Rank Restricted Semidefinite Matrices and Image Closedness*

Ian Davidson† Henry Wolkowicz‡

October 31, 2016

Abstract

We study the closure of the projection of the (nonconvex) cone of rank restricted positive semidefinite matrices onto subsets of the matrix entries. This defines the feasible sets for semidefinite completion problems with restrictions on the ranks. Applications include conditions for low-rank completions using the nuclear norm heuristic.

Keywords: Positive semidefinite (PSD) matrix completion, closedness of projections, low-rank matrix completions.

AMS subject classifications: 90C22, 90C46, 52A99

Contents

*Research supported by The Natural Sciences and Engineering Research Council of Canada.

†Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.

‡Department of Combinatorics and Optimization Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1, `www.math.uwaterloo.ca/~hwolkowi`.

1 Introduction

Consider an *undirected graph,* $G = (V, E)$, with *vertex set,* $V = \{1, \ldots, n\}$, and *index set,* $E \subseteq \{ij : i \leq j\}$. The classical positive semidefinite (PSD) completion problem begins with a given *partial symmetric matrix* $X \in \mathcal{S}^n$, where $X_{ij} = a_{ij}, \forall ij \in E$ and attempts to find the missing entries from the data $a \in \mathbb{R}^E$ so that X is PSD. One of the problems in [8] answers the question of when the projection of the PSD cone $\mathcal{S}^n_+$ onto the matrix entries indexed by E is closed, i.e., when the set of *coordinate shadows,* $\mathcal{P}(\mathcal{S}^n_+)$, is closed. In this paper we add an additional rank restriction and ask the following.

Question 1.1. *When is the projection of the (generally nonconvex) cone of PSD matrices of rank at most r, $\mathcal{P}(\mathcal{S}^n_{r+})$, closed?*

Such questions arise for example in constraint qualifications for guaranteeing strong duality. It is also closely related to the closedness of the sum of sets. See e.g., [1, 3, 8, 10]. In addition, the projection $\mathcal{P}(\mathcal{S}^n_{r+})$ is exactly the data set determining the feasibility of the rank restricted PSD completion problem, e.g., [4, 12].

The paper is organized as follows. We continue with the background and some preliminary results in the remainder of this section. We then show that we can restrict our attention to the connected components of the graphs in Section 2. Specific cases for closure and failure of closure are given in Section 3. In particular we show the importance of bipartite graphs. The concluding Section 4 contains a summary of the results and some open questions and conjectures.

1.1 Background

We work in the space of $n \times n, n \geq 2$, *real symmetric matrices,* $\mathcal{S}^n$, equipped with the *trace inner product* $\langle A, B \rangle = \operatorname{trace} AB, \forall A, B \in \mathcal{S}^n$. We denote the closed convex cone of *PSD matrices,* $\mathcal{S}^n_+$. We focus on the generally nonconvex cone of PSD matrices of rank at most r, $\mathcal{S}^n_{r+}$. We allow for self-loops in the undirected graph, $G = (V, E)$, and denote $\mathcal{L} = \{i : ii \in E\}$ with the complement $\mathcal{L}^c$. A partial symmetric matrix $X \in \mathcal{S}^n$ is is called a *partial PSD matrix* if all the principal submatrices formed by known entries are PSD. The *PSD matrix completion problem with rank restriction at most* r can be stated as completing the partial PSD matrix X to a positive semidefinite matrix such that the rank of the completion is at most r. We assume that $1 \leq r \leq n$.

Our work extends the following.

Theorem 1.1 ([8]). *$\mathcal{P}(S^n_+)$ is closed if and only if $\mathcal{L}, \mathcal{L}^c$ are disconnected.* □

1.2 Preliminary Results

We first add the following related result.

Theorem 1.2 ([2, Thm 1.1], [9], [6, Sect. 31.5]). *Let $\mathcal{A} \subset \mathcal{S}^n$ be an affine subspace such that the intersection $\mathcal{A} \cap \mathcal{S}^n_+$ is non-empty and* $\operatorname{codim}(\mathcal{A}) \leq \binom{r+2}{2} - 1$ *for some non-negative integer r. Then there is a matrix $X \in \mathcal{A} \cap \mathcal{S}^n_+$ such that* $\operatorname{rank}(X) \leq r$. □

We now note the following two results that follow from the above two theorems.

