# A STRENGTHENED SDP RELAXATION via a SECOND LIFTING for the MAX-CUT PROBLEM

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**Abstract**

We present a strengthened semidefinite programming, SDP, relaxation for the Max-Cut, MC, problem. The well known SDP relaxation can be obtained by Lagrangian relaxation and results in a SDP with variable $X \in \mathcal{S}^n$, the space of $n \times n$ symmetric matrices, and $n$ constraints. The strengthened bound is based on applying a *lifting procedure* to this well known semidefinite relaxation while adding nonlinear constraints. The lifting procedure is again done via Lagrangian relaxation. This results in an SDP with $X \in \mathcal{S}^{t(n)+1}$, where $t(n) = n(n+1)/2$, and $2t(n)$ constraints. The new bound obtained this way strictly improves the previous SDP bound.

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## Contents



# 1 Introduction

Semidefinite programming, SDP, has become a very intense area of research in recent years; and, one main reason for this is its success in finding bounds for the Max-Cut problem, MC. The current bounds have proven to be very tight both theoretically and in numerical tests, see e.g. [6, 11, 8, 7].

In this paper we present a strengthened SDP relaxation for MC, i.e. an SDP program that provides a strengthened bound for MC relative to the current well known SDP bound.

One approach to deriving the SDP relaxation is through the Lagrangian dual, see e.g. [13, 12], i.e. one forms the Lagrangian dual of the quadratic constrained quadratic model of MC. The dual of this Lagrangian dual yields the SDP relaxation, MCSDP, i.e. a convex program that consists of matrix inequality constraints. Thus we have lifted/linearized a nonlinear, nonconvex problem to the space of symmetric matrices. The result is a tractable convex problem. The strengthened bound is obtained by adding redundant constraints from the original MC to MCSDP and finding the dual of the Lagrangian dual again, i.e. applying a second lifting. Empirical tests and theory indicate a strict improvement in the strengthened bound.

## 1.1 Background

### 1.1.1 Notation

We work in the space of $n \times n$ symmetric matrices, $\mathcal{S}^n$ , with the trace inner product $\langle A, B\rangle = \operatorname{trace} AB$, and dimension $t(n) = n(n+1)/2$. We let $A \circ B$ denote Hadamard (elementwise) product. For given $Y \in \mathcal{S}^{t(n)+1}$, the $t(n)$ vector $x = Y_{0,1:t(n)}$ denotes the first (zero-th) row of $Y$ after the first element. We let $e$ denote the vector of ones and $E = ee^T$ the matrix of ones. For $S \in \mathcal{S}^n$ , the vector $\operatorname{diag}(S) \in \Re^n$ is the diagonal of $S$, while the adjoint operator $\operatorname{Diag}(v) = \operatorname{diag}^*(v)$ is the diagonal matrix with diagonal formed from the vector $v \in \Re^n$. We use both $\operatorname{Diag}(v)$ and $\operatorname{Diag} v$ if the meaning is clear. (Similarly for diag and other operators.) Also, $s = \operatorname{svec}(S) \in \Re^{t(n)}$, is formed (columnwise) from $S$ while ignoring the strictly lower triangular part of $S$. Its inverse is the operator $S = \operatorname{sMat}(s)$. The adjoint of svec is the operator $\operatorname{hMat}(v)$ which forms a symmetric matrix where the off-diagonal terms are multiplied by a half, i.e. this satisfies

$$\operatorname{svec}(S)^T v = \operatorname{trace} S \operatorname{hMat}(v), \qquad \forall S \in \mathcal{S}^n, v \in \Re^{t(n)}.$$

The adjoint of sMat is the operator $\operatorname{dsvec}(S)$ which works like svec except that the off diagonal elements are multiplied by 2, i.e. this satisfies

$$\operatorname{dsvec}(S)^T v = \operatorname{trace} S \operatorname{sMat}(v), \qquad \forall S \in \mathcal{S}^n, v \in \Re^{t(n)}.$$

