COORDINATE SHADOWS OF SEMI-DEFINITE AND EUCLIDEAN DISTANCE MATRICES

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Abstract. We consider the projected semi-definite and Euclidean distance cones onto a subset of the matrix entries. These two sets are precisely the input data defining feasible semi-definite and Euclidean distance completion problems. We characterize when these sets are closed, and use the boundary structure of these two sets to elucidate the Krislock-Wolkowicz facial reduction algorithm. In particular, we show that under a chordality assumption, the "minimal cones" of these problems admit combinatorial characterizations.

Key words. Matrix completion, semidefinite programming, Euclidean distance matrices, facial reduction, Slater condition, projection, closedness

AMS subject classifications. 90C22, 90C46, 52A99

1. Introduction.

To motivate the discussion, consider an undirected graph G with vertex set $V = \{1, \dots, n\}$ and edge set $E \subset \{ij : i \leq j\}$. The classical *semi-definite (PSD) completion problem* asks whether given a data vector a indexed by E, there exists an $n \times n$ positive semi-definite matrix X completing a, meaning $X_{ij} = a_{ij}$ for all $ij \in E$. Similarly, the *Euclidean distance (EDM) completion problem* asks whether given such a data vector, there exists a Euclidean distance matrix completing it. For a survey of these two problems, see for example [2, 21, 22, 24]. The semi-definite and Euclidean distance completion problems are often mentioned in the same light due to a number of parallel results; see e.g. [20]. Here, we consider a related construction: projections of the PSD cone $\mathcal{S}^n_+$ and the EDM cone $\mathcal{E}^n$ onto matrix entries indexed by E. These "coordinate shadows", denoted by $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$, respectively, appear naturally: they are precisely the sets of data vectors that render the corresponding completion problems feasible. We mention in passing that these sets are interesting types of "spectrahedral shadows" — a hot topic of research in recent years; see e.g. [3, 10, 14, 15].

In this short note, our goal is twofold: (1) we will highlight the geometry of the two sets $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$, and (2) illustrate how such geometric considerations yield a much simplified and transparent analysis of an EDM completion algorithm proposed in [17]. To this end, we begin by asking a basic question:

Under what conditions are the coordinate shadows $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$ closed?

This question sits in a broader context still of deciding if a linear image of a closed convex set is itself closed — a thoroughly studied topic due to its fundamental connection to constraint qualifications and strong duality in convex optimization; see e.g. [8, 9, 27, 30] and references therein. We will show that surprisingly $\mathcal{P}(\mathcal{E}^n)$ is always closed, whereas $\mathcal{P}(\mathcal{S}^n_+)$ is closed if and only if the set of vertices attached to self-loops $L = \{i \in V : ii \in E\}$ is disconnected from its complement L^c (Theorems 3.1, 3.3). Moreover, whenever there is an edge joining L and L^c, one can with

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ease exhibit vectors lying in the closure of $\mathcal{P}(\mathcal{S}^n_+)$, but not in the set $\mathcal{P}(\mathcal{S}^n_+)$ itself, thereby certifying that $\mathcal{P}(\mathcal{S}^n_+)$ is not closed.

To illustrate the algorithmic significance of the coordinate shadows $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$, consider first the feasible region of the PSD completion problem:

$$\{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij} \text{ for } ij \in E\}.$$

For this set to be non-empty, the data vector $a \in \mathbb{R}^E$ must be a partial PSD matrix, meaning all of its principal submatrices are positive semi-definite. This, however, does not alone guarantee the inclusion $a \in \mathcal{P}(\mathcal{S}^n_+)$, unless the restriction of G to L is chordal and L is disconnected from L^c (Corollary 3.2). On the other hand, the authors of [17] noticed that even if the feasible set is nonempty, the Slater condition (i.e. existence of a positive definite completion) will often fail: small perturbations to any specified principal submatrix of a having deficient rank can yield the semi-definite completion problem infeasible. In other words, in this case the partial matrix a lies on the boundary of $\mathcal{P}(\mathcal{S}^n_+)$ — the focus of this short note. An entirely analogous situation occurs for EDM completions

