Anchored Graph Realization and Sensor Localization

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Outline

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Matrix Reformulation, $\boldsymbol{EDMC}$

- Semidefinite Programming connection

SDP Relaxation of Hard Constraint $\bar{Y} = PP^T$

Facial Reduction - Reduced Problem Model

Adjoints/Duality for $\boldsymbol{EDMC - R}$

Primal-Dual Bilinear Optimality Conditions (overdetermined)

Robust Interior-Point algorithm

- Gauss-Newton Direction, crossover, exact p-d feasibility, preconditioning

MATLAB demonstration

Concluding Remarks

Problem

- Ad hoc wireless sensor network
- A few anchors (e.g. with GPS/bulky) have fixed, known locations
- sensors within a given range have some known distance measurements (approximate)
- Problem: Determine positions of all sensors
- Parameters: Radio range, # of anchors, noise level
- Semidefinite Relaxations/Robust Algorithm

Problem Applications

- health, military, home
- natural habitat monitoring, earthquake detection, weather/current monitoring
- random deployment in inaccessible terrains or disaster relief operations

Problem Example - with Radio Range

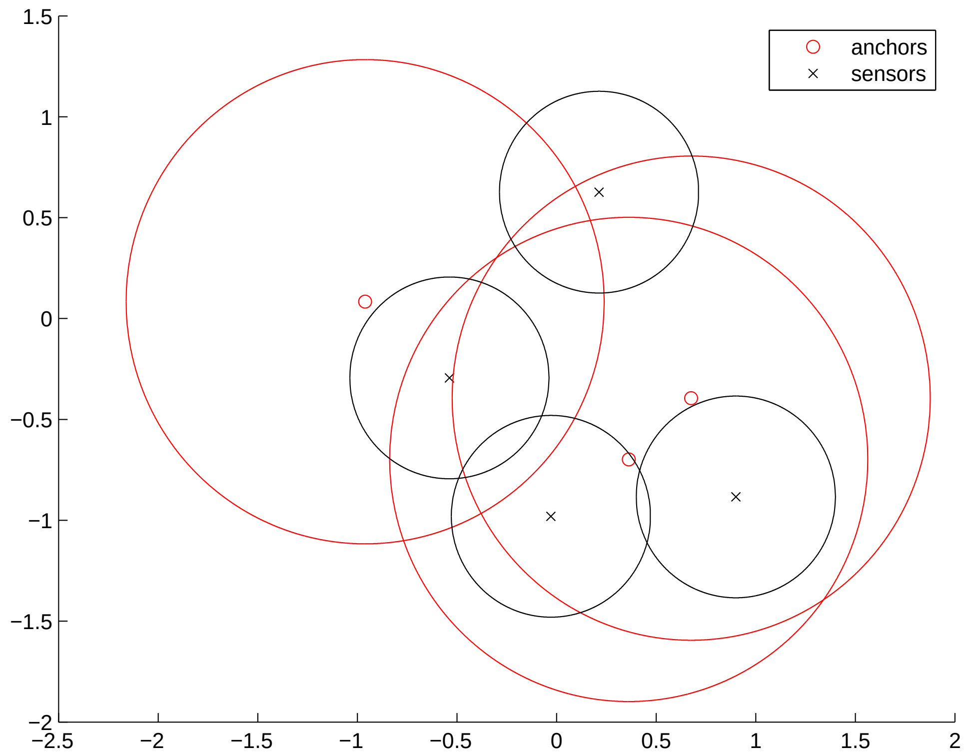

Figure 1: Connected Graph

Problem Formulation

- $p^1, \ldots, p^n \in \Re^r$ unknown (sensor) points
  $a^1, \ldots, a^m \in \Re^r$ known (anchor) points
  r embedding dimension (usually 2 or 3)
- $A^T := [a^1, a^2, \ldots, a^m]$ $X^T := [p^1, p^2, \ldots, p^n]$

$$P^T := (p^1, p^2, \ldots, p^n, a^1, a^2, \ldots, a^m)$$

$$P = \begin{pmatrix} X \\ A \end{pmatrix}$$ rows are sensor/anchor points

Definitions

- index sets of existing values of distances d_{ij} between pairs of sensors, $\{p^i\}_1^n$:
  - $\mathcal{N}_e$ distance values
  - $\mathcal{N}_u$ upper bounds on distances
  - $\mathcal{N}_l$ lower bounds on distances