For notational convenience, we define the vectors $\operatorname{sdiag}(s) := \operatorname{diag}(\operatorname{sMat}(s))$ and $\operatorname{vsMat}(s) := \operatorname{vec}(\operatorname{sMat}(s))$, where vec is the vector formed from the complete columns of the matrix; the adjoint of vsMat is then given by

$$\operatorname{vsMat}^*(v) = \operatorname{dsvec}\left(\left(\operatorname{Mat}(v) + \operatorname{Mat}(v)^T\right)/2\right).$$

In this paper we will have relationships between the following matrices and vectores $X \cong vv^T \cong \operatorname{sMat}(x) \in \mathcal{S}^n$, and $Y \cong \begin{pmatrix} y_0 \\ x \end{pmatrix} \begin{pmatrix} y_0 & x^T \end{pmatrix} \in \mathcal{S}^{t(n)+1}$, $y_0 \in \Re$.

### 1.1.2 Max-Cut Problem

The max-cut problem is the problem of partitioning the node set of an edge-weighted undirected graph into two parts so as to maximize the total weight of edges *cut* by the partition. We tacitly assume that the graph in question is complete (if not, nonexisting edges can be given weight 0 to complete the graph). Mathematically, the problem can be formulated as follows (see e.g [10]). Let the graph be given by its weighted adjacency matrix $A$. Define the matrix $L := \mathrm{Diag}\,(Ae) - A$, where $e$ is the vector of all ones. (The matrix $L$ is called the *Laplacian matrix* associated with the graph.) If a cut $S$ is represented by a vector $v$ where $v_i \in \{-1, 1\}$ depending on whether or not $i \in S$, we get the following formulation for the max-cut problem.

$$\text{(MC)} \quad \begin{array}{rll} \mu^* := & \text{maximize} & \frac{1}{4} v^T L v \\ & \text{s.t.} & v \in \{-1,1\}^n. \end{array}$$

Using $X := vv^T$ and $v^T L v = \mathrm{trace}\, LX$, this is equivalent to

$$\begin{array}{rll} \mu^* = & \text{maximize} & \mathrm{trace}\ \frac{1}{4} LX \\ & \text{s.t.} & \mathrm{diag}\,(X) = e \\ & & \mathrm{rank}(X) = 1 \\ & & X \succeq 0. \end{array}$$

Dropping the rank condition and setting $Q = \frac{1}{4}L$, yields the SDP relaxation with the upper bound $\mu^* \leq \nu^*$, see MCSDP below. This relaxation of MC is now well-known and studied in e.g. [4, 3, 5, 9]. Goemans and Williamson [5] have provided estimates for the quality of the SDP bound for MC. They have shown that the optimal value of this relaxation is at most 14% above the value of the maximum cut, provided there are no negative edge weights. In fact, by randomly rounding a solution to the SDP relaxation, they find a $\rho$-approximation algorithm, i.e. a solution with value at least $\rho$ times the optimal value, where $\rho = .878$. Numerical tests are presented in e.g. [7, 8].

Further results on problems with general quadratic objective functions are presented in [11, 17], e.g. Nesterov [11] uses the SDP bound to provide estimates of the optimal value of MC, with arbitrary $L = L^T$, with constant relative accuracy.

## 2 Lagrangian Relaxation

A quadratic model for MC with a general homogeneous quadratic objective function is

$$MC \quad \begin{array}{rcll} \mu^* & = & \max & v^T Q v \\ & & \text{s.t.} & v_i^2 - 1 = 0, \quad i = 1, \ldots, n. \end{array}$$

Note that if the objective function has a linear term, then we can homogenize using an additional variable similarly constrained. (See below.)

