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A SIMPLE CONSTRAINT QUALIFICATION IN INFINITE DIMENSIONAL PROGRAMMING

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A new, simple, constraint qualification for infinite dimensional programs with linear programming type constraints is used to derive the dual program; see Theorem 3.1. Applications include a proof of the explicit solution of the best interpolation problem presented in [8].

Key words: Infinite dimensional linear programming, semi-infinite programming, constraint qualification, optimality conditions, dual program.

1. Introduction

We consider the following mathematical programming problem:

$$p = \inf\{f(x): Ax = b, x \geq_S 0\}, \tag{P}$$

where $f: X \to R$ is a differentiable functional on X, $A: X \to Y$ is a continuous linear operator, X and Y are normed spaces, $b \in Y$, and $S \subset X$ is a convex cone, i.e. $S + S \subset S$ and $\lambda S \subset S$, for all $\lambda \geq 0$. The cone S induces the *cone partial order*

$$x \geq_S y \quad \text{iff} \quad x - y \in S.$$

Our study of the model (P) was stimulated by the following best interpolation problem considered in [8]: let $\hat{x}, \psi_1, \ldots, \psi_n$ be given in $L_2[0, 1]$ with $\hat{x} \geq 0$; find x^* to solve

$$\min\{\|x\|_2^2: (x, \psi_i) = (\hat{x}, \psi_i), i = 1, \ldots, n, x \geq 0\}, \tag{1.1}$$

where $(\cdot, \cdot)$ and $\|\cdot\|_2$ are the inner product and norm in L_2, respectively. The problem (1.1) is of type (P). In this paper we see that we can derive the explicit solution of (1.1), given in [8], by using a Lagrange multiplier approach.

The property that distinguishes (P) is the linear programming type of constraints. In the case that X and Y are finite dimensional spaces, S is a polyhedral cone, and $f(x) = c'x$ is a linear functional, then there exists a duality theory for (P), see e.g. [1]. For example, if the dual program is defined as

$$d = \sup\{y'b: A'y' \leq_{S^*} c, y' \in Y^*\} \tag{D}$$

where S^+ is the 'polar cone' of S and Y^* is the dual space of Y, then, if p is finite, we get that $p = d$, d is attained, and 'complementary slackness' between the solution vectors holds. The well known ordinary linear program, where S is the nonnegative orthant, falls into this case. Corresponding results for S not necessarily polyhedral are given in [9]. These results require looking at the 'faces' of S unless a constraint qualification holds.

Problems can arise if X and Y are not finite dimensional or if S is not polyhedral.

Example 1.1. Consider the nonpolyhedral problem in finite dimensions, where $X = R^3$,

$$S = \{x \in R^3 : x_1 \geq 0, 2x_1x_2 \geq x_3^2\}, \tag{1.2}$$

and

$$A = [1\ 0\ 0], \quad b = 0, \quad c' = (0\ 0\ 1).$$

Then $p = 0$ and $x^* = (0\ 0\ 0)'$ is clearly an optimum for (P). However, the constraint in the dual program (D) is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y = s^+ \in S^+ = S, \quad y \in R, \tag{1.3}$$

which is clearly inconsistent. This shows that $d = -\infty$ is the optimal value of (D).

Example 1.2. Consider the following best interpolation problem of type (1.1): let $n = 2$ and

$$\psi_1(t) = \begin{cases} 1 - 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < t \leq 1, \end{cases}$$

$$\psi_2(t) = t,$$

$$\hat{x}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

The Kuhn-Tucker conditions for (1.1) for x^* feasible are:

$$\begin{aligned} &x^* = \lambda_1\psi_1 + \lambda_2\psi_2 + s^+, \\ &s^+ \geq 0, \quad s^+x^* = 0. \end{aligned} \tag{1.4}$$

We see that $x^* = 0$ on $[0, \frac{1}{2}]$, the support of ψ_1, since $(x^*, \psi_1) = (\hat{x}, \psi_1)$. Also $\lambda_2 > 0$, since $s^+x^* = 0$ and $0 \neq x^* \geq 0$. Thus $\lambda_1 < 0$, which still leaves $(\lambda_1\psi_1 + \lambda_2\psi_2)(t) > 0$, for t near $\frac{1}{2}$, i.e. the system (1.4) is inconsistent. Note that the constraint $x \geq 0$ is a very simple constraint. We obtain the same difficulty if we replace the objective function $f(x) = \|x\|_2^2$ with the linear objective function $c'x$, where $c' = \frac{1}{2}\nabla f(x^*) = x^*$.

