Symmetrization of Nonsymmetric Quadratic Assignment Problems and the Hoffman-Wielandt Inequality

S.W. Hadley*, F. Rendl** and H. Wolkowicz*

* Department of Combinatorics and Optimization
University of Waterloo
Waterloo, N2L 3G1, Canada

** Technische Universität Graz
Institut für Mathematik
Kopernikusgasse 24, A–8010 Graz, Austria

April 24, 1996

Abstract

A technique is proposed to transform a nonsymmetric Quadratic Assignment Problem (QAP) into an equivalent one, consisting of (complex) Hermitian matrices. This technique provides several new *Hoffman-Wielandt* type eigenvalue inequalities for general matrices and extends the eigenvalue bound for symmetric QAPs to the general case.

Keywords: Hoffman-Wielandt inequality, nonsymmetric quadratic assignment problems, eigenvalue inequalities.

1 Introduction

The *Quadratic Assignment Problem* (QAP) is one of the most difficult combinatorial optimization problems. It is defined as follows:

QAP: For given (real) $n \times n$ matrices A and B, minimize $f(X) := trAXB^tX^t$ over the set of permutation matrices,

where trace denotes trace. This problem is well known to be NP-hard. The QAP is surveyed in e.g. [2, 5, 8]. Lower bounds on $f(X)$ are investigated in [1, 3, 5, 7, 9, 12]. These constitute an essential ingredient in any Branch and Bound approach to solve the QAP. A connection between the range of values of $f(X)$ and the eigenvalues of A and B has been established in [5, 12] for the case of symmetric A and B. This resulted in the *eigenvalue bound* for symmetric QAPs. (A QAP is called *symmetric* if both input matrices A and B are symmetric.) In [6] it is pointed out that this eigenvalue bound is equivalent to the *Hoffman-Wielandt Inequality*, see also [4], in the sense that each can be derived from the other.

In this paper the eigenvalue approach for QAP is extended to the general (nonsymmetric) case. This is achieved by transforming the quadratic form $f(X)$ into an equivalent quadratic form $g(X) :=\mathrm{tr}\tilde{A}_+X\tilde{B}_+^*X^*$ with Hermitian matrices $\tilde{A}_+$ and $\tilde{B}_+$. This allows us to apply the eigenvalue bounds for symmetric QAPs also in the general case. Moreover we show how the eigenvalues of A and $\tilde{A}_+$ are related through majorization. Finally the equivalence between $f(X)$ and $g(X)$ leads to new Hoffman-Wielandt type inequalities for nonnormal matrices.

The paper is organized as follows. In Section 2 we review the Hoffman-Wielandt inequality and the eigenvalue bound for symmetric QAPs. In Section 3 we propose a nontrivial *symmetrization* of QAPs, leading to the main result of the paper, an eigenvalue related bound for general QAPs. The section is concluded by providing majorization relations between the eigenvalues of A and the matrix $\tilde{A}_+$. (The matrix $\tilde{A}_+$ is formed from the Hermitian and skew-Hermitian parts of A.) Several new inequalities of Hoffman-Wielandt type for general matrices are derived in Section 4.

2 The Hoffman-Wielandt Inequality and Symmetric QAPs

The following notation will be used throughout the paper. Π denotes the set of permutations of $\{1, \ldots, n\}$. For two vectors $a, b \in \Re^n$ we define the *minimal* and *maximal scalar product* of a and b by, respectively,

$$< a, b >_- := \min\{\sum_i a_i b_{\pi(i)} : \pi \in \Pi\}, \quad < a, b >_+ := \max\{\sum_i a_i b_{\pi(i)} : \pi \in \Pi\}.$$

Note that $< a, b >_+ = a^t b$ if the components of a and b are both in nondecreasing order. The *distance* $d(a, b)$ between two (possibly complex) vectors a and b is defined by

$$d(a, b) = \min_{\pi \in \Pi} \sum_i |a_i - b_{\pi(i)}|^2.$$

For a and b real this simplifies to $d(a, b) = \|a\|^2 + \|b\|^2 - 2 < a, b >_+$.

If A is a square matrix, then $\lambda(A)$ denotes the vector of eigenvalues of A (in arbitrary order). We denote by $\|K\| = \sqrt{trKK^*}$ the Frobenius norm of the matrix K, where $\cdot^*$ denotes the conjugate transpose.

