Sensor Network Localization, Euclidean Distance Matrix Completions, and Graph Realization

Henry Wolkowicz

with: Yichuan Ding, Nathan Krislock, Veronica Piccialli, Jiawei Qian

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Outline

Ad hoc wireless sensor network

Large Number of Sensor Nodes

- they gather data and communicate among themselves

Anchors

- a few anchors have fixed, known locations (e.g. with GPS/bulky)

Radio Range

- sensors/anchors within given (radio) range have known (approx./noise level) distance measurements

Problem Statement

- Determine (approximate) sensor positions using Semidefinite Relaxations/Robust Algorithm

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Underlying Graph Realization/Partial EDM

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

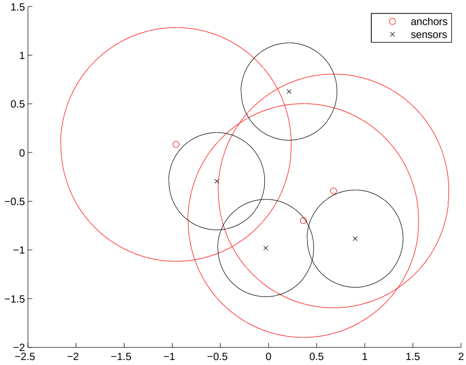
- node set $\mathcal{V} = \{1, \ldots, m+n\}$
- edge set $(i,j) \in \mathcal{E}$; $\omega_{ij} = \|p^i - p^j\|^2$ known approximately
- The anchors form a clique (complete subgraph)
- Realization of $\mathcal{G}$ in $\Re^r$: a mapping of node $v^i \to p^i \in \Re^r$ with squared distances given by ω.

Corresponding Partial Euclidean Distance Matrix, EDM

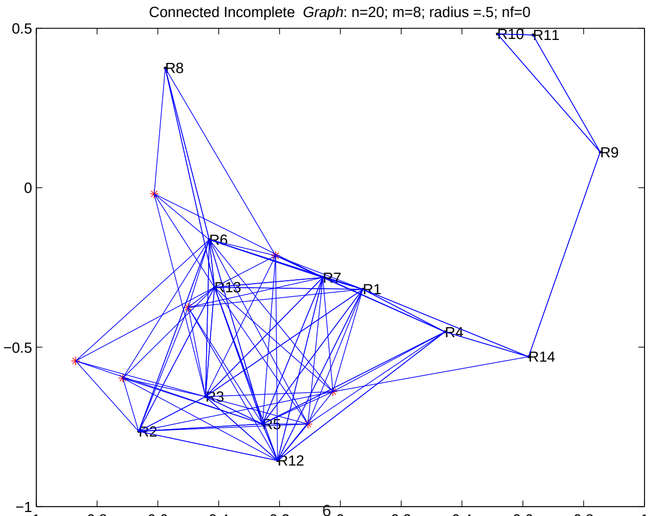
$$E_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise,} \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known (approx) squared Euclidean distances between sensors p^i, p^j; anchors correspond to a clique

Connected Incomplete Graph/Figure



Problem Instance with Radio Range



Applications

Applications for Sensors

- health, military, home
- natural habitat monitoring, earthquake detection, weather/current monitoring
- random deployment in inaccessible terrains or disaster relief operations

Future Applications?

- bicycles, car/computer parts, guns (to prevent theft)
- students/teenagers (prevent class absence)

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SNL with Anchors

SNL Model Formulation

- $p^1, \ldots, p^n \in \Re^r$ unknown (sensor) points
- $a^1 = p^{n+1}, \ldots, a^m = p^{n+m} \in \Re^r$ known (anchor) points
- r embedding dimension (usually 2 or 3)
- $A^T := [a^1, a^2, \ldots, a^m]$ $X^T := [p^1, p^2, \ldots, p^n]$

$$\begin{aligned} P^T &:= (p^1, p^2, \ldots, p^n, a^1, a^2, \ldots, a^m) \\ &= (p^1, p^2, \ldots, p^n, p^{n+1}, p^{n+2}, \ldots, p^{n+m}) \end{aligned}$$