Constraint qualifications for programs of type (P) are given in e.g. [5, 7]. They usually require a 'Slater point', i.e. a feasible point $x \in \operatorname{int} S$. In the above example, S is the nonnegative orthant in L_2 and has empty interior.

In Section 2 we present our constraint qualification and several equivalent formulations. Section 3 contains the main result; see Theorem 3.1 and Corollary 3.1. Here we see that the constraint qualification yields strong duality. In Section 4 we present several examples including a proof of the explicit solution of the best interpolation problem presented in [8]. In Section 5 we show how to extend our results when the constraint qualification fails.

2. Preliminaries and the constraint qualification

In this section we discuss the constraint qualification to be used in our optimality conditions. We also present several equivalent formulations. However, we first introduce the notations and definitions needed in the sequel.

We consider the program

$$p = \inf\{f(x): Ax = b, x \in S\} \tag{P}$$

introduced in Section 1. We let X^* and Y^* denote the *continuous dual spaces* of X and Y respectively, both equipped with the w*-topology. Given any set K in X, the *polar cone* of K is the set

$$K^+ = \{x' \in X^*: x'x \geq 0 \text{ if } x \in K\}. \tag{2.1}$$

Here $x'x$ denotes the bilinear form in the duality between X and X^*. Correspondingly, if K' is in X^*,

$$K'^+ = \{x \in X: x'x \geq 0 \text{ if } x' \in K'\}. \tag{2.2}$$

Note that

$$K^{++} = \overline{\operatorname{cone}}\, K, \tag{2.3}$$

the closure of the convex cone generated by K. The *annihilator* of a set L, in X or X^*, is denoted by $L^\perp$ and is defined by

$$L^\perp = L^+ \cap (-L)^+.$$

The *feasible set* of (P) is

$$F = \{x \in X: Ax = b, x \in S\}.$$

We assume that $F \neq \emptyset$. We let $\mathcal{P}$ be any *generating set* for S^+, i.e.

$$\overline{\operatorname{cone}}\, \mathcal{P} = S^+. \tag{2.4}$$

Then, if S is closed, $x \in S$ is equivalent to the constraints

$$s^+x \geq 0 \quad \text{if } s^+ \in \mathcal{P}.$$

The *equality set* of constraints is

$$\mathscr{P}^{=} = \{s^+ \in \mathscr{P}: Ax = b, x \in S \text{ implies } s^+x = 0\}. \tag{2.5}$$

This notation differs slightly from that in the literature in that the constraint $Ax = b$ is not included in $\mathscr{P}^{=}$. (See e.g. [2, 3, 4].)

We assume that the linear operator A is continuous. We let A', $\mathscr{R}(A)$ and $\mathscr{N}(A)$ denote the adjoint, range space and null space of A respectively. We use dim to denote dimension.

We will employ the following constraint qualification in Section 3.

$$\overline{\text{cone}}(F - S) = X. \tag{CQ}$$

We will refer to it as (CQ).

We now present several equivalent formulations. Here R_+ denotes the nonnegative real line and R_+b is the set of all nonnegative multiples of b.

Proposition 2.1. *The following five statements* (2.6a)–(2.6e) *are equivalent*:

$$\overline{\text{cone}}(F - S) = X, \tag{2.6a}$$

$$\overline{(S \cap A^{-1}(R_+b)) - S} = X, \tag{2.6b}$$

$$P^{=} \subset \{0\}, \tag{2.6c}$$

$$S^+ \cap F^\perp = \{0\}, \tag{2.6d}$$

$$S^+ \cap (-F)^+ = \{0\}. \tag{2.6e}$$

Moreover, each of the above implies the following two equivalent statements:

$$\overline{(S, R_+) + \mathscr{N}(A, -b)} = (X, R), \tag{2.6f}$$

$$\begin{pmatrix} S^+ \\ R_+ \end{pmatrix} \cap \overline{\mathscr{R}\begin{pmatrix} A' \\ -b \end{pmatrix}} = \{0\}. \tag{2.6g}$$

Proof. That (a) and (b) are equivalent is clear. Now suppose that (a) holds. Let $s^+ \in S^+$, $t \geq 0$ and

$$\begin{pmatrix} s^+ \\ t \end{pmatrix} = \lim_n \begin{pmatrix} A'y'_n \\ -by'_n \end{pmatrix}, \quad y'_n \text{ a net in } Y^*.$$