Corollary 1.1. *Let*

$$t = \left\lceil -\frac{3}{2} + \frac{\sqrt{9+8|E|}}{2} \right\rceil. \tag{1.1}$$

Then $t \leq n-1$ *and*

$$\Big(\mathcal{P}(\mathcal{S}^n_{r+}) \text{ is closed for } r = t, t+1, \ldots, n\Big) \iff \Big(\mathcal{L} \text{ and } \mathcal{L}^c \text{ are disconnected}\Big).$$

Proof. Necessity follows from the Theorem 1.1 if we fix $r = n$.

For sufficiency first note that Theorem 1.1 implies $\mathcal{P}(\mathcal{S}^n_+)$ is closed. Moreover, the closure holds if $|\mathcal{L}| = n$, since any sequence of partial PSD matrices $a^i \to a$ with $\mathcal{P}(X^i) = a^i, X^i \in \mathcal{P}(\mathcal{S}^n_{n-1+})$ means that the diagonal elements converge and so we can assume that $X^i \to \bar{X} \succeq 0$. The rank result now follows from lower semi-continuity of the rank function. Therefore, we can assume that $\mathcal{L}^c \neq \emptyset$.

Now suppose that $r = n-1$ and consider a sequence of partial PSD matrices $a^i \to a$ and suppose that $\mathcal{P}(X^i) = a^i, X^i \in \mathcal{P}(\mathcal{S}^n_{n-1+})$. Then there exists $\bar{X} \succeq 0, \mathcal{P}(\bar{X}) = a$. If $\operatorname{rank}(\bar{X}) \leq n-1$ then we are done. If $\operatorname{rank}(\bar{X}) = n$, then we can consider the positive semidefinite completion problem, (PSDC),

$$(PSDC) \qquad \min \operatorname{trace}(CX) \text{ s.t. } X \succeq 0,\ X_{ij} = a_{ij},\ \forall ij \in E. \tag{1.2}$$

This program has a feasible solution $\bar{X} \succ 0$ that is not unique since we have $\mathcal{L}^c \neq \emptyset$. Therefore we can move in the direction $\bar{X} + \alpha D, \alpha \in \mathbb{R}$, for some $0 \neq D \in \mathcal{S}^n$. This means that $\bar{X} + \alpha D \notin \mathcal{S}^n_+$, for some $\alpha \in \mathbb{R}$, and on the line segment $[\bar{X}, \bar{X} + \alpha D]$ we can find a singular feasible point $\bar{X} + \bar{\alpha} D \succeq 0$, for some $\bar{\alpha} \in \mathbb{R}$. The closure follows since feasibility means $\mathcal{P}(\bar{X} + \bar{\alpha} D) = a$.

The key to the sufficiency proof above was in finding a feasible SDP completion with the lower rank. We do this by applying Theorem 1.2. The codimension for a completion problem is exactly $|E|$, the number of constraints or elements that are fixed. Therefore, we have $|E| \leq \frac{(t+2)(t+1)}{2} - 1$ which is equivalent to $2|E| + 2 \leq t^2 + 3t + 2$. The only non-negative root for this quadratic yields the smallest nonnegative integer

$$t = \left\lceil -\frac{3}{2} + \frac{\sqrt{9+8|E|}}{2} \right\rceil.$$

We can combine this with the result for $n-1$ and obtain a feasible solution X, a completion, with rank at most t, i.e.,

$$X \in \mathcal{P}(\mathcal{S}^n_{t+}), \quad \mathcal{P}(X) = a.$$

□

Corollary 1.2. *Suppose that* $|E| < \frac{1}{2}(t^2 + 3t) \left(= \binom{t+2}{2} - 1 \right)$, $t < n$. *Then*

$$\Big(\mathcal{P}(\mathcal{S}^n_{r+}) \text{ is closed for } r = t, t+1, \ldots, n\Big) \iff \Big(\mathcal{L} \text{ and } \mathcal{L}^c \text{ are disconnected}\Big).$$

Proof. We just square both sides in (1.1). □

Corollary 1.3. *Let* $\mathcal{L}$ *and* $\mathcal{L}^c$ *be connected. Then* $\mathcal{P}(\mathcal{S}^n_{r+})$ *is not closed.*