In [10] Hoffman and Wielandt prove the following inequality for the distance between two normal matrices A and B, and the distance between their respective eigenvalues,

$$d(\lambda(A), \lambda(B)) \leq \|A - B\|^2. \qquad (1)$$

This is commonly referred to as the Hoffman-Wielandt (denoted H-W) Inequality. Moreover, there exists a permutation π such that

$$\|A - B\|^2 \leq \sum_i |\lambda_i(A) - \lambda_{\pi(i)}(B)|^2. \qquad (2)$$

The inequalities can fail if A or B is nonnormal. For example, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then (1) fails since

$$d(\lambda(A), \lambda(B)) = 2 > \|A - B\|^2 = 1.$$

Moreover, with A as above and B the 0 matrix, we see that (2) fails since

$$\|A - B\|^2 = 1 > \sum_i |\lambda_i(A) - \lambda_{\pi(i)}(B)|^2 = 0,$$

for all permutations π. But even though (1) and (2) may fail for general matrices, it is still possible to extend the result to a larger class of matrices.

One simple extension for the H-W inequality is to the matrices $A = K\bar{A}$ and $B = K\bar{B}$, where K is positive definite and $\bar{A}$ and $\bar{B}$ are Hermitian. The validity of the inequalities follows from the fact that K has a square root and the eigenvalues of A are the eigenvalues of the Hermitian matrix $K^{\frac{1}{2}}\bar{A}K^{\frac{1}{2}}$. Note that A is normal if and only if K and $\bar{A}$ commute which implies that A is Hermitian.

In Section 4 we will present further generalizations of the H-W inequality to arbitrary square matrices.

We now consider Hermitian A and B in order to show the close relation between the *unitary relaxation* of the QAP and the H-W inequality, see also [6]. First note that in the Hermitian case

$$d(\lambda(A), \lambda(B)) = \|A\|^2 + \|B\|^2 - 2 < \lambda(A), \lambda(B) >_+ .$$

Expanding also shows that

$$\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2trAB^*. \tag{3}$$

Therefore the H-W inequality implies, using (3)

$$< \lambda(A), \lambda(B) >_- \quad \leq \quad trAB^* \quad \leq \quad < \lambda(A), \lambda(B) >_+ . \tag{4}$$

The following theorem was proved in [5] and [12], and is the basis for the *eigenvalue bound* of symmetric QAPs.

Theorem 2.1 *[12] Let A and B be Hermitian matrices. Then*

$$\begin{aligned} \max\{trAXB^*X^* : X \text{ unitary}\} &= < \lambda(A), \lambda(B) >_+, \\ \min\{trAXB^*X^* : X \text{ unitary}\} &= < \lambda(A), \lambda(B) >_- . \end{aligned} \tag{5}$$

Since the permutation matrices are contained in the set of unitary matrices, this result indeed provides bounds on the range of values of a symmetric QAP. Moreover, by comparing (4) and (5) we see that the equivalence of the H-W inequality and the eigenvalue bounds (5) becomes apparent, by observing that $\lambda(B)$ can be assumed to be equal to $\lambda(XBX^*)$ for any unitary X. (The fact that $trAXB^*X^* \in \Re$, even if the matrices involved are complex, follows from (3).)

3 Nonsymmetric Quadratic Assignment Problems

For a square matrix A, let the matrices

$$A_+ = \frac{A + A^*}{2}, \quad A_- = \frac{A - A^*}{2}$$

denote the *Hermitian* and *skew-Hermitian* parts of A, respectively. Consider a general *real* quadratic form $x^t Ax$ in the vector variable x. It is well known that

$$x^t Ax = x^t A_+ x,$$

for all $x \in \Re^n$, i.e. the quadratic form can be represented by an Hermitian matrix. Note that the eigenvalues of A_+ majorize (see below) the real parts of the eigenvalues of A, see [11].

The objective function $f(X) = trAXB^tX^t$ of a QAP with (arbitrary) real matrices A and B can be viewed as a quadratic form in the matrix variable X. It is natural to ask for a *symmetric representation* of $f(X)$, just as in the vector case.