Sensor/Anchor Points

$$P = \begin{pmatrix} X \\ A \end{pmatrix}$$

NLLS

Hard Weighted Nonlinear Least Squares

- with W weight matrix:

$$\begin{array}{ll} \min & f_1(P) := \frac{1}{2}\sum_{i,j} W_{ij}(\|p^i - p^j\|^2 - E_{ij})^2 \\ \text{s.t.} & \|p^i - p^j\|^2 \leq U_{ij} \;\; \forall (i,j) \\ & \|p^i - p^j\|^2 \geq L_{ij} \;\; \forall (i,j) \end{array}$$

- underdetermined, ill-conditioned

Change EDM to SDP

Linear Mapping $\mathcal{K}$: SDP onto EDM

- with $P = \begin{pmatrix} X \\ A \end{pmatrix}$:

$$\begin{aligned} E_{ij} &= \|p^i - p^j\|^2 \\ &= (p^i)^T p^i + (p^j)^T p^j - 2(p^i)^T p^j \\ &= \left(\operatorname{diag}(PP^T)\, e^T + e\, \operatorname{diag}(PP^T) - 2PP^T\right)_{ij} \end{aligned}$$

- $$E = \operatorname{diag}(B)\, e^T + e\, \operatorname{diag}(B) - 2B =: \mathcal{K}(B)$$

Euclidean Distance Matrix, EDMC, Formulation

$\bar{Y} = PP^T$

- $$E_{ij} = \mathcal{K}(\bar{Y})_{ij} = \|p^i - p^j\|^2 \qquad \text{EDM}$$
  $$\begin{aligned} f_1(P) &= \frac{1}{2}\sum_{i,j} W_{ij}(\|p^i - p^j\|^2 - E_{ij})^2 \quad \text{quartic} \\ &= \|W \circ (\mathcal{K}(\bar{Y}) - E)\|_F^2 \quad \text{quadratic} \end{aligned}$$
- quartic objective is changed to a quadratic using additional hard quadratic constraint, $\bar{Y} = PP^T$

Euclidean Distance Matrix, EDMC, Formulation

EDMC

- Using: $\bar{Y} := PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & AA^T \end{pmatrix}$
- EDMC is:

$$
\begin{array}{cllcl}
\min & f_2(\bar{Y}) := \frac{1}{2}\|W \circ (\mathcal{K}(\bar{Y}) - E)\|_F^2 & & \\
\text{subject to} & g_u(\bar{Y}) := H_u \circ (\mathcal{K}(\bar{Y}) - U) & \leq & 0 \\
 & g_l(\bar{Y}) := H_l \circ (\mathcal{K}(\bar{Y}) - L) & \geq & 0 \\
 & \bar{Y} - \begin{pmatrix} Y & XA^T \\ AX^T & AA^T \end{pmatrix} & = & 0 \\
 & Y - XX^T & = & 0
\end{array}
$$

Operator $\mathcal{K}$

$\mathcal{K}$: Centered to Hollow Subspaces

- $$\begin{array}{rcll} \mathcal{S}_C & := & \{B \in \mathcal{S}^n : Be = 0\}, & \text{centered} \\ \mathcal{S}_H & := & \{D \in \mathcal{S}^n : \operatorname{diag}(D) = 0\}, & \text{hollow} \end{array}$$
- $J = I - \frac{1}{n}\text{ONES}$; $\operatorname{offDiag} :=$ zero-out diagonal

$$\begin{array}{rcll} \mathcal{K}(B) & := & \operatorname{diag}(B)e^T + e\operatorname{diag}(B)^T - 2B \\ & := & \mathcal{D}_e(B) - 2(B); \\ \mathcal{K}^\dagger(D) & = & -\frac{1}{2}J\left[\operatorname{offDiag}(D)\right]J & \text{M.P. Gen. Inv.} \\ \mathcal{K}^*(D) & = & \mathcal{D}_e^*(D) - 2D = 2\operatorname{Diag}(De) - 2D \end{array}$$

Properties of $\mathcal{K}$

$\mathcal{K}, \mathcal{K}^\dagger$

- linear operators $\mathcal{K}, \mathcal{K}^\dagger$ are one-one and onto mappings between EDM cone in S_H and the face of the semidefinite cone $\mathcal{F}_{\mathcal{E}_n} := \mathrm{SDP} \cap \mathcal{S}_C$ (centered SDP)
  $\mathcal{K}^\dagger(\mathrm{EDM}) = \mathcal{F}_{\mathcal{E}_n}, \qquad \mathcal{K}(\mathcal{F}_{\mathcal{E}_n}) = \mathrm{EDM}.$
- $\mathcal{F}_{\mathcal{E}_n}$ has empty interior - a problem for interior-point methods!