Proof. The proof is similar to the general case in [8]. We include it for completeness. Without loss of generality, we can assume that $1 \in \mathcal{L}$, $2 \in \mathcal{L}^c$ and $12 \in E$. Taking a sequence of partial matrices a^i with $a^i_{11} = \frac{1}{i}$, $a^i_{12} = 1$ and all other entries of $a^i = 0$. Then we have a sequence of matrices and images

$$X^i = \begin{bmatrix} \frac{1}{i} & 1 & 0 & \dots \\ 1 & ? & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \mathcal{S}^n_{r+}, \quad \mathcal{P}(X^i) = a^i.$$

This sequence of matrices is always rank one completable with $X_{22} = i$. (And thus is rank at most r completable.) Therefore $a^i \in \mathcal{P}(\mathcal{S}^n_{r+})$. But $a^i \underset{i\to\infty}{\longrightarrow} \bar{a}$ with $\bar{a}_{11} = 0$. This is not PSD completable. □

Following Corollaries 1.1 and 1.3 we can add the following.

Assumption 1.1. *In the remainder of this paper we assume that $\mathcal{L}$ and $\mathcal{L}^c$ are disconnected in the undirected graph $G = (V, E)$ and*

$$1 \le r < \left\lceil -\frac{3}{2} + \frac{\sqrt{9 + 8|E|}}{2} \right\rceil \ (\le (n-1)).$$

2 Partitioned Graphs

We now get a rather nice result for closure that allows us to assume, without loss of generality, that our graphs are composed of *two connected components*. As an illustration, we first show the following.

Proposition 2.1. *Suppose $X = \begin{bmatrix} A & ? \\ ? & B \end{bmatrix}$ is a partial PSD matrix, i.e., both A and B are PSD matrices. Then the minimum rank PSD completion of X, denoted $\bar{X}$, has*

$$\operatorname{rank}(\bar{X}) = \max\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

Moreover, the maximum rank PSD completion has rank given by the sum, $\operatorname{rank}(A) + \operatorname{rank}(B)$.

Proof. We use the unique PSD square roots of A, B and get the completion with the correct rank

$$\bar{X} = \begin{bmatrix} A^{1/2} \\ B^{1/2} \end{bmatrix} \begin{bmatrix} A^{1/2} \\ B^{1/2} \end{bmatrix}^T \succeq 0,$$

i.e.,we get $\operatorname{rank}(\bar{X}) = \operatorname{rank}\left(\begin{bmatrix} A^{1/2} \\ B^{1/2} \end{bmatrix} \right)$.

The maximum rank completion is obtained by using zeros in the off-diagonal blocks. □

Theorem 2.1. *Let $\{H_i\}_{i=1}^k$ be a partition of V,*

$$H_1, \dots, H_k \subseteq V,\ \cup_{i=1}^k H_i = V,\ H_i \cap H_j = \emptyset, \forall i \ne j, \quad n_i := |H_i|,\ i = 1, \dots, k.$$

Then the projection $\mathcal{P}(\mathcal{S}^n_{r+})$ is closed if, and only if, the restricted projections to each component $\mathcal{P}_{H_i}(\mathcal{S}^{n_i}_{r+})$ is closed for all $i = 1, \dots, k$.

Proof. Necessity follows by considering the case of setting all the elements in all the components but one to zeros.

For sufficiency we consider the sequence with convergent projections

$$X^j = \begin{bmatrix} X_{11}^j & \dots & X_{1k}^j \\ \vdots & \ddots & \vdots \\ (X_{1k}^j)^T & \dots & X_{kk}^j \end{bmatrix} \in \mathcal{S}_{r+}^n, \qquad x^j = \mathcal{P}(X^j) \to x \in \mathbb{R}^E, \quad j = 1, 2, \dots.$$

We need to find $X \in \mathcal{S}_{r+}^n$ with $x = \mathcal{P}(X)$. Denote the restricted projections

$$x_i^j := \mathcal{P}_{H_i}(X_{ii}^j) \to x_i,\ i = 1, \dots k.$$

From the closure condition, we can now conclude that there exist $X_i \in \mathcal{S}_{r+}^{n_i}$ with $\mathcal{P}_{H_i}(X_i) = x_i, \forall i$. We can now obtain the desired completion with appropriate rank by using

$$X = \begin{bmatrix} X_1^{1/2} \\ X_2^{1/2} \\ \dots \\ X_k^{1/2} \end{bmatrix} \begin{bmatrix} X_1^{1/2} \\ X_2^{1/2} \\ \dots \\ X_k^{1/2} \end{bmatrix}^T,$$

i.e., we apply the idea from Proposition 2.1. □

Corollary 2.1. *The projection $\mathcal{P}(\mathbb{S}_{r+}^n)$ is closed if, and only if, the restricted projections $\mathcal{P}_{H_i}(\mathcal{S}_{r+}^{n_i})$ are closed for all connected components H_i of G, $n_i = |H_i|$.*