### 2.1 SDP Relaxation of MC - First Lifting

The SDP relaxation comes from the Lagrangian dual of the Lagrangian dual of MC, see e.g. [13, 12]. For completeness we include the details of such a derivation. The Lagrangian dual to MC is

$$\mu^* \leq \nu^* := \min_y \max_v v^T Q v - v^T (\operatorname{Diag} y) v + e^T y.$$

Since a quadratic is bounded above only if its Hessian, $2Q - 2\operatorname{Diag} y$, is negative semidefinite, this is equivalent to the following SDP

$$\begin{array}{rll} \nu^* = & \min & e^T y \\ & \text{s.t.} & \operatorname{Diag} y \succeq Q. \end{array}$$

Slater's (strict feasibility) constraint qualification holds for this problem. Therefore its Lagrangian dual satisfies

$$\text{MCSDP} \qquad \begin{array}{rcrll} \mu^* & \leq & \nu^* := & \max & \operatorname{trace} QX \\ & & & \text{s.t.} & \operatorname{diag}(X) = e \\ & & & & X \succeq 0. \end{array}$$

We get the same relaxation as above if we use the relationship or lifting $X = vv^T$ and $v^T Q v = \operatorname{trace} QX$.

The above relaxation is equivalent to the Shor relaxation [13] and the S-procedure in Yakubovitch [15, 16]. For the case that the objective function or the constraints contain a linear term, extra work must be done to include the possibility of inconsistency of the stationarity conditions. Alternatively, this can be done by homogenization and using strong duality of the trust region subproblem, [14]. The latter technique is used below. ???connection to linear reformulation Adams-Sherali???

## 3 Strengthened SDP Relaxation - Second Lifting

Suppose that we want to strengthen the above SDP relaxation. It is not clear what constraints one can add to MC to accomplish this. However, we start with the following lifted program; this is equivalent to program MC, obtained by adding redundant constraints to MCSDP. This is motivated by the work in [2, 1] where it is shown that adding redundant constraints that use terms of the type $XX^T$ can lead to strong duality. It is clear that MC is equivalent to MCSDP if we add the rank 1 constraint on $X$. This rank 1 constraint with $X \succeq 0$ can be replaced by the constraint $X^2 - nX = 0$. We can then add the constraints $X \circ X = E$, even though these are redundant. This yields our starting program.

$$
\text{MC2} \qquad \begin{array}{rll} \mu^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e \\ & & X \circ X = E \\ & & X^2 - nX = 0, \end{array} \tag{3.1}
$$

The last constraint is motivated by $X^2 = vv^Tvv^T$ and $v^Tv = n$. Note that the last constraint implies $X \succeq 0$. Moreover, we can simultaneously diagonalize $X$ and $X^2$. Therefore the eigenvalues satisfy $\lambda^2 - n\lambda = 0$, i.e. the only eigenvalues are 0 and $n$. Since the diagonal constraint implies that the trace is $n$, we conclude that $X$ is rank one, i.e. $MC2$ is equivalent to $MC$ using the factorization $X = vv^T$ and $\operatorname{trace} QX = v^TQv$. In addition, MC is itself a Max-Cut problem but with addional nonlinear constraints, $t(n)$ variables, and with the same optimal objective value as MC.

In order to efficiently apply Lagrangian relaxation and not lose the information from the linear constraint we need to replace the constraint with the norm constraint $||\operatorname{diag}(X) - e||^2 = 0$ and homogenize the problem. We then lift this matrix problem into a higher dimensional matrix space. To keep the dimension as low as possible, we take advantage of the fact that $X = \operatorname{sMat}(x)$ is a symmetric matrix. We then express MC2 as

$$
\text{MC2} \qquad \begin{array}{rll} \mu^* = & \max & \operatorname{trace}\left(Q \operatorname{sMat}(x)\right) y_0 \\ & \text{s.t.} & \operatorname{sdiag}(x)^T \operatorname{sdiag}(x) - 2e^T \operatorname{sdiag}(x) y_0 + n = 0 \\ & & \operatorname{sMat}(x) \circ \operatorname{sMat}(x) = E \\ & & \operatorname{sMat}(x)^2 - n \operatorname{sMat}(x) y_0 = 0 \\ & & 1 - y_0^2 = 0 \\ & & x \in \Re^{t(n)}, y_0 \in \Re. \end{array} \tag{3.2}
$$

Note that this problem is equivalent to the previous formulation since we can change $X$ to $-X$ if $y_0 = -1$. An alternative homogenization would be to change the objective function to $\frac{1}{n}\operatorname{trace}\left(Q\operatorname{sMat}(x)^2\right)$. ???include in an appendix??? It appears that (the eigenvalues of) $Q$ should determine which homogenization would be better, i.e. which would result in a better class of Lagrange multipliers when taking the dual and therefore reduce the duality gap.