Schoenberg Theorem

- An $n \times n$ symmetric D with $\operatorname{diag}(D) = 0$ is EDM iff $Y := -\frac{1}{2}JDJ \succeq 0$, where $J = I - ee^T/n$, $e = (1, \ldots, 1)^T$, and embedding dim of $D = \operatorname{rank} Y$.
- i.e. if and only if $Y = \mathcal{K}^\dagger(D) \succeq 0$
- The points $p^1, \ldots, p^n$ that generate D are given by the rows of P where $Y = PP^T$.

Matrix Reformulation/Cone Optimization

Matrices with Sensors/Anchors

$$P = \begin{pmatrix} X \\ A \end{pmatrix}, \bar{Y} := PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & AA^T \end{pmatrix}$$

SDP Cone Optimization Problem

$$\begin{array}{lrcl} \min & \frac{1}{2}\|W \circ (\mathcal{K}(\bar{Y}) - E)\|_F^2 & & \\ \text{subject to} & H_u \circ (\mathcal{K}(\bar{Y}) - U^b) & \leq & 0 \\ & H_l \circ (\mathcal{K}(\bar{Y}) - L^b) & \geq & 0 \\ \text{hard constraint} & \boxed{\bar{Y} - PP^T = 0} & & \\ & \bar{Y} \succeq 0, \quad \operatorname{sblk}_2(\bar{Y}) = AA^T & & \end{array}$$

NOTE: Discard hard constraint. Relaxation to $\bar{Y} \succeq PP^T$ is redundant/NOT needed

Convexity wrt Löwner partial Order

Generalized Convex Constraint

The constraint $g(P, Y) = PP^T - Y \preceq 0$ is $\succeq$-convex, since each function

$$\phi_Q(P, Y) := \operatorname{trace} Qg(P, Y) \quad \text{is convex } \forall Q \succeq 0.$$

SDP Cone is Self-Polar

$$\begin{aligned} \operatorname{trace} QPP^T &= \operatorname{trace} QPIP^T \\ &= \operatorname{vec}(P)^T (I \otimes Q) \operatorname{vec}(P) \end{aligned}$$

Hessian is $I \otimes Q \succeq 0$; and

$$\text{SDP} = \text{SDP}^+ := \{S \in \mathcal{S}^n : \operatorname{trace} SP \geq 0, \forall P \succeq 0\}$$

Loss of Strict Feasibility

Slater's CQ fails

$$\bar{Y} = \begin{pmatrix} Y_{11} & Y_{21}^T \\ Y_{21} & AA^T \end{pmatrix} \succeq 0 \quad \Rightarrow \quad Y_{21}^T = XA^T, \text{ for some } X$$

$\mathrm{sblk}_2(\bar{Y}) = AA^T$ or equiv: $\bar{Y}_{22} = AA^T$
implies that $\bar{Y}$ is singular (positive semidefinite)

Take advantage/PROJECT onto Minimal Face

Use:
$\mathrm{sblk}_2(\mathcal{K}(\bar{Y})) = \mathcal{K}(AA^T)$ fixed clique of distances

Minimal Face

Minimal Face of SDP cone

Suppose $\mathcal{F}$ is a face of SDP^{n+m}.
$Z \in \operatorname{relint} \mathcal{F}$ iff $t := \operatorname{rank} Z = \leq n+m$; t is max rank in $\mathcal{F}$.
Let $Z = UDU^T$, $\mathcal{R}(Z) = \mathcal{R}(U)$ (same range).

