 & Y_1 & Y_3^T \\ Y_2 & Y_3 & Y_4 \end{bmatrix} [Q_P|Q_N|Q_D]^T,$$

where $Y_1 \succ 0$, Y_4 symmetric, and $Q_D Y_2 \neq 0$.
- Choose $A_i \in \mathbb{S}^n$, with $\{A_1 Q_P, A_2 Q_P, \ldots, A_m Q_P\}$ lin. indep. (Note $A_1 Q_P = Q_D Y_2 \neq 0$.)
- Set $b := \mathcal{A}(x)$, $c := \mathcal{A}^*(y) + s$, with $y \in \Re^m$ random.

Theorem for Generating Hard Instances

Theorem

The data $(\mathcal{A}, b, c)$ constructed in the above algorithm gives a *hard* SDP instance with a strict complementarity nullity g.

Proof Outline

- Step 2 guarantees $s, x \succeq 0, sx = 0$ but strict complementarity fails.
- step 5 guarantees primal-dual feasibility, i.e. s,x are an optimal pair.
- Steps 3,4 guarantee s, x are a maximal complementary solution pair.

Generating Hard Instances with Slater Condition

Corollary

With data $(\mathcal{A}, b, c)$ constructed using above algorithm:

1. If the following additional condition on A_2 is satisfied $[Q_P|Q_N]^T A_2 [Q_P|Q_N] \succ 0$, then Slater's CQ holds for (P).
2. If the following additional conditions on $A_i, i = 1, \dots, m$, are satisfied,

$$\begin{array}{c} \operatorname{trace} Y_4 = -\operatorname{trace} Y_1 \\ \alpha > 0, \quad \operatorname{trace} A_i x = \alpha \operatorname{trace} A_i, \quad i = 2, \dots, m, \end{array}$$

then $\hat{x} = \alpha I \succ 0$ is feasible for (D)

Empirical Observations

Numerical Difficulties Correlate with Large Nullity

- There is a strong correlation between the iteration number to achieve the desired stopping tolerance and the size of the complementarity nullity, when the accuracy requirement is high.
- Large nullity instances cause problems for SDPT3 solver.
- Local asymptotic convergence rate is slower when nullity is larger.

Theoretical Connections Complementarity/Duality?

Numerical Difficulties

(Both) loss of Slater CQ (strict feasibility) and loss of strict complementarity independently result in numerical difficulties for interior-point methods.

Theoretical Connection?

Is there a theoretical connection between loss of duality (from loss of a CQ) and loss of strict complementarity?

Complementarity Partition

Recall Faces of Recession Directions

$$f_P^0 := \text{face}\left(\mathcal{L}^\perp \cap K\right), \qquad f_D^0 := \text{face}\left(\mathcal{L} \cap K^*\right)$$

The pair f_P^0, f_D^0 define a Complementarity Partition

$\text{face}(f_P^0) \subset \text{face}(f_D^0)^c$ and $\text{face}(f_D^0) \subset \text{face}(f_P^0)^c$.
it is a strict complementarity partition if both
$[\text{face}(f_P^0)]^c = \text{face}(f_D^0)$ and $[\text{face}(f_D^0)]^c = \text{face}(f_P^0)$;
it is proper if f_P^0 and f_D^0 are both nonempty.

SDP Picture

For SDP (after a rotation)

$$\begin{bmatrix} f_D^0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_P^0 \end{bmatrix}$$

Form Primal-Dual Pair

$$\tilde{x} = \tilde{s} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v \succ 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \langle s, x \rangle \geq \|v\|_F^2,$$

for all feasible pairs s, x. (gap is dimension of v)

Strict Complementarity and Nonzero Gaps

Theorem: K is a proper cone

(1) If f_P^0, f_D^0 define a proper complementarity partition with a gap of dimension 1, so, the partition is not a strict complementarity partition, then there exists $\bar{s}$ and $\bar{x}$ such that $(\mathbb{P})$–$(\mathbb{D})$ with data $(\mathcal{L}, K, \bar{s}, \bar{x})$ has a finite nonzero duality gap.

(Partial Converse)

(2) If (a) $(\mathbb{P})$–$(\mathbb{D})$ with data $(\mathcal{L}, K, \bar{s}, \bar{x})$ has a finite nonzero duality gap with both optimal values attained, and (b) the objective functions are constant along all recession directions of $(\mathbb{P})$ and $(\mathbb{D})$, then f_P^0, f_D^0 has a proper complementarity partition but not a strict complementarity partition.

Conclusion

- Minimal Representations of the data regularize (P) min. face f_P and/or the min. L.T. $\mathcal{A}_{PM}$ or $\mathcal{L}^*_{PM}$
- presented a stable algorithm to solve (feasible) conic problems for which Slater's CQ fails
- Failure of strict complementarity for the associated recession problems is closely related to the existence of instances having a finite nonzero duality gap; provides a means of generating instances for testing.