Proof. From Theorem 2.1 we can restrict to components. From Assumption 1.1 we can restrict to connected components. □

We can now focus on the connected components of a graph; equivalently, we can deal with each component separately and so assume we are dealing with a connected graph.

Assumption 2.1. *Assume that Assumption 1.1 holds and that the graph G is connected with*

$$|\mathcal{L}| = n, \text{ or } |\mathcal{L}| = 0.$$

2.1 Closure for Loop Graphs, $|\mathcal{L}| = n$

This result follows from a similar proof to the main result in [8] or as a corollary to Theorem 1.1.

Theorem 2.2. *Let $|\mathcal{L}| = n$. Then $\mathcal{P}(\mathcal{S}_{r+}^n)$ is closed.*

Proof. Suppose we have a sequence of matrices, $\{X^j\} \subset \mathcal{S}_{r+}^n$ with $\mathcal{P}(X^j) = x^j \to x$. Then the diagonal elements of X^j converge and therefore the off-diagonal elements are bounded. Therefore, without loss of generality $X^j \to X$. The result now follows from the closure of $\mathcal{S}_+^n$ and the lower semi-continuity of rank. □

2.2 Examples of Failure for Loopless Graphs, $|\mathcal{L}| = 0$

We note that the following follows from the above results. We include a proof since it emphasizes the elementary nature for $r = n$ and the difficulty that might arise for $r < n$.

Theorem 2.3. *Let $r \in \{0, n\}$, $|\mathcal{L}| = 0$. Then $\mathcal{P}(\mathcal{S}^n_{r+})$ is closed.*

Proof. The $r = 0$ follows from $\mathcal{P}(0) = 0$. For $r = n$, we can always set the unspecified off-diagonal elements to 0; and then we set the diagonal elements large enough to ensure positive definiteness. □

From our results we now only have one case to consider: $0 < r < n$ and all the vertices of our connected graph are loopless. Unfortunately, we do not have simple results for closedness.

We will look at exclusions to begin with, and then provide theorems for closure and completion.

2.2.1 Rank One Case, $r = 1$

We begin by looking at examples of the simplest case, the rank one case, $r = 1$. In fact, the following two instances characterize failure of closure for the rank one case, see Corollary 3.2, below.

Lemma 2.1. *If the graph G has a triangle, a cycle of length 3, then $\mathcal{P}(\mathcal{S}^n_{1+})$ is not closed.*

Proof. Without loss of generality, we can let the triangle be formed by the vertices $\{1, 2, 3\}$. Let

$$v^j = \begin{pmatrix} \frac{1}{\sqrt{j}} & \frac{1}{\sqrt{j}} & \sqrt{j} & 0 & \dots & 0 \end{pmatrix}^T \in \mathbb{R}^n, \quad X^j = v^j (v^j)^T \in \mathcal{S}^n_{1+}.$$

Then, with $E = \{12, 13, 23, \dots\}$, we have

$$\mathcal{P}(X^j) = \begin{pmatrix} \frac{1}{j} & 1 & 1 & 0 & \dots & 0 \end{pmatrix}^T \to \begin{pmatrix} 0 & 1 & 1 & 0 & \dots & 0 \end{pmatrix}^T,$$

and $\begin{bmatrix} ? & 0 & 1 & \dots \\ 0 & ? & 1 & \dots \\ 1 & 1 & ? & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ has no rank one completion. □

Lemma 2.2. *If G has a path of length 3 that is not a cycle (of length 4), then $\mathcal{P}(\mathcal{S}^n_{1+})$ is not closed.*