Project to Lower Dimension

$$\mathcal{F} = U(\mathrm{SDP}^t)U^T, \quad U \text{ is } (n+m) \times t$$

Here we have

$$0 \preceq Z = \bar{Y} = \begin{pmatrix} I_n & 0 \\ 0 & AA^T \end{pmatrix}, \quad \operatorname{rank}(Z) = t \leq n+r.$$

Project onto Minimal Face

Minimal Face of SDP cone

$A = U\Sigma_r V^T$ $m \times r$, rank r, compact SVD

$\bar{Y} = \begin{pmatrix} Y_{11} & Y_{21}^T \\ Y_{21} & Y_{22} \end{pmatrix} \succeq 0, \bar{Y}_{22} = AA^T, \bar{Y} \in \mathcal{S}^{n+m}$

<u>IFF</u>

$$\bar{Y} = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix} Z \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}^T \succeq 0, Z_{22} = I_r, Z \in \mathcal{S}^{n+r}$$

<u>IFF</u>

$$\bar{Y} = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} Z \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}^T \succeq 0, Z_{22} = \Sigma_r, Z \in \mathcal{S}^{n+r}$$

Summary of Projection

Feasible $\bar{Y}$

$$\bar{Y} = \begin{pmatrix} Y_{11} & XA^T \\ AX^T & Y_{22} \end{pmatrix} \succeq 0, \bar{Y}_{22} = AA^T$$

Structure of Minimal Face of SDP cone

$$\bar{Y} = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{21}^T \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}^T \succeq 0, Z_{22} = I_r$$

Z_{21} plays role of X^T.

$E = \mathcal{K}(\bar{Y})$

$E_{22} = \mathcal{K}(AA^T)$ (clique)

Equivalent Reduced Problem

With

$$Z := \begin{pmatrix} Z_{11} & Z_{21}^T \\ Z_{21} & Z_{22} \end{pmatrix}, \qquad Z_{22} := I_r$$

$$\mathcal{Y}(Z) := \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix} Z \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}^T$$

$$\text{EDMC}-\boldsymbol{R} \qquad \begin{array}{ll} \min & \frac{1}{2}\|W \circ \mathcal{K}(\mathcal{Y}(Z)) - \bar{E}\|_F^2 \\ \text{s.t.} & H_u \circ \mathcal{K}(\mathcal{Y}(Z)) - \bar{U}^b \le 0 \\ & \bar{L}^b - H_l \circ \mathcal{K}(\mathcal{Y}(Z)) \le 0 \\ & Z \succeq 0 \end{array}$$

ℓ_2 rather than ℓ_1 in the literature, e.g. H. Jin(05), A. So, Y. Ye(05), P. Biswas, T. Liang, K. Toh, T. Wang, Y. Ye(06).

Problem/Questions for Relaxation

Problem/Questions

1. $\{(\bar{Y}, P) : \bar{Y} = PP^T\} \subset \{(\bar{Y}, P) : \bar{P}P^T - \bar{Y} \preceq 0\}$ (But, is Lagrangian relaxation stronger?)
2. Least squares problem is (usually) underdetermined/ill-conditioned.
3. linearization (using Schur complement) results in constraint *NOT* onto, i.e. two relaxations *NOT* numerically equivalent quadratic constraint $XX^T - Y \preceq 0$ better!
4. Exploit EDM model?

Clique Reductions

Clique: Sensors with Known Distances

$S_c := \{p^{t+1}, \dots, p^n\}$
all distances $\|p^i - p^j\|$ known $\forall t+1 \le i,j \le n$ (clique)

$\mathcal{N}(\mathcal{K}) = \mathcal{R}(\mathcal{D}_e)$, $\quad \mathcal{D}_e(y) = ye^T + ey^T$

$$E = \begin{pmatrix} E_1 & \cdot & \cdot \\ \cdot & \underline{E_2} & \cdot \\ \cdot & \cdot & \underline{E_3} \end{pmatrix} \quad \begin{pmatrix} \cdot \\ \underline{\text{clique for sensors}} \\ \underline{\text{clique for anchors}} \end{pmatrix} \qquad \text{EDM}$$

$$\bar{Y} = \begin{pmatrix} Y_{11} & \cdot & \cdot \\ \cdot & Y_{22} \in \mathcal{K}^\dagger(E_2) + \mathcal{D}_e(y) & \cdot \\ \cdot & \cdot & Y_{33} \end{pmatrix}, \quad E = \mathcal{K}(\bar{Y})$$