We now take the Lagrangian dual of this strengthened formulation, i.e.

we use Lagrange multipliers $w \in \Re, T, S \in \mathcal{S}^n$ and get

$$\begin{array}{rl}\mu^* \leq \nu_2^* := \min\limits_{w,T,S} \max\limits_{x,y_0^2=1} & \operatorname{trace}\,(Q\mathrm{sMat}\,(x))\,y_0 \\ & + w(\mathrm{sdiag}\,(x)^T \mathrm{sdiag}\,(x) - 2e^T \mathrm{sdiag}\,(x) y_0 + n) \\ & + \operatorname{trace} T(E - \mathrm{sMat}\,(x) \circ \mathrm{sMat}\,(x)) \\ & + \operatorname{trace} S((\mathrm{sMat}\,(x))^2 - n\,\mathrm{sMat}\,(x) y_0).\end{array} \tag{3.3}$$

We can now move the variable $y_0$ into the Lagrangian without increasing the duality gap, i.e. this is a trust region subproblem and the Lagrangian relaxation of it is tight, [14]. This yields

$$\begin{array}{c}\nu_2^* = \min\limits_{t,w,T,S} \max\limits_{x,y_0} \operatorname{trace}\,(Q\mathrm{sMat}\,(x))\,y_0 \\ + w(\mathrm{sdiag}\,(x)^T \mathrm{sdiag}\,(x) - 2e^T \mathrm{sdiag}\,(x) y_0 + n) \\ + \operatorname{trace} T(E - \mathrm{sMat}\,(x) \circ \mathrm{sMat}\,(x)) \\ + \operatorname{trace} S((\mathrm{sMat}\,(x))^2 - n\,\mathrm{sMat}\,(x) y_0) \\ + t(1 - y_0^2).\end{array} \tag{3.4}$$

The inner maximization of the above relaxation is an unconstrained pure quadratic maximization, i.e. the optimal value is infinity unless the Hessian is negative semidefinite is which case $x = 0$ is optimal. Thus we need to calculate the Hessian.

Using $\operatorname{trace} Q\mathrm{sMat}\,(x) = x^T \mathrm{dsvec}\,(Q)$, and adding a 2 for convenience, we get the constant part (no Lagrange multipliers) of the Hessian:

$$2H_c := 2 \begin{pmatrix} 0 & \frac{1}{2}\mathrm{dsvec}\,(Q)^T \\ \frac{1}{2}\mathrm{dsvec}\,(Q) & 0 \end{pmatrix}. \tag{3.5}$$

The nonconstant part of the Hessian is made up of a linear combination of matrices, i.e. it is a linear operator on the Lagrange multipliers. To make the quadratic forms in (refeq:maxcutlagr2) easier to differentiate we note that $\mathrm{dsvec}\,\mathrm{Diag}\,\mathrm{diag}\,\mathrm{sMat} = \mathrm{sdiag}^*\mathrm{sdiag} = \mathrm{Diag}\,\mathrm{svec}\,(I)$; and rewrite the quadratic forms as follows:

$$\begin{array}{rcl}\mathrm{sdiag}\,(x)^T \mathrm{sdiag}\,(x) & = & x^T\,(\mathrm{dsvec}\,\mathrm{Diag}\,\mathrm{diag}\,\mathrm{sMat}\,)\,x; \\ e^T \mathrm{sdiag}\,(x) & = & (\mathrm{dsvec}\,\mathrm{Diag}\,e)^T\,x;\end{array}$$