If we let $x = vec(X)$ be the vector formed from unravelling X rowwise, and we let $K = A \otimes B$ be the Kronecker product of A and B, then it is easily verified that

$$trAXB^tX^t = x^t Kx.$$

Thus a trivial way to symmetrize $f(X)$ would be to use $x^t K_+ x$ instead of $f(X)$. As a consequence we would have to work with the $n^2 \times n^2$ matrix K_+ instead of the two $n \times n$ matrices A and B. This seems computationally intractable, e.g. even storing K_+ is nontrivial for larger values of n.

In the following we propose a different approach to symmetrize $f(X)$, that keeps the factored Kronecker product form of $f(X)$. This approach is based on the fact that $trAXB^tX^t = 0$ if A is (real) symmetric and B is skewsymmetric.

Lemma 3.1 *Let A and B be real $n \times n$ matrices with $A = A^t$ and $B = -B^t$. Then for any real $n \times n$ matrix X*

$$trAXB^tX^t = 0.$$

Proof.

$$trA(XB^tX^t) = trA^t(XBX^t) = -trAXB^tX^t.$$

The first equality follows from $trMN = trM^tN^t$, the second from the properties of A and B. $\square$

Note that the lemma is wrong if we allow complex matrices A and B, or if X is allowed to be complex.

Let

$$\tilde{A}_+ = (A_+ + iA_-),\ \tilde{A}_- = A_+ - iA_- \tag{6}$$

denote the *positive* and *negative Hermitian* parts of A, respectively. Note that both $\tilde{A}_+$ and $\tilde{A}_-$ are Hermitian. Using the positive Hermitian parts of A and B we can symmetrize $f(X)$.

Theorem 3.1 *Let A and B be two real $n \times n$ matrices. For any real $n \times n$ matrix X*

$$trAXB^*X^* = tr\tilde{A}_+X\tilde{B}_+^*X^*. \tag{7}$$

Proof.

$$\begin{aligned} trAXB^*X^* &= tr(A_+ + A_-)X(B_+ - B_-)X^* \\ &= trA_+XB_+X^* - trA_-XB_-X^*. \end{aligned}$$

The last equality follows from the previous lemma.

$$\begin{aligned} tr\tilde{A}_+X\tilde{B}_+^*X^* &= tr(A_+ + iA_-)X(B_+ + iB_-)X^* \\ &= trA_+XB_+X^* - trA_-XB_-X^*. \end{aligned}$$

The last equality follows again from the previous lemma. □

As a consequence we can bound the range of an arbitrary QAP by the minimal and maximal scalar product of $\lambda(\tilde{A}_+)$ and $\lambda(\tilde{B}_+)$.

Theorem 3.2 *Let a QAP with real matrices A and B be given. Then for all permutation matrices X*

$$< \lambda(\tilde{A}_+), \lambda(\tilde{B}_+) >_- \quad \leq\ trAXB^tX^t\ \leq \quad < \lambda(\tilde{A}_+), \lambda(\tilde{B}_+) >_+ .$$

Proof. By Theorem (3.1) we have for all permutation matrices X

$$trAXB^tX^t = trAXB^*X^* = tr\tilde{A}_+X\tilde{B}_+^*X^*$$

because A, B and X are real. The bounds follow from Theorem (2.1) by observing that permutation matrices are unitary. □

Relation (3) also provides a bound on the range of values of an arbitrary QAP.

Theorem 3.3 *Let a QAP with real matrices A and B be given. Then for all permutation matrices X*

$$\frac{-\|A\|^2 - \|B\|^2}{2} \leq trAXB^tX^t \leq \frac{\|A\|^2 + \|B\|^2}{2} \tag{8}$$

Proof. We have

$$0 \leq \|A \pm XBX^t\|^2 = \|A\|^2 + \|B\|^2 \pm 2trAXB^tX^t$$

for all permutation matrices X. $\square$

It was already pointed out that the eigenvalues of A_+ majorize the real parts of those of A. We conclude this section by providing similar majorization relations for the eigenvalues of A and $\tilde{A}_+$. Following the notation in [11] we denote by

$$x_{[1]} \geq \ldots \geq x_{[n]}$$

the components of a given vextor $x = (x_1, \ldots, x_n) \in \Re^n$ in nonincreasing order. For given $x, y \in \Re^n$, we say that x *majorizes* y (denoted $x \succ y$) if

$$\sum_{i=1}^{k} x_{[i]} \geq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, \ldots, n-1, \tag{9}$$