Exploit Loss of Slater CQ

Solve and Reduce

Let: $B = \mathcal{K}^\dagger(E_{22}), B \in \mathcal{S}_+^{n-t}, Be = 0$
Then: $\exists y$: $\bar{Y}_{22} = P_2 P_2^T = B + ye^T + ey^T$ is rank $\leq r$
And $\hat{B} := B + ee^T + ee^T$ is rank $r_2 \leq r + 1$

If $r_2 < n - t$, then Slater's CQ fails! Reduction.

$\hat{B} = U_2 D_2 U_2^T \in \operatorname{relint} \mathcal{F}_2, \mathcal{R}(\hat{B}) = \mathcal{R}(U_2)$

$$\bar{Y} = \begin{pmatrix} I_t & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix} Z \begin{pmatrix} I_t & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix}^T, \quad Z \in \mathcal{S}^{(t+r_2)+r}.$$

Best Feasible Solution, I

Once optimal $\bar{Y}$ found/Simple Approach

can find sensor positions X using relaxation $\bar{Y} \succeq PP^T$.
Equivalently: $Y \succeq XX^T$ or $\begin{pmatrix} Y & X^T \\ X & I \end{pmatrix} \succeq 0$.

Best Feasible Solution, II

But can also use EDM

factor $\bar{Y}$ (use Schoenberg Thm); largest r eigenvalues/best rank r approximation $\mathcal{K}(\bar{Y}) \approx \mathcal{K}(\bar{Y}_r) = \mathcal{K}(\sum_{i=1}^{r} \lambda_i(\bar{Y}) v_i v_i^T)$

$$QQ^T = I,\ V = \left[\sqrt{\lambda_1} v_1 \dots \sqrt{\lambda_r} v_r\right] = \begin{pmatrix} \bar{X} \\ \bar{A} \end{pmatrix}$$

$$\bar{Y}_r = (VQ)(VQ)^T \approx \begin{pmatrix} X \\ A \end{pmatrix} \begin{pmatrix} X \\ A \end{pmatrix}^T$$

Analytic Solution

$Q^* := V_Q U_Q^T \in \operatorname{argmin}_{Q^T Q = I} \|\bar{A}Q - A\|_F^2$; where $U_Q \Sigma_Q V_Q$ is SVD of $A^T \bar{A}$

Compare two methods

Method 1

Recover X from optimal $\bar{Y}$, $X = \bar{Y}_{21}^T A(A^T A)^{-1}$.

Method 2

Use SVD to compute best rank r approximation to optimal $\bar{Y}$: $\bar{Y}_r = P_r P_r^T$. Then find an orthogonal Q so that $P = P_r Q$ has the anchors in the correct positions.

distances of X estimates from true sensor locations X^*

	test 1	test 2	test 3	test 4	test 5	test 6	test 7	mean
Method 1	1.2780	1.4200	1.4801	1.3696	1.1820	1.4317	1.3912	1.3647
Method 2	0.1887	0.1630	0.1050	0.1394	0.0778	0.0808	0.3881	0.1633

GN Framework

Outline of GN Method

- primal-dual strictly feasible starting point is available
- Use Wolfe Dual and eliminate primal-dual feasibility equations
- exact primal-dual feasibility at each iteration
- NLF (crossover - asymptotic quadratic convergence - high accuracy)

Lagrangian of EDMC -R

Form Lagrangian of EDMC -R

$$\begin{aligned} L(x,y,\Lambda_u,\Lambda_l,\Lambda) \;=\; & \tfrac{1}{2}\|W \circ \mathcal{K}(\mathcal{Y}(x,y) - \bar{E}\|_F^2 \\ & +\langle \Lambda_u, H_u \circ \mathcal{K}(\mathcal{Y}(x,y)) - \bar{U}\rangle \\ & +\langle \Lambda_l, \bar{L} - H_l \circ \mathcal{K}(\mathcal{Y}(x,y))\rangle \\ & -\langle \Lambda, \mathrm{sBlk}_1(I) + \mathcal{Z}_s(x,y)\rangle, \end{aligned}$$