  Similarly, the index sets $(\mathcal{M}_e, \mathcal{M}_u, \mathcal{M}_l)$ are for pairs from $\{p^i\}_1^n$ (sensors) and $\{a^k\}_1^m$ (anchors)
- partial EDM matrix E of squared distances

$$E_{ij} := \begin{cases} d_{ij}^2 & \text{if} \quad ij \in \mathcal{N}_e \cup \mathcal{M}_e \\ 0 & \text{otherwise.} \end{cases}$$

Definitions

Similarly, we define

- the (partial) matrix of (squared distances) upper bounds
  U, using $ij \in \mathcal{N}_u \cup \mathcal{M}_u$
- and the (partial) matrix of (squared distances) lower bounds
  L, using $ij \in \mathcal{N}_l \cup \mathcal{M}_l$

Weighted Least Squares Error

In the case E_{ij} have errors:

Let W_p, W_a be weight matrices. We minimize the weighted least squares error. ($\boldsymbol{EDMC}$)

$$
\begin{aligned}
f_1(P) \;:=\; & \sum_{(i,j)\in\mathcal{N}_e} (W_p)_{ij}(\|p^i - p^j\|^2 - E_{ij})^2 \\
& + \sum_{(i,k)\in\mathcal{M}_e} (W_a)_{ik}(\|p^i - a^k\|^2 - E_{ik})^2
\end{aligned}
$$

HARD (nonconvex) Constrained LS

$EDMC$ Problem:

$$
\begin{array}{lll}
\text{min} & f_1(P) & \text{(weighted least squares)} \\
\text{s.t.} & \|p^i - p^j\|^2 \le U_{ij} \;\; \forall (i,j) \in \mathcal{N}_u & \left(n_u = \frac{|\mathcal{N}_u|}{2}\right) \\
& \|p^i - a^k\|^2 \le U_{ik} \;\; \forall (i,k) \in \mathcal{M}_u & \left(m_u = \frac{|\mathcal{M}_u|}{2}\right) \\
& \|p^i - p^j\|^2 \ge L_{ij} \;\; \forall (i,j) \in \mathcal{N}_l & \left(n_l = \frac{|\mathcal{N}_l|}{2}\right) \\
& \|p^i - a^k\|^2 \ge L_{ik} \;\; \forall (i,k) \in \mathcal{M}_l & \left(m_l = \frac{|\mathcal{M}_l|}{2}\right)
\end{array}
$$

$\mathcal{K}(\boldsymbol{SDP}) = \boldsymbol{EDM}$

$B = PP^T$ **($\boldsymbol{SDP}$)**. $B_{ii} = (p^i)^T p^i$; $B_{ij} = (p^i)^T p^j$

The squared distance

$$\begin{array}{rcccccc} D_{ij} & = & \|p^i - p^j\|^2 & & \boldsymbol{(EDM)} & & \\ & = & (p^i)^T p^i & + & (p^j)^T p^j & - & 2(p^i)^T p^j \\ & = & \updownarrow & & \updownarrow & & \updownarrow \\ & = & (\mathrm{diag}\,(B) e^T & + & e\,\mathrm{diag}\,(B)^T & - & 2B)_{ij} \\ & =: & (\mathcal{K}(B))_{ij} & & & & \end{array}$$

$D = \mathcal{K}(B)$ change $\boldsymbol{EDM}\, D \leftrightarrow \boldsymbol{SDP}\, B$

Löwner Partial Order

matrix inner-product $\boxed{\langle M, N \rangle = \operatorname{trace} M^T N}$ and Frobenius norm $\boxed{\|M\|^2 = \operatorname{trace} M^T M}$.