If $x \in F$, then

$$s^+x = (\lim_n A'y'_n)x = \lim_n y'_n Ax = \lim_n y'_n b = -t \leq 0.$$

But $s^+x \geq 0$, since $x \in F$, so $t = 0 = s^+x$. Thus

$$s^+(F - S) \leq 0, \tag{2.7}$$

which, by (a) implies that $s^+=0$, i.e. we have shown that (a) implies (g). That (f) and (g) are equivalent follows from the fact that

$$(K\cap L)^+=\overline{K^++L^+}$$

for closed convex cones K and L. Note that $K^\perp=K^+$, when K is a subspace, and that $\mathcal{R}(B')^\perp=\mathcal{N}(B)$, for any continuous linear operator. Now (a) fails if and only if there exists $0\neq x'\in X^*$ such that

$$x'(F-S)\leq 0, \tag{2.8}$$

if and only if $x'F\leq 0$ and $x'S\geq 0$ if and only if (c) fails. That (c) is equivalent to both (d) and (e) is clear from (2.5). Recall that $F\subset S$. □

It is not true that (g) (or (f)) implies (c) (or its equivalents) in general. For, let $A=\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right]$, $b=0$, and $S=\operatorname{span}\{(1,0)'\}$. Then $\mathcal{R}(A')\cap S^+=\{0\}$ while

$$S^+\cap F^\perp=S^+\cap(S\cap\mathcal{N}(A))^\perp=\operatorname{span}\{(0,1)'\}\cap\mathbb{R}^2\neq\{0\}.$$

The set $\mathcal{P}^=$ in (2.6c) is defined differently in the literature. For, consider the abstract convex program

$$\min\{f(x):\ g(x)\leq_S 0,\ x\in\Omega\}$$

where $g:X\to Y$ is S-convex and $\Omega\subset X$ is convex. Then, see e.g. [3],

$$\mathcal{P}^==\{s^+\in\mathcal{P}:\ x\in\Omega,\ g(x)\leq_S 0 \text{ implies } s^+g(x)=0\}. \tag{2.9}$$

(If we set $g=I$, the identity operator, and $\Omega=\{x:\ Ax=b\}$, then the two definitions coincide.) For the ordinary convex program in finite dimensions, $\mathcal{P}^==\emptyset$ is equivalent to Slater's condition (see e.g. [2]), i.e. there exists $\hat{x}\in\Omega$ such that $g(\hat{x})<0$.

In finite dimensions we conclude the following stronger statement.

Corollary 2.1. *If X is finite dimensional, or S^+ is w*-compactly based, then each of the five statements* (2.6a) *to* (2.6e) *is equivalent to*

$$\exists\hat{x}\in\operatorname{int}S\quad\textit{such that}\quad A\hat{x}=b. \tag{2.10}$$

Proof. Let us show the equivalence with (2.6d). First suppose X is finite dimensional. If (2.10) holds, then $s^+\hat{x}>0$ for all $0\neq s^+\in S^+$, i.e. (2.6d) holds. Conversely, if (2.10) fails, then, since X is finite dimensional, either $\operatorname{int}S=\emptyset$ and there exists $0\neq\phi\in S^\perp\subset S^+\cap F^\perp$, or $\operatorname{int}S\cap F=\emptyset$ and we can apply the Hahn-Banach Theorem to find $0\neq\phi\in S^+\cap(-F)^+$. Since $F\subset S$, we conclude that $\phi\in S^+\cap F^\perp$.

Now, if S^+ is w*-compactly based, i.e. $S^+=\operatorname{cone}(\mathcal{P})$, with $0\notin\mathcal{P}$ convex and w*-compact, and X^* is not necessarily finite dimensional, then the result follows from Lemma 2.2 and Corollary 2.1 in [4]. □

Note that the finite dimension assumption can be replaced by nonempty relative interior.

3. Main results

We now develop the duality theory for the program (P). We first consider the special case when $f(x)$ is linear, i.e. $f(x) = c'x$, for some c' in X^*. In this case the dual program we consider is

$$d = \sup\{by' : A'y' \leq_{S^+} c', y' \in Y^*\}. \tag{D}$$

We obtain conditions for the optimal values to be equal, i.e. $p = d$, and for d to be attained (see Theorem 3.1). The following lemma uses the closure condition (3.2) as a constraint qualification.