Proof. Without loss of generality, we can let the path be defined by the first four distinct vertices $1, 2, 3, 4$. Let

$$v^j = \begin{pmatrix} \sqrt{j} & \frac{1}{\sqrt{j}} & \frac{1}{\sqrt{j}} & \sqrt{j} & 0 & \dots & 0 \end{pmatrix}^T \in \mathbb{R}^n, \quad X^j = v^j (v^j)^T \in \mathcal{S}^n_{1+}.$$

Then, with $E = \{12, 23, 34 \dots\}$[1]

$$\mathcal{P}(X^j) = \begin{pmatrix} 1 & \frac{1}{j} & 1 & 0 & 0 & \dots & 0 \end{pmatrix}^T \to \begin{pmatrix} 1 & 0 & 1 & 0 & \dots & 0 \end{pmatrix}^T,$$

[1] We could choose $E = \{12, 13, 23, 24, 34 \dots\}$, $E = \{12, 23, 24, 34 \dots\}$, or $E = \{12, 13, 23, 34 \dots\}$.

and $\begin{bmatrix} ? & 1 & ? & ? & \dots \\ 1 & ? & 0 & ? & \dots \\ ? & 0 & ? & 1 & \dots \\ ? & ? & 1 & ? & \dots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$ has no rank one completion. □

Note that the path in the instance in the proof of Lemma 2.2 cannot be a cycle since the $(1,4)$ entry in X^j diverges to $+\infty$.

Remark 2.1. *Note that we could extend the above two lemmas to higher rank using orthogonal vectors. For example, for Lemma 2.1 with rank* 3*, we could use a cycle of length* 5 *and use two orthogonal vectors*

$$v_\pm^j = \begin{pmatrix} \frac{1}{\sqrt{j}} & \pm\frac{1}{\sqrt{j}} & \frac{1}{\sqrt{j}} & \pm\frac{1}{\sqrt{j}} & \sqrt{j} & \pm\sqrt{j} & 0 & \dots & 0 \end{pmatrix}^T \in \mathbb{R}^n, \quad X^j = \sum_{\pm} v_\pm^j (v_\pm^j)^T \in \mathcal{S}_{2+}^n.$$

The limit yields a partial matrix with no rank 2 *PSD completion. We could similarly extend Lemma 2.2.*

3 Bipartite Graphs, Independent Sets, Cliques

We now look at first sufficient and then necessary conditions for closure. Recall that Assumption 2.1 holds.

3.1 Complete Bipartite Graphs

The graphs in the above two examples in Lemmas 2.1 and 2.2 where closure can fail for $\mathcal{P}(\mathcal{S}_{1+}^n)$ are both complete bipartite graphs. Recall that a graph is *bipartite* means that it is 2-*colourable*, i.e., the vertices can be coloured using two colours with no two adjacent nodes having the same colour. We now see that G complete bipartite provides a sufficient condition for closure for all r and characterizes closure for $r = 1$.

Proposition 3.1 ([7, Prop. 1.6.1])**.** *A graph is bipartite if, and only if, it contains no odd cycle.* □

Lemma 3.1. *A graph G is complete bipartite if, and only if, G has no triangle and every path of length* 3 *forms a cycle (of length* 4*).*

Proof. For sufficiency suppose that G has an odd cycle. Then it is either a triangle or it must contain a path of (at least) length 3. That G is bipartite now follows from the characterization in Proposition 3.1.

Now, if G was not complete, then there exists x, y in different partitions that are not adjacent. Then, consider the shortest path from x to y: $x, z_1, z_2 \dots, z_k, y$. This path has length at least 4. Moreover, $z_{k-2}z_{k_1}z_k y$ is a path that does not form a cycle.

For necessity, we immediately see that G cannot have a triangle from Proposition 3.1. And, if G has a path of length 3, then without loss of generality the nodes are $1, 2, 3, 4$. Then looking at all possible cases means that completeness implies there is a cycle, i.e., we have a contradiction. □

Corollary 3.1. *Suppose that $\mathcal{P}(\mathcal{S}_{1+}^n)$ is closed. Then G is a complete bipartite graph.*

Proof. If $\mathcal{P}(\mathcal{S}^n_{1+})$ is closed, then the conditions in Lemmas 2.1 and 2.2 fail which by Lemma 3.1 imply that G is a complete bipartite graph. □

We now use a characterization of the minimum rank PSD completion of a complete bipartite graph to obtain sufficiency for closure for all r.