In $\mathcal{S}^n$, $n \times n$ symmetric matrices:

$$B \succeq 0 \quad \text{(is positive semidefinite)}$$

$$\Longleftrightarrow$$

$$\exists P \text{ with } B = PP^T, \ \operatorname{rank}(B) = \operatorname{rank}(P)$$

the positive semidefinite (Löwner) partial order is:

$$A \succeq B \ (A \succ B) \text{ if } A - B \succeq 0 \ (A - B \succ 0)$$

Matrix Reformulation of $EDMC$

Let $\bar{Y} := PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & AA^T \end{pmatrix}$

We get the equivalent $\boldsymbol{EDMC}$

$$\begin{array}{rrcl} \min & f_2(\bar{Y}) := \frac{1}{2}\|W \circ (\mathcal{K}(\bar{Y}) - E)\|_F^2 & & \\ \text{subject to} & g_u(\bar{Y}) := H_u \circ (\mathcal{K}(\bar{Y}) - U) & \leq & 0 \\ & g_l(\bar{Y}) := H_l \circ (\mathcal{K}(\bar{Y}) - L) & \geq & 0 \\ & \text{hard constraint } \boxed{\bar{Y} - PP^T = 0} & & \end{array}$$

SDP Relaxation of Hard Constraint

$$\bar{Y} = PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & \boxed{AA^T} \end{pmatrix}$$ holds

$$\Longleftrightarrow$$

$\bar{Y}_{11} = XX^T$ and $\bar{Y}_{21} = AX^T, \boxed{\bar{Y}_{22} = AA^T}.$

Relax $\bar{Y} = PP^T$ to (Löwner partial order)

$\boxed{\bar{Y}_{22} = AA^T}$ $\quad PP^T - \bar{Y} \preceq 0$ $\quad$ quadr convex constr

(But .... why this relaxation?)

Convex wrt Löwner Partial Order

The constraint $g(P,Y) = PP^T - Y \preceq 0$ is $\succeq$-convex, since each function

$$\phi_Q(P,Y) = \operatorname{trace} Qg(P,Y) \quad \text{is convex } \forall Q \succeq 0.$$

Note

$$\begin{aligned} \operatorname{trace} QPP^T &= \operatorname{trace} QPIP^T \\ &= \operatorname{vec}(P)^T (I \otimes Q) \operatorname{vec}(P) \end{aligned}$$

Hessian is $I \otimes Q \succeq 0$;
and the cone $\boldsymbol{SDP}$ is self-polar.

Linearization of SDP Relaxation

$$PP^T - \bar{Y} \preceq 0, \qquad P = \begin{pmatrix} X \\ A \end{pmatrix}, \boxed{\bar{Y}_{22} = AA^T}$$

$\Longleftrightarrow$ (by Schur complement)

$$Z = \begin{pmatrix} I & P^T \\ P & \bar{Y} \end{pmatrix} \succeq 0, \quad P = \begin{pmatrix} X \\ A \end{pmatrix}, \boxed{\bar{Y}_{22} = AA^T}$$

$\Longleftrightarrow$ (ignore $\bar{\cdot}$)

$$Z = \begin{pmatrix} I & X^T & A^T \\ X & Y & Y_{21}^T \\ A & Y_{21} & \boxed{AA^T} \end{pmatrix} \succeq 0, \qquad \begin{pmatrix} \text{NOT! } \succ 0 \\ \Rightarrow Y_{21} = AX^T \end{pmatrix}$$

Facial Reduction

$$Z_s := \begin{pmatrix} I & X^T \\ X & Y \end{pmatrix}$$

(NE 2×2 block)
(Lin.Tr. but NOT onto)

THEOREM:

$$Z = \begin{pmatrix} I & X^T & A^T \\ X & Y & Y_{21}^T \\ A & Y_{21} & \boxed{AA^T} \end{pmatrix} \succeq 0$$

$$\Longleftrightarrow$$

$Z_s \succeq 0$ and $Y_{21} = AX^T$

Matrix $\leftrightarrow$ Vector Notation I

vector $\boxed{v = \operatorname{vec} V}$ is matrix V taken columnwise

$\boxed{\operatorname{sblk}_{21}\begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix} = \sqrt{2}X}$ pulls out the 21 block -

the $\sqrt{2}$ is for isometry in Frobenius norm

$$x := \operatorname{vec}\left(\operatorname{sblk}_{21}\begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix}\right) = \sqrt{2}\operatorname{vec}(X)$$