Lemma 3.1. *Suppose that p is finite and $f(x) = c'x$, $c' \in X^*$, in* (P). *Define*

$$B = (A, -b), \quad K = \begin{pmatrix} S \\ R_+ \end{pmatrix}, \tag{3.1}$$

and let

$$\mathcal{R}(B') + K^+ \quad \text{be closed.} \tag{3.2}$$

Then

$$p = d \text{ and } d \text{ is attained by some } y^* \in Y^*.$$

Moreover, if $p = c'x^*$ with $x^* \in F$, then

$$(A'y^* - c')x^* = 0.$$

Proof. We homogenize the problem. The optimality of p yields

$$Ax - tb = 0, \ (x, t) \in (S, R_+) \quad \text{implies} \quad c'x - tp \geq 0. \tag{3.3}$$

Note that if $t = 0$, $Ax = 0$, $0 \neq x \in S$, and $c'x < 0$, then $p = -\infty$; while if $t > 0$, $x \in S$, then $A(t^{-1}x) = b$ which implies $c'(t^{-1}x) \geq p$. Now let $a = (c', -p)$, $y = \binom{x}{t}$. Then the above becomes

$$By = 0, \ y \in K \quad \text{implies} \quad ay \geq 0.$$

Thus

$$\begin{aligned} a \in (\mathcal{N}(B) \cap K)^+ = \overline{\overline{\mathcal{R}(B')} + K^+} &\subset \overline{\mathcal{R}(B') + K^+} \quad \text{since } 0 \in K^+ \\ &= \mathcal{R}(B') + K^+ \quad \text{by (3.2).} \end{aligned} \tag{3.4}$$

Therefore

$$c' = A'y' + s^+, \ -p = -by' + t, \ (s^+, t) \in (S^+, R_+), \ y' \in Y^*,$$

i.e. $c' \geq_{S^+} A'y'$ and $p \leq by'$. That $p \geq by'$ is clear by weak duality, so we conclude $p = d = by'$. Moreover, if p is attained at x^*, then

$$\begin{aligned} d = p = c'x^* &= c'x^* - y'(Ax^* - b) \quad \text{since } x^* \in F, \\ &= (c' - A'y')x^* + y'b \geq d, \end{aligned}$$

i.e. $(c' - A'y')x^* = 0$. $\square$

To apply the lemma we need to find conditions which guarantee the closure condition in (3.2). We will apply the following closure conditions; see e.g. [6, pp. 104, 105].

Lemma 3.2. *Let X be a Hausdorff linear topological space and let A, B be closed convex subsets of X. If A is locally compact and the recession cones $C_A \cap C_B = \{0\}$, then $B - A$ is closed in X. (Here the recession cone of a set A is $C_A = \{x \in X : x + A \subset A\}$.)* □

Corollary 3.1. *Let N be a finite dimensional subspace of X and P a closed convex cone in X such that $N \cap P = \{0\}$. Then $N + P$ is closed in X.* □

We now present several conditions which guarantee the closure condition (3.2). We denote:

$$\begin{array}{ll} \text{I} & \overline{\text{cone}}(F - S) = X, \\ \text{II} & S^+ \text{ is locally compact}, \\ \text{III} & \dim \mathcal{R}(A) \text{ is finite}, \\ \text{IV} & S \text{ is polyhedral (the intersection of a finite number of closed halfspaces)}, \\ \text{V} & \mathcal{R}(A') \text{ is closed}. \end{array} \tag{3.5}$$

Theorem 3.1. *Suppose that p is finite and $f(x) = c'x$, $c' \in X^*$ in (P). If one of the following three statements holds:*

(a) I, II *and* V,
(b) I *and* III,
(c) III *and* IV,

then

$$p = d \text{ and } d \text{ is attained by } y^* \in Y^*. \tag{3.6}$$

In addition, if $p = c'x^*$ with $x^* \in F$, then

$$(A'y^* - c')x^* = 0 \text{ (complementary slackness)}. \tag{3.7}$$

Proof. By Lemma 3.1, we need only show that the closure condition (3.2) holds. By Proposition 2.1, (d) condition I implies that

$$\overline{\mathcal{R}(B')} \cap K^+ = \{0\}. \tag{3.8}$$

For, suppose that $\binom{z}{t} \in \overline{\mathcal{R}(B')} \cap K^+$, i.e. $A'y_n' = z_n \to z \in S^+$ and $-by_n' = t_n \to t \geq 0$. But then if $x \in F$, we see that

$$0 \leq xz = \lim_n xz_n = \lim_n (Ax)y_n' = \lim_n by_n' = -t \leq 0,$$

i.e. $z \in F^\perp$ which, by (2.6d), implies $z = 0$.