Theorem 3.1. *Let G be a complete bipartite graph. Then $\mathcal{P}(\mathcal{S}^n_{r+})$ is closed for all $0 \le r \le n$.*

Proof. Let a^i be a sequence of partial matrices with $\mathcal{P}(X^i) = a^i \to a$ and $\operatorname{rank}(X_i) \le r$. Since G is complete bipartite we can permute the vertices of so that one partition is $\{1 \dots k\}$ while the other is $\{k+1 \dots n\}$. This leads to matrices of the form $\begin{bmatrix} ? & B \\ B^T & ? \end{bmatrix}$, where $B^i \to B$ is a complete matrix with $\operatorname{rank}(B^i) \le r$. By the lower semi-continuity of the rank function, we get that $\operatorname{rank}(B) \le r$. Let $B = PQ^T$ be a full rank decomposition of B and let

$$X = \begin{bmatrix} P \\ Q \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}^T.$$

We conclude that

$$X \in \mathcal{P}(\mathcal{S}^n_{r+}), \qquad \mathcal{P}(X) = a, \quad a_{ij} = B_{ij}, \forall ij \in E.$$

□

Corollary 3.2. *Let $r = 1$. Then G is a complete bipartite graph if, and only if, $\mathcal{P}(\mathcal{S}^n_{r+})$ is closed.* □

Remark 3.1. *Let $Z \in \mathbb{R}^{m \times n}$ be given rank one data matrices that are sampled at the coordinates $ij \in \Omega$, i.e., Z_Ω is given data. Then Corollary 3.2 indicates that the nuclear norm (sum of the singular values) heuristic cannot recover all instances unless all of Z is sampled. Recall, e.g., [11], that the nuclear norm heuristic for rank minimization is equivalent to solving the SDP*

$$\begin{array}{ll} \min & \frac{1}{2} \operatorname{trace} Y \\ s.t. & Y = \begin{bmatrix} A & Z \\ Z^T & B \end{bmatrix} \succeq 0 \\ & Y_E = Z_\Omega, \end{array}$$

where E are the edges corresponding appropriately to the coordinates Ω. From the above results we can find classes of examples where closure fails and no rank one completion can be found. This means that we have classes of examples where the nuclear norm heuristic fails for data matrices Z with rank one.

3.2 Independent Sets

Finding independent sets provide sufficient conditions for closure. First we need the following.

Lemma 3.2. *Let $Y_C := \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0$ be given. Then the minimum rank PSD completion problem*

$$\begin{array}{ll} \min_X & \operatorname{rank}\left(\begin{bmatrix} A & B \\ B^T & X \end{bmatrix} \right) \\ s.t. & Y_X = \begin{bmatrix} A & B \\ B^T & X \end{bmatrix} \succeq 0 \end{array}$$

has optimal solution $X^* = B^T A^\dagger B$*, where* $\cdot^\dagger$ *denotes the Moore-Penrose generalized inverse. Moreover,* $\operatorname{rank}(Y_{X^*}) = \operatorname{rank}(A)$.

Then X *is positive semidefinite and* $\operatorname{rank}(X) = \operatorname{rank}(A)$. $X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0.$

where A *is positive definite and* $C = B^T A^{-1} B$. *Then* X *is positive semidefinite and* $\operatorname{rank}(X) = \operatorname{rank}(A)$.

Proof. From the full rank factorization using the unique PSD square roots, we have

$$Y_C = \begin{bmatrix} A^{1/2} \\ C^{1/2} \end{bmatrix} \begin{bmatrix} A^{1/2} \\ C^{1/2} \end{bmatrix}^T, \qquad B = A^{1/2} C^{1/2}.$$

This means that $\operatorname{range}(B) \subseteq \operatorname{range}(A)$. The range condition implies that the projection can be discarded $B = A^{1/2}(A^{1/2})^\dagger B$. We now define

$$\begin{aligned} Y_{X^*} &= \begin{bmatrix} A^{1/2} \\ B^T (A^{1/2})^\dagger \end{bmatrix} \begin{bmatrix} A^{1/2} \\ B^T (A^{1/2})^\dagger \end{bmatrix}^T \\ &= \begin{bmatrix} A & A^{1/2}(A^{1/2})^\dagger B \\ B^T (A^{1/2})^\dagger A^{1/2} & B^T (A^{1/2})^\dagger (A^{1/2})^\dagger B \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ B^T & B^T A^\dagger B \end{bmatrix}. \end{aligned}$$

□

Recall that an *independent set* (or stable set) is a set vertices in a graph, no two of which are adjacent. We can always complete the corresponding matrix to rank at most $n - k$.