$\boxed{y := \operatorname{svec}(Y)}$ $(Y = Y^T,$ isometry$)$

Matrix $\leftrightarrow$ Vector Notation II

(adjoints: $\mathrm{sblk}^*_{21}(X) =$)

$$\mathrm{sBlk}_{21}(X) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix}$$

$$\mathrm{svec}^{-1}(\cdot) = \mathrm{svec}^*(\cdot) = \mathrm{sMat}(\cdot)$$

$$\mathcal{Z}_s^x(x) := \mathrm{sBlk}_{21}(\mathrm{Mat}(x)), \quad \mathcal{Z}_s^y(y) := \mathrm{sBlk}_2(\mathrm{sMat}($$

$$\mathcal{Z}_s(x,y) := \mathcal{Z}_s^x(x) + \mathcal{Z}_s^y(y), \quad Z_s := \mathrm{sBlk}_1(I) + \mathcal{Z}_s(x$$

to build $\quad Z_s = \begin{pmatrix} I & X^T \\ X & Y \end{pmatrix} \in \mathcal{S}^{r+n}$

Matrix $\leftrightarrow$ Vector Notation III

$$\mathcal{Y}^x(x) = \operatorname{sBlk}_{21}(A\operatorname{Mat}(x)^T),\ \mathcal{Y}^y(y) = \operatorname{sBlk}_1(\operatorname{sMat}$$

$$\mathcal{Y}(x,y) = \mathcal{Y}^x(x) + \mathcal{Y}^y(y),\ \boxed{\bar{Y} = \operatorname{sBlk}_2(AA^T) + \mathcal{Y}(x}$$

$$\begin{aligned}
\bar{E} &:= W \circ \left[E - \mathcal{K}(\operatorname{sBlk}_2(AA^T))\right] \\
\bar{U} &:= H_u \circ \left[\mathcal{K}(\operatorname{sBlk}_2(AA^T)) - U\right] \\
\bar{L} &:= H_l \circ \left[L - \mathcal{K}(\operatorname{sBlk}_2(AA^T))\right]
\end{aligned}$$

The unknown matrix $\bar{Y}$ is equal to $\mathcal{Y}(x,y)$ (with additional constant $2,2$ block), i.e. unknowns are the vectors x, y.

Equivalent Reduced Problem Model

$(\boldsymbol{EDMC-R})$

$$
\begin{array}{lrcl}
\min & f_3(x,y) := \frac{1}{2}\|W \circ (\mathcal{K}(\mathcal{Y}(x,y))) - \bar{E}\|_F^2 & & \\
\text{s.t.} & g_u(x,y) := H_u \circ \mathcal{K}(\mathcal{Y}(x,y)) - \bar{U} & \leq & 0 \\
 & g_l(x,y) := \bar{L} - H_l \circ \mathcal{K}(\mathcal{Y}(x,y)) & \leq & 0 \\
 & \mathrm{sBlk}\,_1(I) + \mathcal{Z}_s(x,y) & \succeq & 0
\end{array}
$$

(objective is ℓ_2 rather than ℓ_1 in the literature, e.g. H. Jin(05), A. So, Y. Ye(05), P. Biswas, T. Liang, K. Toh, T. Wang, Y. Ye(06).)

Problems with Relaxation

1. $\{(\bar{Y}, P) : \bar{Y} = PP^T\} \subset \{(\bar{Y}, P) : \bar{P}P^T - \bar{Y} \preceq 0\}$
   (But, is Lagrangian relaxation stronger?)
2. linearization (using Schur complement) results in a constraint that is *NOT* onto, i.e. two relaxations *NOT* numerically equivalent
3. Least squares problem is (usually) underdetermined.