$$\begin{array}{rcl}\operatorname{trace} S(\mathrm{sMat}\,(x))^2 & = & \operatorname{trace}\mathrm{sMat}\,(x)\,S\,\mathrm{sMat}\,(x) \\ & = & x^T \mathrm{dsvec}\,(S\mathrm{sMat}\,(x)) \\ & = & x^T\,(\mathrm{dsvec}\,S\mathrm{sMat}\,)\,x;\end{array}$$

$$\begin{array}{rcl}\operatorname{trace} T(\mathrm{sMat}\,(x) \circ \mathrm{sMat}\,(x)) & = & x^T\,\{\mathrm{dsvec}\,(T \circ \mathrm{sMat}\,(x))\} \\ & = & x^T\,(\mathrm{dsvec}\,(T \circ \mathrm{sMat}\,))\,x.\end{array}$$

For notational convenience, we use the *negative* of the Hessian and split it into four linear operators with the factor 2:

$$
\begin{array}{rcl}
\nabla^2 & = & 2\mathcal{H}(w,T,S,t) \\
& := & 2\mathcal{H}_1(w) + 2\mathcal{H}_2(T) + 2\mathcal{H}_3(S) + 2\mathcal{H}_4(t) \\
2\mathcal{H}(w,T,S,t) & := & 2\mathcal{H}_1(w) + 2\mathcal{H}_2(T) + 2\mathcal{H}_3(S) + 2\mathcal{H}_4(t) \\
& := & 2w \begin{pmatrix} 0 & (\mathrm{dsvec}\,\mathrm{Diag}\,e)^T \\ (\mathrm{dsvec}\,\mathrm{Diag}\,e) & -\mathrm{sdiag}^*\mathrm{sdiag} \end{pmatrix} \\
& & \quad + 2\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{dsvec}\,(T \circ \mathrm{sMat}\,) \end{pmatrix} \\
& & \quad\quad + 2\begin{pmatrix} 0 & \frac{1}{n}\mathrm{dsvec}\,(S)^T \\ \frac{1}{n}\mathrm{dsvec}\,(S) & -\mathrm{dsvec}\,S\mathrm{sMat} \end{pmatrix} \\
& & \quad\quad\quad + 2t\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{array}
\tag{3.6}
$$

The elements of the above matrices may need clarification. The matrix $\mathrm{sdiag}^*\mathrm{sdiag} \in \mathcal{S}^{t(n)}$ is diagonal with elements determined using

$$
\begin{array}{rcl}
e_i^T\,(\mathrm{sdiag}^*\mathrm{sdiag}\,)\,e_j & = & \mathrm{sdiag}\,(e_i)^T\mathrm{sdiag}\,(e_j) \\
& = & \begin{cases} 1 & \text{if i=j=t(k)} \\ 0 & \text{otherwise.} \end{cases}
\end{array}
$$

Similarly, we find that, for $T = \sum_{ij} t_{ij}E_{ij}$, where the matrices $E_{ij}$ are the elementary matrices $e_ie_j^T + e_je_i^T$, we have

$$
\mathrm{dsvec}\,(T \circ \mathrm{sMat}\,) = \sum_{ij} t_{ij}\mathrm{dsvec}\,(E_{ij} \circ \mathrm{sMat}\,).
$$

Then the matrix $\mathrm{dsvec}\,(E_{ij} \circ \mathrm{sMat}\,)$ is found from using $e_k^T\,[\mathrm{dsvec}\,(E_{ij} \circ \mathrm{sMat}\,)\,(e_l)]$. Similarly, we can find the elements of $\mathrm{dsvec}\,S\mathrm{sMat}$.