If III holds then $\dim \mathcal{R}(A')$ is also finite and so $\mathcal{R}(A')$ is closed. Since III or V must hold, let us show that $\mathcal{R}(A')$ closed implies $\mathcal{R}(B')$ closed. Suppose that $y_n' \in Y^*$ and $A'y_n' \to z$, $by_n' \to t$, i.e. $B'y_n \to \binom{z}{t}$. Then there exists $y' \in Y^*$ such that $A'y' = z$.

First, suppose that $A\hat{x} = b$, $\hat{x} \in X$. (Recall that we have assumed $F \neq \emptyset$.) Then

$$t = \lim_n by'_n = \lim_n (A\hat{x})y'_n = \lim_n \hat{x}(A'y'_n) = \hat{x}A'y' = by',$$

i.e. $B'y_n \to B'y = z$. Then $\mathcal{R}(B')$ is closed.

Since $\mathcal{R}(B')$ is closed, we see that (3.8) together with II or III satisfies the hypotheses of Lemma 3.2 or Corollary 3.1. Thus (3.2) holds. If statement (c) holds, then the closure condition is always satisfied since K^+ is finitely generated. □

Note that we do not have to assume $F \neq \emptyset$ in the above. For if $b \notin \mathcal{R}(A)$, then $\mathcal{N}(A') \subset \mathcal{N}(b)$. Let $\bar{t} = by'$ and let $n' \in \mathcal{N}(A')/\mathcal{N}(b)$ such that $bn' = t - \bar{t}$. Then we again see that $B(y' + n') = \binom{z}{t}$.

To obtain symmetric duality between the primal and dual programs (P) and (D), we need an additional assumption. We shall employ the following generalized Farkas' lemma of Craven and Koliha.

Lemma 3.2 [5, Theorem 2]. *Suppose that $A(S)$ is closed in Y. Then the following are equivalent:*

$$Ax = b, \quad x \in S \quad \textit{is consistent}; \tag{3.9a}$$

$$A'y' \in S^+ \quad \textit{implies} \quad by' \geq 0. \quad \square \tag{3.9b}$$

Theorem 3.3. *Suppose that $f(x) = c'x$, $c' \in X$, and that $A(S)$ is closed. If one of the three statements* (a), (b) *or* (c) *in Theorem* 3.1 *holds, then*

(i) *if one of the problems is inconsistent, then the other is inconsistent or unbounded*;

(ii) *let the two problems be consistent with $x \in F$ and y' feasible for* (D), *then*

$$c'x \geq by' \quad (\textit{weak duality}); \tag{3.10}$$

(iii) *if both* (P) *and* (D) *are consistent, then their optimal values are equal and* (D) *has an optimal solution* (*strong duality*);

(iv) *if x and y' are feasible for* (P) *and* (D) *respectively, then they are optimal if and only if*

$$(c' - A'y')x = 0.$$

Proof. Suppose that (P) is inconsistent. Then by Lemma 3.3, there exists $\phi' \in Y^*$ such that $A'\phi' \in -S^+$ and $b\phi' > 0$. Thus, if (D) is consistent, then it must be unbounded. Conversely, if (P) is consistent and bounded, then Theorem 3.1 implies that D is consistent and bounded. This proves (i).

If both programs have feasible solutions x and y' respectively, then

$$c'x = c'x + y'(b - Ax) = (c' - A'y')x + y'b \geq y'b, \tag{3.11}$$

which proves (ii).

Now, if x and y' are feasible solutions as above, then both programs are bounded by (ii) and so Theorem 3.1 implies that $p = d$, d is attained in (D) and complementary slackness holds. This proves (iii) and necessity in (iv). Sufficiency in (iv) follows from (ii) and

$$\begin{aligned} p &\leq c'x = c'x + y'(b - Ax) \quad \text{since } x \in F, \\ &= (c' - A'y')x + y'b \\ &= y'b \qquad\qquad\qquad \text{by (3.11)}, \\ &\leq d. \quad \square \end{aligned}$$

The general symmetric dual pair, with closed cones S and T,

$$\min\{c^*x \colon Ax \geq_T b, x \geq_S 0\}, \tag{$\bar{\text{P}}$}$$

$$\max\{by^* \colon A'y^* \leq_{S^+} c^*, y^* \geq_{T^+} 0\} \tag{$\bar{\text{D}}$}$$

can be treated by the above results by replacing the constraints in ($\bar{\text{P}}$) with $[A - I] \times \binom{x}{y} = b$, $x \geq_S 0$, $y \geq_T 0$. For ($\bar{\text{P}}$) to be the dual of $\bar{\text{D}}$ one needs to assume that X and Y are reflexive Banach spaces, see e.g. [7].