$$x_1 + \cdots + x_n = y_1 + \cdots + y_n. \tag{10}$$

Theorem 3.4 *Let A be an arbitrary $n \times n$ matrix. Then*

$$\lambda(\tilde{A}_+) \succ Re(\lambda(A)) - Im(\lambda(A)).$$

Proof. Let $M = (1+i)A$. Then $M_+ = \tilde{A}_+$. Using

$$\lambda(M_+) \succ Re(\lambda(M))$$

see [11, p.237], we conclude

$$\lambda(\tilde{A}_+) \succ Re(\lambda((1+i)A)).$$

Since $Re((1+i)z) = Re(z) - Im(z)$, the result follows. $\square$

Theorem 3.5 *Let A be an arbitrary $n \times n$ matrix. Then*

$$\lambda(A_+) + Im(\lambda(A_-)) \succ \lambda(\tilde{A}_+).$$

Proof. Note that $\tilde{A}_+$ can be written as the sum of the two Hermitian matrices A_+ and iA_-. Using

$$\lambda(M) + \lambda(N) \succ \lambda(M+N)$$

for Hermitian matrices M and N, see [11, p.241], the result follows. (In a slight abuse of notation, we assumed here that for a Hermitian matrix M, $\lambda(M)$ denotes the sequence of eigenvalues of M in nonincreasing order.) $\square$

Finally we provide a majorization result between the singular values of A and the eigenvalues of $\tilde{A}_+$. Let $\sigma_k(A)$ denote the k^{th} largest singular value of A and $\lambda_k(\tilde{A}_+)$ denote the k^{th} largest eigenvalue of $\tilde{A}_+$.

Theorem 3.6 *Let A be an arbitrary $n \times n$ matrix. Then*

$$\lambda_k(\tilde{A}_+) \leq \sqrt{2}\sigma_k(A), \quad k = 1, \ldots, n,$$

and

$$(|\lambda_1(\tilde{A}_+)|, \ldots, |\lambda_n(\tilde{A}_+)|) \prec_w \sqrt{2}(\sigma_1(A), \ldots, \sigma_n(A)),$$

where $\prec_w$ denotes weak majorization, i.e."$\leq$" replaces "$=$" in (10).

Proof. In [11, p.240], it is shown that

$$\lambda_k(M_+) \leq \sigma_k(M)$$

and

$$(|\lambda_1(M_+)|, \ldots, |\lambda_n(M_+)|) \prec_w (\sigma_1(M), \ldots, \sigma_n(M)).$$

The result follows using

$$\tilde{A}_+ = ((1+i)A)_+.$$

$\square$

It should be pointed out that similar results as those above can be obtained by using the negative Hermitian parts $\tilde{A}_-$ and $\tilde{B}_-$ instead of the positive Hermitian parts.

4 New Hoffmann-Wielandt Type Inequalities

We conclude by providing inequalities between the distance of two general matrices, based on the symmetrization derived in Section 3. First we relate the distance between two matrices to the distance between the eigenvalues of the respective positive Hermitian parts.

Theorem 4.1 *Let A and B be two real $n \times n$ matrices. Then*

$$d(\lambda(\tilde{A}_+), \lambda(\tilde{B}_+)) \leq \|A - B\|^2.$$

Moreover, there exists a permutation π such that

$$\|A - B\|^2 \leq \sum_i (\lambda_i(\tilde{A}_+) - \lambda_{\pi(i)}(\tilde{B}_+))^2.$$

Proof. Note that by Theorems (3.1) and (3.2) we have

$$\begin{aligned}
\|A-B\|^2 &= trAA^* + trBB^* - 2trAB^* \\
&= tr\tilde{A}_+\tilde{A}_+ + tr\tilde{B}_+\tilde{B}_+ - 2tr\tilde{A}_+\tilde{B}_+. \\
&\geq \sum \lambda_i^2(\tilde{A}_+) + \lambda_i^2(\tilde{B}_+) - 2 < \lambda(\tilde{A}_+), \lambda(\tilde{B}_+) >_+ . \\
&= d(\lambda(\tilde{A}_+), \lambda(\tilde{B}_+)).
\end{aligned}$$

The remaining part of the theorem is proved similarly using the minimal scalar product of the eigenvalues. □

Finally we provide a lower bound on the distance between the eigenvalues of two arbitrary matrices.