Corollary 3.3. *Let* G *be a graph with an independent set of size* k*. Then* $\mathcal{P}(\mathcal{S}^n_{(n-k)+})$ *is closed.*

Proof. An independent set of size k means that we can apply Lemma 3.2 with the free block C of size k. More precisely, we can pick the diagonal elements of the A and C blocks large enough so that $Y_C \succeq 0$ exists. Then we can always find a completion with rank at most $\operatorname{rank}(A) \le n - k$. □

Of course, determining if a graph has an independent set of certain size is generally a hard problem. However some nice corollaries follow. The following was already given in Corollary 1.1.

Corollary 3.4. $\mathcal{P}(\mathcal{S}^n_{(n-1)+})$ *is closed.*

Proof. Every vertex is by itself is an independent set. □

Corollary 3.5. $\mathcal{P}(\mathcal{S}^n_{(n-2)+})$ *is not closed if and only if* G *is the complete graph* K_n*.*

Proof. The only graph without an independent set of size at least two is the complete graph K_n. □

3.3 Cliques

Lemma 3.3. *Let G have a clique of size $k > 2$. Then $\mathcal{P}(\mathcal{S}^n_{(j-2)+})$, $j = 3, \dots, k$, is not closed.*

Proof. Let $\{1, 2, \dots k\}$ be the clique. Let $x^i \in \mathbb{R}^n$ be a sequence of vectors defined by:

$$x^i_j = \begin{cases} \frac{1}{i}, & \text{if } j < k \\ i, & \text{if } j = k \\ 0, & \text{if } j > k. \end{cases}$$

Define the rank one sequence of PSD matrices

$$X^i = x^i {x^i}^T = \begin{bmatrix} \frac{1}{i^2}J & e & 0 \\ e^T & i^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & e & 0 \\ e^T & \infty & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where e is the vector of ones and J is the matrix of ones. Then $a^i := \mathcal{P}(X^i) \in \mathcal{P}(\mathcal{S}^n_{1+})$. But, as $a^i \to \bar{a}$ we have that $\bar{a}$ has a $(k-1) \times (k-1)$ submatrix with all 0 off-diagonal entries, and free diagonal entries. Moreover the (j,k) and (k,j) entries are one, non-zero, for $j < k$. Since the diagonal is free, we see that there is a rank $k-1$ completion but no smaller rank completion, i.e., $\bar{a} \notin \mathcal{P}(\mathcal{S}^n_{(k-2)+})$. □

Theorem 3.2. *Let G have disjoint cliques C_i with cardinalities k_i and integers j_i satisfying $|C_i| = k_i \geq j_i > 2$, $i = 1, \dots, t$. Let $\mathcal{I} \subseteq \{1, \dots, t\}$, and $t = \sum_{i \in \mathcal{I}} (j_i - 2)$. Then*

$$\mathcal{P}(\mathcal{S}^n_{t+}) \text{ is not closed.}$$

Proof. From Lemma 3.3 we have that a completion can lose rank $j_i - 2$ for each clique. □

4 Conclusion

In this paper we have studied the problem when the rank restricted coordinate shadows $\mathcal{P}(\mathcal{S}^n_{r+})$ are closed.

4.1 Summary of Closure Conditions

1. $\mathcal{P}(\mathcal{S}^n_{0+})$ is trivially closed.
2. $\mathcal{P}(\mathcal{S}^n_{+})$ is closed if, and only if, $\mathcal{L}$ and $\mathcal{L}^c$ are <u>disconnected</u>.
3. $\mathcal{L}$ and $\mathcal{L}^c$ are <u>connected</u> implies $\mathcal{P}(\mathcal{S}^n_{r+})$ is <u>not</u> closed for all r.
4. $\mathcal{P}(\mathbb{S}^n_{r+})$ is closed if, and only if, the restricted projections $\mathcal{P}_{H_i}(\mathcal{S}^{n_i}_{r+})$ are closed for all <u>connected</u> components H_i of G, $n_i = |H_i|$.
5. We can now assume that $\mathcal{L}$ and $\mathcal{L}^c$ are <u>not connected</u>.
   (a) $\mathcal{P}(\mathcal{S}^n_{r+})$ <u>is</u> closed for all r such that

$$\min\left\{ n-1, \left\lceil -\frac{3}{2} + \frac{\sqrt{9+8|E|}}{2} \right\rceil \right\} \leq r \leq n.$$

(b) $|\mathcal{L}| = n$, a loop graph, implies that $\mathcal{P}(\mathcal{S}^n_{r+})$ is closed for all r.