Lagrangian of $EDMC-R$

$$
\begin{aligned}
&L(x,y,\Lambda_u,\Lambda_l,\Lambda) = \\
&\quad \tfrac{1}{2}\|W \circ \mathcal{K}(\mathcal{Y}(x,y) - \bar{E}\|_F^2 \\
&\qquad + \langle \Lambda_u, H_u \circ \mathcal{K}(\mathcal{Y}(x,y)) - \bar{U} \rangle \\
&\qquad\quad + \langle \Lambda_l, \bar{L} - H_l \circ \mathcal{K}(\mathcal{Y}(x,y)) \rangle \\
&\qquad\qquad - \langle \Lambda, \mathrm{sBlk}_1(I) + \mathcal{Z}_s(x,y) \rangle ,
\end{aligned}
$$

where $0 \leq \Lambda_u, 0 \leq \Lambda_l \in \mathcal{S}^{m+n}, \quad 0 \preceq \Lambda \in \mathcal{S}^{m+n}$

$$
\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_{21}^T \\ \Lambda_{21} & \Lambda_2 \end{pmatrix}, \qquad \langle A, B \rangle = \mathrm{trace}\, A^T B.
$$

Matrix $\leftrightarrow$ Vector Dual Variable Notation

$$\lambda_u := \operatorname{svec}(\Lambda_u), \quad \lambda_l := \operatorname{svec}(\Lambda_l),$$
$$h_u := \operatorname{svec}(H_u), \quad h_l := \operatorname{svec}(H_l),$$
$$\lambda := \operatorname{svec}(\Lambda), \quad \lambda_1 := \operatorname{svec}(\Lambda_1),$$
$$\lambda_2 := \operatorname{svec}(\Lambda_2), \quad \lambda_{21} := \operatorname{vec}\operatorname{sblk}_{21}(\Lambda).$$

Adjoints

To differentiate the Lagrangian, we need the adjoints of the various linear transformations, e.g. part of $\mathcal{K}$:

$$
\begin{aligned}
\mathcal{D}_e(B) &= \operatorname{diag}(B)\, e^T + e \operatorname{diag}(B)^T \\
\mathcal{D}_e^*(D) &= 2\operatorname{Diag}(De) \\
\langle \mathcal{D}_e(B), D \rangle &= \operatorname{trace}\left(\operatorname{diag}(B) e^T D + e \operatorname{diag}(B)^T D\right) \\
&= \operatorname{trace}\left(De (\operatorname{diag} B)^T + De (\operatorname{diag} B)^T\right) \\
&= 2\operatorname{trace}\ (\operatorname{diag} B)^T (De) \\
&= \langle B, \mathcal{D}_e^*(D) \rangle, \forall D, B
\end{aligned}
$$

Primal-Dual Optimal. Conditions 1

THEOREM: The primal-dual variables $x, y, \Lambda, \lambda_u, \lambda_l$ are optimal for $\boldsymbol{EDMC - R}$ if and only if:

1. Primal Feasibility:

The slack variables satisfy

$$S_u = \bar{U} - H_u \circ (\mathcal{K}(\mathcal{Y}(x,y))), \quad s_u = \operatorname{svec} S_u \geq 0$$

$$S_l = H_l \circ (\mathcal{K}(\mathcal{Y}(x,y))) - \bar{L}, \quad s_l = \operatorname{svec} S_l \geq 0$$

and

$$\begin{aligned} Z_s &= \mathrm{sBlk}_1(I) + \mathrm{sBlk}_2 \mathrm{sMat}(y) + \mathrm{sBlk}_{21} \mathrm{Mat} \\ &\succeq 0 \end{aligned}$$

Primal-Dual Optimal. Conditions 2a

2a. Dual Feasibility:

The stationarity equations ($\Rightarrow$ exact p-d feas.)

$$
\begin{aligned}
(\mathcal{Z}_s^x)^*(\Lambda) &= \lambda_{21} \quad \text{(eliminated)} \\
&= \left[W \circ (\mathcal{K}\mathcal{Y}^x)\right]^* \big(W \circ \mathcal{K}(\mathcal{Y}(x,y)) - \\
&\qquad + \left[H_u \circ (\mathcal{K}\mathcal{Y}^x)\right]^* (\Lambda_u) \\
&\qquad\quad - \left[H_l \circ (\mathcal{K}\mathcal{Y}^x)\right]^* (\Lambda_l) \\
(\mathcal{Z}_s^y)^*(\Lambda) &= \lambda_{2} \quad \text{(eliminated)} \\
&= \left[W \circ (\mathcal{K}\mathcal{Y}^y)\right]^* \big(W \circ \mathcal{K}(\mathcal{Y}(x,y)) - \\
&\qquad + \left[H_u \circ (\mathcal{K}\mathcal{Y}^y)\right]^* (\Lambda_u) \\
&\qquad\quad - \left[H_l \circ (\mathcal{K}\mathcal{Y}^y)\right]^* (\Lambda_l)
\end{aligned}
$$