Theorem 4.2 *Let A and B be two arbitrary $n \times n$ matrices. Then*

$$\frac{(Re(trA) - Re(trB))^2}{n} + \frac{(Im(trA) - Im(trB))^2}{n} \leq d(\lambda(A), \lambda(B)). \quad (11)$$

Proof. Let $a = Re(\lambda(A))$, $b = Re(\lambda(B))$, $c = Im(\lambda(A))$, $d = Im(\lambda(B))$, $e = a - b$, and $f = c - d$. Then a lower bound on the distance of the eigenvalues of A and B is given by the (global) minimum of the following program.

$$\min_{e,f} \sum_i e_i^2 + f_i^2$$

such that

$$\begin{aligned}
\sum_i e_i &= Re(trA) - Re(trB) \\
\sum_i f_i &= Im(trA) - Im(trB).
\end{aligned}$$

The objective function is convex, and

$$e_i = \frac{Re(trA) - Re(trB)}{n}$$

and

$$f_i = \frac{Im(trA) - Im(trB)}{n}$$

satisfy the first and second order sufficient optimality conditions. Substitution into the objective function yields the result. □

We leave it as an open problem to derive good upper bounds on $d(\lambda(A), \lambda(B))$.

5 Discussion and Summary

We have shown that an arbitrary QAP can be expressed using (possibly complex) Hermitian matrices. This allowed us to derive eigenvalue related bounds on the range of values of general QAPs. We do not claim that these bounds, taken as they are, will be competitive with existing bounding rules for general QAPs. To make these bounds better, further work, as in the symmetric case is necessary. In [12] the concept of "reductions" is used to improve the eigenvalue bound for symmetric problems. This involved nonsmooth optimization and turned out to be very successful. Since "reductions" can also be applied in the general case, the improvement techniques apply here as well. On the other hand, a projection technique is used in [9] to improve the eigenvalue bound of Theorem (2.1) by constraining the set of unitary matrices to an affine subspace. A similar technique can be applied also for general QAPs. Future research will have to demonstrate the practical quality of the bounds proposed in this paper.

The close connection between the Hoffman-Wielandt inequality and the eigenvalue bound for symmetric QAPs on one hand, and the symmetrization of general QAPs on the other hand suggested several extensions of the Hoffman-Wielandt inequality for general matrices. The key role is played here by $\tilde{A}_+$, the positive Hermitian part of A.

References

[1] A.A. Assad and W. Xu, On lower bounds for a class of (0,1) programs, Operations Research Letters 4, 175–180, 1985.

[2] R.E. Burkard, Locations with spatial interactions: the quadratic assignment problem, Discrete Location Theory (eds.: P.B. Mirchandani and R.L. Francis), Wiley Intersection Series in Discrete Mathematics and Optimization, 387–437, 1990.

[3] P. Carraresi and F. Malucelli, A new lower bound for the quadratic assignment problem, Technical Report TR-7/88, Pisa, 1988.

[4] W.E. Donath and A.J. Hoffman, Lower bounds for the partitioning of graphs, IBM J. Res. Dev. 17, 420–425, 1973.

[5] G.Finke, R.E.Burkard and F.Rendl, Quadratic assignment problems, Annals of Discrete Mathematics 31, 61–82, 1987.

[6] G. Finke and E.B. Medova-Dempster, Approximation approach to combinatorial optimization problems, Technical Report, Technical University of Nova Scotia, Halifax, 1987.

[7] A.M. Frieze and J. Yadegar, On the quadratic assignment problem, Discrete Applied Mathematics 5, 89–98, 1983.

[8] S.W. Hadley, Continuous optimization approaches for the quadratic assignment problem, Ph.D. thesis, University of Waterloo, Canada, 1989.

[9] S.W. Hadley, F. Rendl and H. Wolkowicz, A new lower bound using projection for the quadratic assignment problem, Report 1989, TU-Graz.

[10] A.J. Hoffman and H.W. Wielandt, The variation of the spectrum of a normal matrix, Duke Mathematics 20, 37–39, 1953.

[11] A.W. Marshall and I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.

[12] F.Rendl and H.Wolkowicz, Applications of parametric programming and eigenvalue maximization to the quadratic assignment problem, Report TU Graz, 1989, to appear in: Mathematical Programming.