The above results immediately yield a characterization of optimality for (P) if f is a convex function.

Corollary 3.2. *Suppose that p is finite and f is convex and Fréchet differentiable in* (P). *If one of the three statements* (a), (b) *or* (c) *of Theorem* 3.1 *hold, then $x^* \in F$ solves* (P) *if and only if the* (*Kuhn–Tucker type*) *system*

$$\begin{aligned} &\nabla f(x^*) - A'y^* = s^+, \\ &s^+x^* = 0, \quad s^+ \in S^+, \quad y^* \in Y^*, \end{aligned} \tag{3.12}$$

is consistent, where $\nabla f(x^*)$ denotes the derivative at x^*.

Proof. Let $c' = \nabla f(x^*)$. The feasible point x^* solves (P) if and only if it solves the linearized program

$$\inf\{c'(x - x^*) \colon Ax = b, x \in S\}. \tag{3.13}$$

By Theorem 3.1, we conclude that the system (3.12) is consistent. Note that we can replace $c'(x - x^*)$ in (3.13) with $c'x$, since $c'x^*$ is a fixed constant.

Conversely, if (3.12) holds and $x \in F$, then

$$\begin{aligned} c'x &= c'x + y^*(b - Ax) = (c' - A'y^*)x + y^*b \\ &\geq y^*b = y^*b + (c' - A'y^*)x^* \\ &= c'x^* + y^*(b - Ax^*) = c'x^*, \end{aligned}$$

i.e. x^* solves (3.13) and so also (P). $\square$

4. Examples and applications

In this section we present several examples illustrating the theory including examples where the (CQ) in Theorem 3.1 fails. The model program (P) appears naturally in many situations. Our work was stimulated by the approximation theory problem presented in Example 1.2 and solved in Example 4.1.

Example 4.1. We consider the problem of finding the 'best' interpolant discussed in [8], where a nonnegativity constraint is added to eliminate undesirable inflection points. The problem reduces to the following model in $L_2[0, 1]$. Let $\hat{x}, \psi_1, \psi_2, \ldots, \psi_n$ in $L_2[0, 1]$ be given with $\hat{x} \geq 0$. Find x^* in $L_2[0, 1]$ which solves

$$\min\{\|x\|^2: (x, \psi_i) = (\hat{x}, \psi_i),\ i = 1, \ldots, n, \text{ and } x \geq 0\}. \tag{4.1}$$

Define the linear operator $A: L_2 \to R^n$ as

$$Ax = (y_i), \qquad y_i = (x, \psi_i), \tag{4.2}$$

and let $b = (b_i)$, $b_i = (\hat{x}, \psi_i)$.

Thus we see that (4.1) is a problem of type (P). Let us assume that $\mathcal{P}^- \subset \{0\}$. Note that a solution x^* exists, since it is the closest point in a convex set to the origin, and also note that Corollary 3.2 is applicable. Thus $x^* \in F$ solves (4.1) if and only if

$$x^* = A'y^* + s^+, \quad s^+x^* = 0, \quad s^+ \in S^+, \quad y^* \in R^n.$$

Now this says that x^* is the positive part of $A'y^*$. In other words, if we define, almost everywhere,

$$(A'y^*)_+(t) = \begin{cases} A'y^*(t) & \text{if } A'y^*(t) > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{4.3}$$

then we have shown, in the case $\mathcal{P}^- \subset \{0\}$, that the solution x^* of (4.1) is the unique solution of the system

$$A(A'y^*)_+ = b, \quad y^* \in Y^*. \tag{4.4}$$

We complete this problem by considering the case $P^- \not\subset \{0\}$ in Example 5.1. Note that $P^- \not\subset \{0\}$ in Example 1.2.