Here $0 \leq \Lambda_u, 0 \leq \Lambda_l \in \mathcal{S}^{m+n}, \quad 0 \preceq \Lambda \in \mathcal{S}^{m+n}$

$$\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_{21}^T \\ \Lambda_{21} & \Lambda_2 \end{pmatrix}, \qquad \langle A, B\rangle = \mathrm{trace}\, A^T B.$$

Primal-Dual Optimality Conditions

Theorem

The primal-dual variables $x, y, \Lambda, \lambda_u, \lambda_l$ are optimal for EDMC-R if and only if the following holds: Primal Feasibility; Dual Feasibility; Complementary Slackness

Primal Feasibility

The slack variables satisfy

$$S_u = \bar{U} - H_u \circ (\mathcal{K}(\mathcal{Y}(x,y))), \; s_u = \operatorname{svec} S_u \geq 0$$
$$S_l = H_l \circ (\mathcal{K}(\mathcal{Y}(x,y))) - \bar{L}, \quad s_l = \operatorname{svec} S_l \geq 0$$
$$Z_s := \operatorname{sBlk}_1(I) + \operatorname{sBlk}_2 \operatorname{sMat}(y) + \operatorname{sBlk}_{21} \operatorname{Mat}(x) \succeq 0$$

Primal-Dual Optimality Conditions IIa

Dual Feasibility - Part a

$$
\begin{aligned}
(\mathcal{Z}_s^x)^*(\Lambda) &= \lambda_{21} \\
&= [W \circ (\mathcal{K}\mathcal{Y}^x)]^* \left(W \circ \mathcal{K}(\mathcal{Y}(x,y)) - \bar{E}\right) \\
&\quad + [H_u \circ (\mathcal{K}\mathcal{Y}^x)]^* (\Lambda_u) \\
&\quad - [H_l \circ (\mathcal{K}\mathcal{Y}^x)]^* (\Lambda_l) \\
(\mathcal{Z}_s^y)^*(\Lambda) &= \lambda_2 \\
&= [W \circ (\mathcal{K}\mathcal{Y}^y)]^* \left(W \circ \mathcal{K}(\mathcal{Y}(x,y)) - \bar{E}\right) \\
&\quad + [H_u \circ (\mathcal{K}\mathcal{Y}^y)]^* (\Lambda_u) \\
&\quad - [H_l \circ (\mathcal{K}\mathcal{Y}^y)]^* (\Lambda_l)
\end{aligned}
$$

Primal-Dual Optimality Conditions IIb

Dual Feasibility - Part b

$$\begin{aligned}\Lambda &= \mathrm{sBlk}_1\mathrm{sMat}(\lambda_1) + \mathrm{sBlk}_2\mathrm{sMat}(\lambda_2)\\ &\quad + \mathrm{sBlk}_{21}\mathrm{Mat}(\lambda_{21}) \succeq 0;\end{aligned}$$

$$\lambda_u \geq 0; \lambda_l \geq 0$$

$$\Lambda = \Lambda(\lambda_1, x, y, \lambda_u, \lambda_l)$$

Primal-Dual Optimality Conditions III

Complementary Slackness

$$\begin{aligned} \lambda_u \circ s_u &= 0 \\ \lambda_l \circ s_l &= 0 \\ \Lambda Z_s &= 0 \qquad (\text{equivalently } \operatorname{trace} \Lambda Z_s = 0\ ) \end{aligned}$$

Perturbed Complementary Slackness

After Substitutions: Overdetermined Bilinear Optimality System

$$F_\mu(x, y, \lambda_u, \lambda_l, \lambda_1) := \begin{pmatrix} \lambda_u \circ s_u - \mu_u e \\ \lambda_l \circ s_l - \mu_l e \\ \boxed{\Lambda Z_s - \mu_c I} \end{pmatrix} = 0,$$

an $\boxed{\text{overdetermined}}$ bilinear system with
$(m_u + n_u) + (m_l + n_l) + (n + r)^2$ equations
$nr + t(n) + (m_u + n_u) + (m_l + n_l) + t(r)$ variables.