$$\{X \in \mathcal{E}^n : X_{ij} = a_{ij} \text{ for } ij \in E\},$$

with the rank of each principal submatrix of $a \in \mathbb{R}^E$ replaced by its "embedding dimension". In [17], the authors propose a preprocessing strategy utilizing the cliques in the graph G to systematically decrease the size of the EDM completion problem. Roughly speaking, the authors use each clique to find a face of the EDM cone containing the entire feasible region, and then iteratively intersect such faces. The numerical results in [17] were impressive. In the current work, we provide a much simplified and transparent geometric argument behind their algorithmic idea, with the boundary of $\mathcal{P}(\mathcal{E}^n)$ playing a key role. As a result, we put their techniques in a broader setting unifying the PSD and EDM cases. Moreover, we show that when G is chordal and all cliques are considered, the preprocessing technique discovers the minimal face of $\mathcal{E}^n$ (respectively $\mathcal{S}^n_+$) containing the feasible region; see Theorems 4.5 and 4.9. This in part explains the observed success of the method [17]. In particular, this shows that in contrast to general semi-definite programming, the minimal face of the PSD cone containing the feasible region of the PSD completion problem (one of the simplest semi-definite programming problems) admits a purely combinatorial description.

The outline of the manuscript is as follows. In Section 2 we record basic results on convex geometry and PSD and EDM completions. In Section 3, we characterize when the coordinate shadows $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$ are closed, while in Section 4 we discuss the aforementioned clique facial reduction strategy.

2. Preliminaries.

2.1. Basic elements of convex geometry. We begin with some notation, following closely the classical text [30]. Consider a Euclidean space $\mathbb{E}$ with the inner product $\langle \cdot, \cdot \rangle$. The adjoint of a linear mapping $\mathcal{M} \colon \mathbb{E} \to \mathbb{Y}$, between two Euclidean spaces $\mathbb{E}$ and $\mathbb{Y}$, is written as $\mathcal{M}^*$, while the range and kernel of $\mathcal{M}$ is denoted by $\operatorname{rge} \mathcal{M}$ and $\ker \mathcal{M}$, respectively. We denote the closure, boundary, interior, and relative interior of a set Q in $\mathbb{E}$ by $\operatorname{cl} Q$, $\operatorname{bnd} Q$, $\operatorname{int} Q$, and $\operatorname{ri} Q$, respectively. Consider a convex cone C in $\mathbb{E}$. The linear span and the orthogonal complement of the linear span of C will be denoted by $\operatorname{span} C$ and $C^\perp$, respectively. For a vector v, we let $v^\perp := \{v\}^\perp$. We associate with C the *nonnegative polar* cone

$$C^* = \{y \in \mathbb{E} : \langle y, x \rangle \geq 0 \ \text{ for all } x \in C\}.$$

The second polar $(C^*)^*$ coincides with the original C if, and only if, C is closed. A convex subset $F \subseteq C$ is a *face of* C, denoted $F \trianglelefteq C$, if F contains any line segment in C whose relative interior intersects F. The *minimal face* containing a set $S \subseteq C$, denoted $\mathrm{face}(S, C)$, is the intersection of all faces of C containing S. When S is itself a convex set, then $\mathrm{face}(S, C)$ is the smallest face of C intersecting the relative interior of S. A face F of C is an *exposed face* when there exists a vector $v \in C^*$ (the exposing vector) satisfying $F = C \cap v^\perp$. The cone C is *facially exposed* when all faces of C are exposed. In particular, the cones of positive semi-definite and Euclidean distance matrices, which we will focus on shortly, are facially exposed. With any face $F \trianglelefteq C$, we associate a face of the polar C^*, called the *conjugate face* $F^\triangle := C^* \cap F^\perp$. Equivalently, $F^\triangle$ is the face of C^* exposed by any point $x \in \mathrm{ri}\, F$, that is $F^\triangle := C^* \cap x^\perp$. Thus, in particular, conjugate faces are always exposed. Not surprisingly then equality $(F^\triangle)^\triangle = F$ holds if, and only if, $F \trianglelefteq C$ is exposed.

Fix a point x of a closed, convex cone C. We will use the following two basic constructions: the *cone of feasible directions* of C at x is the set

$$\mathrm{dir}(x, C) := \{v : x + \epsilon v \in C \text{ for some } \epsilon > 0\},$$

and the *tangent cone* of C at x is

$$\mathrm{tcone}(x, C) := \mathrm{cl}\,\mathrm{dir}(x, C).$$

Both of the cones above can conveniently be described in terms of the minimal face $F := \mathrm{face}(x, C)$ as follows (for details, see [27, Lemma 1]):

$$\mathrm{dir}(x, C) = C + \mathrm{span}\, F \qquad \text{and} \qquad \mathrm{tcone}(x, C) = (F^\triangle)^*.$$

A central (and classical) question in convex analysis is when a linear image of a closed convex cone is itself closed. In a recent paper [26], the author showed that there is a convenient characterization for “nice cones” — those cones C for which $C^* + F^\perp$ is closed for all faces $F \trianglelefteq C$ [5, 26]. Reassuringly, most cones which we can efficiently optimize over are nice; see the discussion in [26]. For example, the cones of positive semi-definite and Euclidean distance matrices are nice. Theorem 2.1 below, originating in [26, Theorem 1.1, Corollary 3.1] and [27, Theorem 3], plays a central role in our work.