Example 4.2. Consider the simple program

$$p = \inf\left\{c'x: \int_0^1 x(t)\,dt = 1, x(t) \geq 0\right\}.$$

Here $X = L_2[0, 1]$ and $S = \{x: x \geq 0\}$ has no interior. But our constraint qualification holds, i.e. we see that $\mathcal{P}^- \subset \{0\}$ by considering the functional $e(t) = 1$. We can apply Theorem 3.1 and solve the trivial dual program

$$d = \max\{y \in R: y - c(t) \leq 0, \text{a.e. in } [0, 1]\},$$

i.e. $p=d$ is the (essential infimum) ess-inf $c(t)$ in $[0,1]$. We can now conclude several facts. First $p=d$ is finite if and only if c is essentially bounded below. Also, since complementary slackness holds, i.e. since $(y^*-c(t))x^*(t)=0$, $y^*=\text{ess-inf } c(t)$ must hold if p is attained. We see that p is attained if and only if the measure

$$\mu\{t: c(t)=y^*\}=\alpha>0$$

and in this case

$$x^*(t)=\begin{cases}1/\alpha & \text{if } c(t)=y^*,\\ 0 & \text{otherwise.}\end{cases}$$

Example 4.3. Let $X=L_p[0,1]$, $1\le p\le\infty$, and $S=\{x\in X: x\ge 0 \text{ a.e.}\}$ be the nonnegative orthant in X. Consider the mixed equality-inequality constrained program.

$$p=\inf\{c'x: A_1x=b_1, A_2x\ge b_2, x\ge_s 0\}, \tag{PEI}$$

where $A_i: X\to R^{m_i}$, and m_i is a positive integer $i=1,2$. Then, the 'Slater type' condition

$$\exists \hat{x}\in X \text{ s.t. } A_1\hat{x}=b_1, A_2\hat{x}>b_2, \hat{x}>0 \text{ a.e.} \tag{4.5}$$

is a constraint qualification for (PEI). This can be seen by adding slack variables to the second constraint, thus changing (PEI) to a program of type (P), and then applying Theorem 3.1. Note that if E is a measurable subset of $[0,1]$ with positive measure, then the Lebesgue integral

$$\int_E |x(t)|^q \, d\mu(t)>0,$$

for $1\le q\le\infty$. This implies that $P^-\subset\{0\}$, i.e. (CQ) holds.

The Slater constraint qualification, which requires $\hat{x}\in\text{int } S$, is more restrictive than (4.5) even in L_∞, where int S is nonempty. Also, it can be shown that (CQ) is in fact equivalent to (4.5). These facts remain true when the interval $[0,1]$ is replaced by a σ-finite measure space.

5. Duality without the constraint qualification

We now extend our results to include the case when our constraint qualification (CQ) may fail, i.e. when $P^-\not\subset\{0\}$. The structure of our problem allows for the finite dimensional type of approach used in [3]. This approach uses the faces of the cone S. We also complete Example 4.1.

Definition 5.1. (a) K is a *face* of a convex cone S if K is a convex cone, and

$$s_1, s_2\in S,\ s_1+s_2\in K \text{ implies } s_1, s_2\in K. \tag{5.1}$$

(b) S^f denotes the (unique) smallest face of S which contains the feasible set F. (Note that S^f is just the intersection of all the faces of S which contain F.)

Now consider the 'enlarged' dual program

$$(\hat{\mathrm{D}}) \quad \hat{d} = \sup\{by' : A'y' \leq_{S^{f+}} c',\ y' \in Y^*\}, \tag{5.2}$$

i.e. we replace S^+ with the larger dual cone S^{f+}.

We shall see that using $(\hat{\mathrm{D}})$ rather than (D) enables us to allow $\mathscr{P}^- \not\subset \{0\}$. We first present the following preliminary result connecting $\mathscr{P}^-$ and S^f.

Proposition 5.1. *The statement*

$$S^f = S \tag{5.3a}$$

implies

$$\mathscr{P}^- \subset S^\perp. \tag{5.3b}$$

If the relative interior (*in* $S-S$) $\operatorname{ri} S \neq \emptyset$, *then* (b) *implies* (a).