We can cancel the 2 in (3.6) and (3.5) and get the (equivalent to the Lagrangian dual) semidefinite program

$$
\text{MCDSDP2} \quad \nu_2^* = \begin{array}{cc} \min & nw + \mathrm{trace}\,ET + \mathrm{trace}\,0S + t \\ \text{s.t.} & \mathcal{H}(w,T,S,t) \succeq H_c \end{array}
\tag{3.7}
$$

If we take $T$ sufficiently positive definite and $t$ sufficiently large, then we can guarantee Slater's constraint qualification. Therefore the dual of this SDP has the same optimal value $\nu_2^*$ and it provides the strengthened SDP relaxation of MC:

$$\text{MCPSDP2} \qquad \begin{array}{rll} \nu_2^* = & \max & \operatorname{trace} H_c Y \\ & \text{s.t.} & \mathcal{H}_1^*(Y) = n \\ & & \mathcal{H}_2^*(Y) = E \\ & & \mathcal{H}_3^*(Y) = 0 \\ & & \mathcal{H}_4^*(Y) = 1 \\ & & Y \succeq 0. \end{array} \tag{3.8}$$

Thus we need to calculate the adjoint operators and also remove redundant constraints in MCDSDP2. To evaluate the adjoint operators we write

$$Y = \begin{pmatrix} 1 & x^T \\ x & \bar{Y}. \end{pmatrix}$$

The adjoint operators can be quickly described. $\mathcal{H}_1^*(Y)$ is twice the sum of the elements in the first row of $Y$ corresponding to the positions of the diagonal of $\operatorname{sMat}(x)$ minus the sum of the same elements in the diagonal of $Y$. If these elements were all 1, then clearly the result would be $n$. $\mathcal{H}_2^*(Y) = \operatorname{sMat}\operatorname{diag}(\bar{Y})$. $\mathcal{H}_3^*(Y)$ consists of the sums in MCPSDP2 below. This describes the linearization of $X^2 - nX$. $\mathcal{H}_4^*(Y)$ is just the top left element of $Y$.

## 3.1 Redundant Constraints - Direct Second Lifting

We can see the SDP relaxation directly for MC2 using the relationship

$$Y \cong \begin{pmatrix} y_0 \\ x \end{pmatrix} \begin{pmatrix} y_0 & x^T \end{pmatrix}, \quad X = \operatorname{sMat}(x).$$

The advantage in this is that we can use the origin of $X$ from MC, e.g. $\operatorname{diag}(X) = e$ and the elements of $X$ are $\pm 1$. Thus we get

$$\operatorname{diag}(Y) = e, \quad \text{and} \quad Y_{0,t(i)} = 1, \forall i = 1, \ldots, n.$$

We can also express the $t(n+1)$ constraints from $X^2 - nX = 0$. Several of the constraints become redundant. The result is the simplified SDP relaxation:

$$
\text{MCPSDP2} \qquad
\begin{array}{rll}
\nu_2^* = & \max & \operatorname{trace} H_c Y \\
& \text{s.t.} & \operatorname{diag}(Y) = e \\
& & Y_{0,t(i)} = 1, \quad \forall i = 1, \ldots, n \\
& & \sum\limits_{k=1}^{i} Y_{t(i-1)+k,t(j-1)+k} + \sum\limits_{k=i+1}^{j} Y_{t(k-1)+i,t(j-1)+i} \\
& & \quad + \sum_{k=j+1}^{n} Y_{t(i-1)+i,t(k-1)+j} - nY_{0,t(j-1)+i} = 0 \\
& & \qquad \forall 1 \leq i < j \leq n \\
& & Y \succeq 0, Y \in S^{t(n)+1}.
\end{array}
\tag{3.9}
$$

This problem has $2t(n) - 1$ constraints. ????full row rank - onto ???? The dual is ????

One surprising result is that the projection of the first row of a feasible $Y$ results in $X \succeq 0$, even though this constraint was discarded in the relaxation.