(c) We can now assume that G is connected and $|\mathcal{L}| = 0$, a loopless graph.

   i. for complete bipartite:

      A. If G is complete bipartite, then $\mathcal{P}(\mathcal{S}^n_{r+})$ is closed for all r.

      B. G is complete bipartite if, and only if, $\mathcal{P}(\mathcal{S}^n_{1+})$ is closed.

   ii. for independent set:

      A. If G has an independent set of size k, then $\mathcal{P}(\mathcal{S}^n_{n-k+})$ is closed.

      B. $\mathcal{P}(\mathcal{S}^n_{(n-1)+})$ is closed.

      C. $\mathcal{P}(\mathcal{S}^n_{(n-2)+})$ is not closed if and only if G is K_n.

   iii. for clique:

      A. If G has a clique of size $k > 2$, then $\mathcal{P}(\mathcal{S}^n_{(k-2)+})$ is not closed. (And extensions to more disjoint cliques.)

4.2 Open Questions

We saw that complete bipartite characterized closure for rank one and was sufficient for all r. A reasonable conjecture is that for non-bipartite graphs, we get that tripartite characterizes closure for rank 2 and is sufficient for $r \geq 3$. This naturally leads to the corresponding conjecture for higher ranks and higher multipartite graphs. Note that a simple proof for sufficiency for the tripartite case follows if the matrices A, B, C in the partial symmetric matrix $\begin{bmatrix} ? & A & B \\ A^T & ? & C \\ B^T & C^T & ? \end{bmatrix}$ are all rank 2 and all 2×2. We could then explicitly solve for P, Q, R in the three equations $A = PQ^T, B = PR^T, C = QR^T$ to obtain the rank 2 PSD completion $\begin{bmatrix} P \\ Q \\ R \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}^T$.

In Remark 3.1 we have emphasized that there are instances where the nuclear norm fails to recover the data matrix Z of the correct rank. This leads to questions about the measure of the sets where failure occurs and is related to the conditions on sampling for high probability completions, see [5].

Index

References

[1] A. Auslender. Closedness criteria for the image of a closed set by a linear operator. *Numer. Funct. Anal. Optim.*, 17(5-6):503–515, 1996. 2

[2] A. Barvinok. A remark on the rank of positive semidefinite matrices subject to affine constraints. *Discrete Comput. Geom.*, 25(1):23–31, 2001. 2

[3] J.M. Borwein and W.B. Moors. Stability of closedness of convex cones under linear mappings ii. *J. Nonlinear Anal. and Optim.*, 1:1–7, 2010. 2

[4] A.A. Bostian and H.J. Woerdeman. Unicity of minimal rank completions for tri-diagonal partial block matrices. *Linear Algebra Appl.*, 325(1-3):23–55, 2001. 2

[5] E.J. Candès and B. Recht. Exact matrix completion via convex optimization. *Found. Comput. Math.*, 9(6):717–772, 2009. 11

[6] M.M. Deza and M. Laurent. *Geometry of cuts and metrics.* Springer-Verlag, Berlin, 1997. 2

[7] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 2000. 7

[8] D. Drusvyatskiy, G. Pataki, and H. Wolkowicz. Coordinate shadows of semidefinite and Euclidean distance matrices. *SIAM J. Optim.*, 25(2):1160–1178, 2015. 2, 4, 5

[9] G. Pataki. On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues. *Math. Oper. Res.*, 23(2):339–358, 1998. 2

[10] G. Pataki. On the closedness of the linear image of a closed convex cone. *Math. Oper. Res.*, 32(2):395–412, 2007. 2

[11] B. Recht, M. Fazel, and P. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Rev.*, 52(3):471–501, 2010. 8

[12] H.J. Woerdeman. Minimal rank completions for block matrices. *Linear Algebra Appl.*, 121:105–122, 1989. Linear algebra and applications (Valencia, 1987). 2