Proof. Suppose (b) fails, i.e. $0 \neq \phi \in \mathscr{P}^-$, $\phi \notin S^\perp$. Then $K = \{\phi\}^\perp \cap S$ is a proper face of S. Thus

$$S^f \subset K \subsetneq S.$$

Conversely, suppose that $\operatorname{ri} S \neq \emptyset$ and S^f is a proper face of S. Since S^f is a proper face, we see that

$$S^f \cap \operatorname{ri} S = \emptyset.$$

The Hahn-Theorem yields $0 \neq \phi \in X^*$ such that

$$\phi s > 0 \quad \text{for all } s \in \operatorname{ri} S, \quad \phi s \leq 0 \quad \text{for all } \phi \in S^f.$$

Since $S^f \subset S$, we conclude that

$$S^f \subset \{\phi\}^\perp \cap S \subsetneq S, \tag{5.4}$$

i.e. $\phi \in \mathscr{P}^-$, $\phi \notin S^\perp$. $\square$

If S is finite dimensional, then the above Proposition states that (5.3)(a) and (b) are equivalent. We now present the duality result without (CQ).

Theorem 5.1. *Suppose that* (CQ), *i.e. statement* (3.5)I, *is replaced by the following*:

I′ replace the dual program (D) with $(\hat{\mathrm{D}})$, the optimal value d with $\hat{d}$, and S^+ with S^{f+}.

Then Theorem 3.1 *and Corollary* 3.2 *still hold.*

Proof. Since $F \subset S^f$, we see that the optimal value p, of the original program (P), satisfies the new program

$$p = \inf\{\bar{f}(x): \bar{A}x = b, x \in S^f, x \in U\}, \tag{$\bar{\text{P}}$}$$

where the subspace $U = S^f - S^f$ replaces the space X. We now see that $\mathcal{P}^= \subset \{0\}$ as a subset of U^*. For if $\phi \in \mathcal{P}^= \subset U^*$, then $S^f \subset \{\phi\}^\perp$ which implies $\phi = 0$ since $U = S^f - S^f$. Therefore, we can apply our results to ($\bar{\text{P}}$), where we consider $\bar{f}$ and $\bar{A}$ to be f and A restricted to U. let us only consider Theorem 3.1. Corollary 3.2 follows similarly.

We get the dual program

$$\bar{d} = \sup\{by': \bar{A}'y' \leq_{S^{f+}} \bar{c}', y' \in Y^*\}. \tag{$\bar{\text{D}}$}$$

Here $\bar{c}'$ is c' restricted to U and $\bar{A}': Y^* \to U^*$. We know from Theorem 3.1, when $p = \bar{d}$, $\bar{d}$ is attained and complementary slackness holds. The proof is completed if we show that $\bar{A}'y' \leq_{S^{f+}} \bar{c}'$ if and only if $A'y' \leq_{S^{f+}} c'$, where we consider S^{f+} both in X^* and in U^* depending on the context. But this clearly holds since $(A'y')|_U$ (restricted to U) equals $(A|_U)'y'$. (Note that $\bar{c}' - \bar{A}'y' = s^+ \in S^{f+} \subset U^*$ iff $s(c' - \bar{A}'y') \geq 0$, for all $s \in S^f$, iff $sc' - (\bar{A}s)y' \geq 0$, for all $s \in S^f$, iff $s(c' - A'y') \geq 0$, for all $s \in S^f$, since $\bar{A} = A$ on S^f.) $\square$

Corollary 5.1. *Suppose that* ri $S \neq \emptyset$ *and that statement* (CQ), *i.e.* (3.5)I, *is replaced by the following*:

$$\text{I}'' \quad \mathcal{P}^= \subset S^\perp.$$

Then Theorem 3.1 *and Corollary* 3.2 *still hold.*

Proof. The result follows from the Theorem since Proposition 5.1 yields $S^f = S$. $\square$

Example 5.1. We now complete the derivation of the explicit solution of the best interpolation problem begun in Example 4.1. Theorem 5.1 implies that the solution x^* satisfies

$$x^* = A'y' + s^+, s^+ \in S^{f+}, \quad s^+x^* = 0. \tag{5.5}$$

Let $T^=$ be the maximal set, by set inclusion, such that it has positive measure and $x \in F$ implies $x = 0$ a.e. on $T^=$. Then

$$S^f = \{x \geq 0: x = 0 \text{ a.e. on } T^=\}, \qquad S^{f+} = \{x: x \geq 0 \text{ a.e. on } [0,1]/T^=\}.$$

Since $x^* = 0$ on $T^=$, and since, as in Example 4.1, $x^* = (A'y')_+$ on $[0,1]/T^=$, we see that

$$x^*(t) = (A'y')_+(t)\chi_T(t) \tag{5.6}$$

where χ_T is the characteristic function of $T = [0,1]\backslash T^=$, i.e. (5.6) and $Ax^* = b$ characterizes the solution x^*. This corresponds to the solution given in [8].

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