Primal-Dual Optimal. Conditions 2b

2b. Dual Feasibility:

Nonnegativity

$$\begin{aligned}\Lambda &= \mathrm{sBlk}_1\mathrm{sMat}\left(\lambda_1\right) + \mathrm{sBlk}_2\mathrm{sMat}\left(\lambda_2\right)\\ &\quad + \mathrm{sBlk}_{21}\mathrm{Mat}\left(\lambda_{21}\right) \succeq 0;\end{aligned}$$

$$\lambda_u \geq 0; \lambda_l \geq 0$$

$\Lambda = \Lambda(\lambda_1, x, y, \lambda_u, \lambda_l)$ (from stationarity)

Primal-Dual Optimal. Conditions 3 (C.S.)

3. Complementary Slackness:

$$
\begin{aligned}
\lambda_u \circ s_u &= 0 \\
\lambda_l \circ s_l &= 0 \\
\Lambda Z_s &= 0 \qquad \text{(equivalently } \operatorname{trace} \Lambda Z_s = 0\text{)}
\end{aligned}
$$

Perturbed Compl. Slack. Conditions

$$F_\mu(x, y, \lambda_u, \lambda_l, \lambda_1) := \begin{pmatrix} \lambda_u \circ s_u - \mu_u e \\ \lambda_l \circ s_l - \mu_l e \\ \boxed{\Lambda Z_s - \mu_c I} \end{pmatrix} = 0,$$

where $s_u = s_u(x, y)$, $s_l = s_l(x, y)$,
$\Lambda = \Lambda(\lambda_1, x, y, \lambda_u, \lambda_l)$, $Z_s = Z_s(x, y)$
an overdetermined bilinear system with
$(m_u + n_u) + (m_l + n_l) + (n + r)^2$ equations
$nr + t(n) + (m_u + n_u) + (m_l + n_l) + t(r)$ variables.

Gauss-Newton Search Direction

$$\Delta s := \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \lambda_u \\ \Delta \lambda_l \\ \Delta \lambda_1 \end{pmatrix}$$

overdetermined linearized system is:

$$F'_\mu(\Delta s) \cong F'_\mu(x, y, \lambda_u, \lambda_l, \lambda_1)(\Delta s) = -F_\mu(x, y, \lambda_u, \lambda_l,$$

Notation - Compos. of Lin. Tr.

$$\begin{aligned}
\mathcal{K}_H^x(x) &:= H \circ (\mathcal{K}(\mathcal{Y}^x(x))),\\
\mathcal{K}_H^y(y) &:= H \circ (\mathcal{K}(\mathcal{Y}^y(y))),\\
\mathcal{K}_H(x,y) &:= H \circ (\mathcal{K}(\mathcal{Y}(x,y))).
\end{aligned}$$

GN: Three blocks of Equations

1. $\lambda_u \circ \operatorname{svec} \mathcal{K}_{H_u}(\Delta x, \Delta y) + s_u \circ \Delta\lambda_u = \mu_u e - \lambda_u \circ s_u$
2. $\lambda_l \circ \operatorname{svec} \mathcal{K}_{H_l}(\Delta x, \Delta y) + s_l \circ \Delta\lambda_l = \mu_l e - \lambda_l \circ s_l$
3.