Here
$s_u = s_u(x, y)$, $s_l = s_l(x, y)$, $\Lambda = \Lambda(\lambda_1, x, y, \lambda_u, \lambda_l)$,
$Z_s = \mathcal{Z}_s(x, y)$

Least Squares Problem for Search Direction

Gauss-Newton Search Direction

$$\Delta s := \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \lambda_u \\ \Delta \lambda_l \\ \Delta \lambda_1 \end{pmatrix}$$

overdetermined linearized system

$$F'_\mu(\Delta s) \cong F'_\mu(x, y, \lambda_u, \lambda_l, \lambda_1)(\Delta s) = -F_\mu(x, y, \lambda_u, \lambda_l, \lambda_1)$$

Blocks of Equations

1. $\lambda_u \circ \operatorname{svec} \mathcal{K}_{H_u}(\Delta x, \Delta y) + s_u \circ \Delta\lambda_u = \mu_u e - \lambda_u \circ s_u$

2. $\lambda_l \circ \operatorname{svec} \mathcal{K}_{H_l}(\Delta x, \Delta y) + s_l \circ \Delta\lambda_l = \mu_l e - \lambda_l \circ s_l$

3.
$$\begin{aligned}
&\Lambda \mathcal{Z}_s(\Delta x, \Delta y) + [\operatorname{sBlk}_1(\operatorname{sMat}(\Delta\lambda_1)) \\
&\quad +\operatorname{sBlk}_2\Big(\operatorname{sMat}\big\{(\mathcal{K}^y_W)^*\mathcal{K}_W(\Delta x, \Delta y) \\
&\qquad +(\mathcal{K}^y_{H_u})^*(\operatorname{sMat}(\Delta\lambda_u)) - (\mathcal{K}^y_{H_l})^*(\operatorname{sMat}(\Delta\lambda_l))\big\}\Big) \\
&\qquad\quad +\operatorname{sBlk}_{21}\Big(\operatorname{Mat}\big\{(\mathcal{K}^x_W)^*\mathcal{K}_W(\Delta x, \Delta y) \\
&\qquad\qquad +(\mathcal{K}^x_{H_u})^*(\operatorname{sMat}(\Delta\lambda_u)) - (\mathcal{K}^x_{H_l})^*(\operatorname{sMat}(\Delta\lambda_l))\big\}\Big)\Big] Z_s \\
&= \mu_c I - \Lambda Z_s
\end{aligned}$$

Strictly Feasible Starting Point

Initial Str. Feas. Start Heuristic

If the graph is connected, we can use the stationarity equations and get a strictly feasible primal-dual starting point and maintain exact numerical primal-dual feasibility throughout the iterations.

Explicit Preconditioning

Diagonal Preconditioning

Given $A \in \mathcal{M}^{m\times n}$, $m \geq n$ full rank matrix;
use condition number for $K \succ 0$:

$$\omega(K) = \frac{\operatorname{trace}(K)/n}{\det(K)^{1/n}}$$

Optimal Diagonal Scaling

$$D^* := \operatorname{Diag}(1/\|A_{:,i}\|) \in \operatorname{argmin}_{D\succ 0}\omega\left((AD)^T(AD)\right)$$

(cite Dennis-W.)
$\|A_{:,i}\|$, the norms of the columns of $F'_\mu(\cdot)$ can be found explicitly/efficiently.