THEOREM 2.1 (Image closedness of nice cones). *Let $\mathcal{M} : \mathbb{E} \to \mathbb{Y}$ be a linear transformation between two Euclidean spaces $\mathbb{E}$ and $\mathbb{Y}$, and let $C \subseteq \mathbb{Y}$ be a nice, closed convex cone. Consider a point $x \in \mathrm{ri}(C \cap \mathrm{rge}\,\mathcal{M})$. Then the following two statements are equivalent.*

1. *The image $\mathcal{M}^* C^*$ is closed.*
2. *The implication*

$$v \in \mathrm{tcone}(x, C) \cap \mathrm{rge}\,\mathcal{M} \quad \Longrightarrow \quad v \in \mathrm{dir}(x, C) \quad \textit{holds.} \tag{2.1}$$

Moreover, suppose that implication (2.1) fails and choose an arbitrary vector $v \in (\mathrm{tcone}(x, C) \cap \mathrm{rge}\,\mathcal{M}) \setminus \mathrm{dir}(x, C)$. Then for any point

$$a \in (\mathrm{face}(x, C))^\perp \quad \textit{satisfying} \quad \langle a, v \rangle < 0, \tag{2.2}$$

the point $\mathcal{M}^* a$ lies in $(\mathrm{cl}\,\mathcal{M}^* C^*) \setminus \mathcal{M}^* C^*$, thereby certifying that $\mathcal{M}^* C^*$ is not closed.

Remark 2.2. Following notation of Theorem 2.1, it is shown in [26, Theorem 1.1, Corollary 3.1] that for any point $x \in \operatorname{ri}(C \cap \operatorname{rge}\mathcal{M})$, we have equality

$$(\operatorname{tcone}(x,C) \cap \operatorname{rge}\mathcal{M}) \setminus \operatorname{dir}(x,C) \;=\; (\operatorname{tcone}(x,C) \cap \operatorname{rge}\mathcal{M}) \setminus \operatorname{span}\operatorname{face}(x,C).$$

Hence for any point $x \in \operatorname{ri}(C \cap \operatorname{rge}\mathcal{M})$ and any vector $v \in (\operatorname{tcone}(x,C) \cap \operatorname{rge}\mathcal{M}) \setminus \operatorname{dir}(x,C)$, there indeed exists some point a satisfying (2.2).

The following sufficient condition for image closedness is now immediate.

Corollary 2.3 (Sufficient condition for image closedness).
Let $\mathcal{M} : \mathbb{E} \to \mathbb{Y}$ be a linear transformation between two Euclidean spaces $\mathbb{E}$ and $\mathbb{Y}$, and let $C \subseteq \mathbb{Y}$ be a nice, closed convex cone. If for some point $x \in \operatorname{ri}(C \cap \operatorname{rge}\mathcal{M})$, the inclusion $\operatorname{rge}(\mathcal{M}) \subseteq \operatorname{span}\operatorname{face}(x,C)$ holds, then $\mathcal{M}^*C^*$ is closed.

Proof. Define $F := \operatorname{face}(x,C)$ and note $\operatorname{rge}(\mathcal{M}) \subseteq \operatorname{span} F \subseteq \operatorname{dir}(x,C)$. We deduce

$$\operatorname{tcone}(x,C) \cap \operatorname{rge}(\mathcal{M}) \quad\subseteq\quad \operatorname{tcone}(x,C) \cap \operatorname{dir}(x,C) = \operatorname{dir}(x,C).$$

The result now follows from Theorem 2.1, since implication (2.1) holds. □

2.2. Semi-definite and Euclidean distance matrices. We will focus on two particular realizations of the Euclidean space $\mathbb{E}$: the n-dimensional vector space $\mathbb{R}^n$ with a fixed basis and the induced *dot-product* $\langle \cdot, \cdot \rangle$ and the vector space of $n \times n$ real symmetric matrices $\mathcal{S}^n$ with the *trace inner product* $\langle A, B \rangle := \operatorname{trace} AB$. The symbols $\mathbb{R}_+$ and $\mathbb{R}_{++}$ will stand for the non-negative orthant and its interior in $\mathbb{R}^n$, while $\mathcal{S}^n_+$ and $\mathcal{S}^n_{++}$ will stand for the set of positive semi-definite and positive definite matrices in $\mathcal{S}^n$ (or *PSD* and *PD* for short), respectively. We let $e \in \mathbb{R}^n$ be the vector of all ones and for any vector $v \in \mathbb{R}^n$, the symbol $\operatorname{Diag} v$ will denote the $n \times n$ diagonal matrix with v on the diagonal.