**Lemma 3.1** *Suppose that $Y$ is feasible in MCPSDP2. Then*

$$
\operatorname{sMat}\left(Y_{0,1:t(n)}\right) \succeq 0.
$$

**Proof.** Let $x = Y_{0,2:t(n)}$. The $X^2 = nX$ constraint can be viewed as

$$
n\operatorname{sMat}(x) = \operatorname{sMat}(x)\operatorname{sMat}(x) = \operatorname{sMat}(x)\operatorname{sMat}(x)^T = \operatorname{sMat}(x)x^T\operatorname{sMat}^*.
$$

Since we identify $xx^T$ with the lower right block of $Y$, we see that this is a congruence of a positive semidefinite matrix and so must be positive semidefinite itself. Alternatively, using $\mathcal{H}_3^*$, and the fact that $\operatorname{dsvec}(\cdot)\operatorname{sMat}$ is a self-adjoint operator, we get $n\operatorname{dsvec}^*(x) = \operatorname{dsvec}\bar{Y}\operatorname{sMat} = \left(\operatorname{dsvec}\bar{Y}\operatorname{sMat}\right)^*$, where $\bar{Y}$ is the bottom right block of $Y$, i.e. we again have a congruence of a positive semidefinite matrix.

□

**Corollary 3.1** *The optimal values satisfy*

$$
\nu_2^* = \nu^* \Rightarrow \nu_2^* = \nu^* = \mu^*.
$$

**Proof.** Suppose that $\nu_2^* = \nu^*$ and

$$
Y^* = \begin{pmatrix} 1 & (x^*)^T \\ x^* & \bar{Y}^* \end{pmatrix}
$$

solves MCSDP2. It is clear that sMat $(x^*)$ is feasible for MCSDP. Moreover, from the structure of the objective function of MCSDP2, we see that $\nu_2^* = \operatorname{trace} Q \operatorname{sMat}(x^*)$, i.e. it must be optimal as well. If $\operatorname{sMat}(x^*) \succ 0$, then we are done, since the objective function of MCSDP is linear, i.e. we get a contradiction to $\nu_2^* = \nu^*$. Therefore we can assume that $X^* = \operatorname{sMat}(x^*)$ is singular and optimal for MCSDP.

We can modify MCSDP by making the objective quadratic, i.e. adding $y_0$, and not changing the optimality of $X^*$. We can now perturb the objective function to get a contradiction.

□

# 4 Numerical Tests

Both bounds were tested on a variety of problems. The matrix $X$ in this section is obtained by applying sMat to the last $\frac{n(n+1)}{2}$ elements of the first column of the matrix $Y$ obtained from solving MCPSDP2 (i.e. only the first element of that column is ignored). In all the test problems we used, the resulting matrix $X$ was always found to be positive semidefinite. Some typical results follow.

In the following we include the numerical rank of $X$, i.e. the number of eigenvalues that appear to be nonzero.

These results show the strengthened bound MCPSDP2 yielding a strict improvement over MCSDP every time. Since $L$ was integer valued, we see that, in all but the last instance, the optimal solution was actually found.

# 5 Conclusion

We have presented an SDP that provides a strengthened bound for MC relative to the current well know SDP bound for SDP. Though the time to solve this new SDP is large compared to the time for solving the current SDP, it is hoped that exploiting structure will improve this situation and that this new bound will be competitive both in time and in quality. In addition, attempts to get proveable quality estimates need to be done.

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| $n$ | Weight of optimal cut | MCSDP bound (% rel. error) | MCPSDP2 bound (% rel. error) | Numerical rank of X |
|---|---|---|---|---|
| 5 | 4 | 4.5225 (13.06%) | 4.2890 (7.22%) | 2 |
| 7 | 56 | 56.4055 (0.72%) | 56.0954 (0.17%) | 3 |
| 8 | 30 | 30.2015 (0.67%) | 30.0000 (0.0000000075%) | 1 |
| 9 | 58 | 58.9361 (1.61%) | 58.1182 (0.20%) | 3 |
| 10 | 64 | 64.08 (0.1268%) | 64 (-2.228e-08%) | 3 |
| 12 | 88 | 90.3919 (2.72%) | 89.5733 (1.79%) | 4 |

Table 1: The first line of results corresponds to solving both MC relaxations for a 5-cycle with unit edge-weights; the others come from randomly generated weighted graphs.