Numerics: Increasing Noise Factor

density $W = .75$; density $L = .8$; $n = 15, m = 5, r = 2$

nf	optvalue	relaxation	cond.number F'_μ	sv($\mathcal{Z}_s$)	sv(F'_μ)
0.0000e+000	3.9909e-009	1.1248e-005	3.8547e+006	15	19
5.0000e-002	7.5156e-004	4.4637e-002	1.0244e+011	6	27
1.0000e-001	3.7103e-003	1.1286e-001	1.9989e+010	5	25
1.5000e-001	6.2623e-003	1.3125e-001	1.0065e+010	6	14
2.0000e-001	1.3735e-002	1.3073e-001	6.8833e+009	7	12
2.5000e-001	2.3426e-002	2.4828e-001	2.4823e+010	8	6
3.0000e-001	6.0509e-002	2.3677e-001	3.4795e+010	7	7
3.5000e-001	5.5367e-002	3.7260e-001	2.3340e+008	6	4
4.0000e-001	7.6703e-002	3.6343e-001	8.9745e+010	8	3
4.5000e-001	1.2493e-001	6.9625e-001	3.2590e+010	6	9
5.0000e-001	1.3913e-001	3.9052e-001	2.2870e+005	8	0
5.5000e-001	8.8552e-002	3.8742e-001	5.8879e+007	8	2
6.0000e-001	4.2425e-001	4.1399e-001	4.9251e+012	8	4
6.5000e-001	2.0414e-001	6.6054e-001	2.4221e+010	7	4
7.0000e-001	1.2028e-001	3.4328e-001	1.9402e+010	7	6
7.5000e-001	2.6590e-001	7.9316e-001	1.3643e+011	7	4
8.0000e-001	4.7155e-001	3.7822e-001	6.6910e+009	8	2
8.5000e-001	1.8951e-001	5.8652e-001	1.4185e+011	6	7
9.0000e-001	2.1741e-001	9.8757e-001	2.9077e+005	8	0
9.5000e-001	4.4698e-001	4.6648e-001	2.7013e+006	9	2

Decreasing density W and L

$n = 15, m = 5, r = 2$

densityW	optvalue	$\|Y - XX\| / \|XX^T\|$	$\|\bar{Y} - PP^T\| / \|PP^T\|$	sv(Z_S)	sv(F'_μ)
9.9900e-001	9.4954e-009	2.8991e-005	2.3742e-005	15	6
9.5405e-001	5.1134e-009	1.5913e-005	1.3441e-005	15	10
9.0910e-001	5.9131e-009	2.0394e-005	1.8625e-005	15	12
8.6415e-001	3.9076e-009	1.4313e-005	1.0464e-005	15	18
8.1920e-001	4.0578e-009	2.7828e-005	1.7884e-005	15	18
7.7425e-001	6.8738e-009	4.0661e-005	2.7265e-005	15	15
7.2930e-001	3.6859e-009	8.4506e-006	6.1864e-006	15	21
6.8435e-001	4.6248e-009	1.9904e-005	1.3379e-005	15	22
6.3940e-001	4.6809e-009	2.0534e-005	1.5896e-005	15	29
5.9445e-001	9.6305e-009	3.3416e-005	2.6535e-005	15	18
5.4950e-001	8.2301e-010	1.0282e-005	8.8234e-006	15	91
5.0455e-001	6.3663e-009	1.2188e-004	1.0635e-004	13	51
4.5960e-001	6.4292e-010	4.6191e-004	3.7895e-004	12	95
4.1465e-001	3.0968e-009	7.4456e-005	6.1611e-005	14	54
3.6970e-001	9.7092e-010	3.5303e-004	2.8009e-004	14	98
3.2475e-001	5.5651e-011	2.7590e-002	2.2310e-002	7	101
2.7980e-001	1.3733e-015	4.7867e-001	3.6324e-001	5	104
2.3485e-001	9.4798e-012	2.0052e+000	1.1612e+000	6	110
1.8990e-001	2.3922e-010	7.5653e-001	6.2430e-001	4	119
1.4495e-001	3.2174e-029	2.5984e+000	1.7237e+000	0	122

Concluding Remarks

- Sensor localization is a special case of EDM where a clique of distances are known.
- different formulations/models for SDP relaxation influence stability of algorithms.
- It can be disadvantageous to linearize the convex constraint $XX^T - Y \preceq 0$ to the LMI $\begin{pmatrix} I & X^T \\ X & Y \end{pmatrix}$.
- Using Wolfe dual, get a stable GN algorithm; with exact primal-dual feasibility and high accuracy.

Thanks!! Enjoy the Weekend

Thanks for your attention!

Sensor Network Localization, Euclidean Distance Matrix Completions, and Graph Realization

Henry Wolkowicz

with: Yichuan Ding, Nathan Krislock, Veronica Piccialli, Jiawei Qian