$$
\begin{aligned}
&\Lambda \mathcal{Z}_s(\Delta x, \Delta y) + [\operatorname{sBlk}_1(\operatorname{sMat}(\Delta\lambda_1)) \\
&\quad + \operatorname{sBlk}_2(\operatorname{sMat}\{(\mathcal{K}_W^y)^* \mathcal{K}_W(\Delta x, \Delta y) + \\
&\qquad (\mathcal{K}_{H_u}^y)^*(\operatorname{sMat}(\Delta\lambda_u)) - (\mathcal{K}_{H_l}^y)^*(\operatorname{sMat}(\Delta\lambda_l))\}) \\
&\qquad\quad + \operatorname{sBlk}_{21}(\operatorname{Mat}\{(\mathcal{K}_W^x)^* \mathcal{K}_W(\Delta x, \Delta y) \\
&\qquad\qquad + (\mathcal{K}_{H_u}^x)^*(\operatorname{sMat}(\Delta\lambda_u)) - (\mathcal{K}_{H_l}^x)^*(\operatorname{sMat}(\Delta\lambda_l)) \\
&= \mu_c I - \Lambda Z_s
\end{aligned}
$$

Initial Str. Feas. Start Heuristic

If the graph is connected, we can use the stationarity equations and get a strictly feasible primal-dual starting point and *maintain exact numerical primal-dual feasibility* throughout the iterations.

Diagonal Preconditioning

Given $A \in \mathcal{M}^{m\times n}$, $m \geq n$ full rank matrix; and using condition number of $K \succ 0$:

$\omega(K) = \frac{\operatorname{trace}(K)/n}{\det(K)^{1/n}}$, the optimal diagonal scaling

$$\min_{D \succ 0} \omega\left((AD)^T(AD)\right), \quad D^* = \operatorname{Diag}\left(1/\|A_{:,i}\|\right)$$

(cite Dennis-W.) Therefore, need to evaluate columns of $F'_\mu(\cdot)$ (can be done explicitly/efficiently)

(Partial block Cholesky precondioning)

dens: W .75,L .8; n 15, m 5, r 2

nf	optvalue	relaxation	cond.number	sv($\mathcal{Z}_s$)	sv(F'_μ)
0.0000e+000	3.9909e-009	1.1248e-005	3.8547e+006	15	19
5.0000e-002	7.5156e-004	4.4637e-002	1.0244e+011	6	27
1.0000e-001	3.7103e-003	1.1286e-001	1.9989e+010	5	25
1.5000e-001	6.2623e-003	1.3125e-001	1.0065e+010	6	14
2.0000e-001	1.3735e-002	1.3073e-001	6.8833e+009	7	12
2.5000e-001	2.3426e-002	2.4828e-001	2.4823e+010	8	6
3.0000e-001	6.0509e-002	2.3677e-001	3.4795e+010	7	7
3.5000e-001	5.5367e-002	3.7260e-001	2.3340e+008	6	4
4.0000e-001	7.6703e-002	3.6343e-001	8.9745e+010	8	3
4.5000e-001	1.2493e-001	6.9625e-001	3.2590e+010	6	9
5.0000e-001	1.3913e-001	3.9052e-001	2.2870e+005	8	0
5.5000e-001	8.8552e-002	3.8742e-001	5.8879e+007	8	2
6.0000e-001	4.2425e-001	4.1399e-001	4.9251e+012	8	4

dens: W .75,L .8; n 15, m 5, r 2

nf	optvalue	relaxation	cond.number	sv($\mathcal{Z}_s$)	sv(F'_μ)
0.0000e+000	3.9909e-009	1.1248e-005	3.8547e+006	15	19
5.0000e-002	7.5156e-004	4.4637e-002	1.0244e+011	6	27
5.5000e-001	8.8552e-002	3.8742e-001	5.8879e+007	8	2
6.0000e-001	4.2425e-001	4.1399e-001	4.9251e+012	8	4
6.5000e-001	2.0414e-001	6.6054e-001	2.4221e+010	7	4
7.0000e-001	1.2028e-001	3.4328e-001	1.9402e+010	7	6
7.5000e-001	2.6590e-001	7.9316e-001	1.3643e+011	7	4
8.0000e-001	4.7155e-001	3.7822e-001	6.6910e+009	8	2
8.5000e-001	1.8951e-001	5.8652e-001	1.4185e+011	6	7
9.0000e-001	2.1741e-001	9.8757e-001	2.9077e+005	8	0
9.5000e-001	4.4698e-001	4.6648e-001	2.7013e+006	9	2

Table 1: Robust Algorithm for Ill-posed Problem