It is well-known that all faces of $\mathcal{S}^n_+$ can be expressed as

$$F = \left\{ U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^T : A \in \mathcal{S}^r_+ \right\},$$

for some orthogonal matrix U and some integer $r = 0, 1, \ldots, n$. Such a face can equivalently be written as $F = \{X \in \mathcal{S}^n_+ : \operatorname{rge} X \subset \operatorname{rge} \overline{U}\}$, where $\overline{U}$ is formed from the first r columns of U. The conjugate face of such a face F is then

$$F^{\triangle} = \left\{ U \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} U^T : A \in \mathcal{S}^{n-r}_+ \right\}.$$

For any convex set $Q \subset \mathcal{S}^n_+$, the set $\operatorname{face}(Q, \mathcal{S}^n_+)$ coincides with $\operatorname{face}(X, \mathcal{S}^n_+)$ where X is any maximal rank matrix in Q.

A matrix $D \in \mathcal{S}^n$ is a *Euclidean distance matrix* (or EDM for short) if there exist n points p_i (for $i = 1, \ldots, n$) in some Euclidean space $\mathbb{R}^k$ satisfying $D_{ij} = \|p_i - p_j\|^2$, for all indices i, j. The smallest integer k for which this realization of D by n points is possible is the *embedding dimension* of D and will be denoted by $\operatorname{embdim} D$. We let $\mathcal{E}^n$ be the set of $n \times n$ Euclidean distance matrices. There is a close relationship between PSD and EDM matrices. Indeed $\mathcal{E}^n$ is a closed convex cone that is linearly isomorphic to $\mathcal{S}^{n-1}_+$. To state this precisely, consider the mapping

$$\mathcal{K} : \mathcal{S}^n \to \mathcal{S}^n$$

defined by

$$\mathcal{K}(X)_{ij} := X_{ii} + X_{jj} - 2X_{ij}.$$

Then the adjoint $\mathcal{K}^*\colon \mathcal{S}^n \to \mathcal{S}^n$ is given by

$$\mathcal{K}^*(D) = 2(\mathrm{Diag}(De) - D)$$

and the equations

$$\operatorname{rge}\mathcal{K} = \mathcal{S}_H, \qquad \operatorname{rge}\mathcal{K}^* = \mathcal{S}_c \tag{2.3}$$

hold, where

$$\mathcal{S}_c := \{X \in \mathcal{S}^n : Xe = 0\}; \qquad \mathcal{S}_H := \{D \in \mathcal{S}^n : \mathrm{diag}(D) = 0\}, \tag{2.4}$$

are the *centered* and *hollow matrices*, respectively. It is known that $\mathcal{K}$ maps $\mathcal{S}^n_+$ onto $\mathcal{E}^n$, and moreover the restricted mapping

$$\mathcal{K} : \mathcal{S}_c \to \mathcal{S}_H \text{ is a linear isomorphism carrying } \mathcal{S}_c \cap \mathcal{S}^n_+ \text{ onto } \mathcal{E}^n. \tag{2.5}$$

In turn, it is easy to see that $\mathcal{S}_c \cap \mathcal{S}^n_+$ is a face of $\mathcal{S}^n_+$ isomorphic to $\mathcal{S}^{n-1}_+$; see the discussion after Lemma 4.7 for more details. These and other related results have appeared in a number of publications; see for example [1, 12, 13, 18, 19, 31–34].

2.3. Semi-definite and Euclidean distance completions. The focus of the current work is on the PSD and EDM completion problems, see e.g., [16, Chapter 49]. Throughout the rest of the manuscript, we fix an undirected graph $G = (V, E)$, with a vertex set $V = \{1, \ldots, n\}$ and an edge set $E \subset \{ij : 1 \le i \le j \le n\}$. Observe that we allow self-loops. These loops will play an important role in what follows, and hence we define L to be the set of all vertices i satisfying $ii \in E$, that is those vertices that are attached to a loop.

Any vector $a \in \mathbb{R}^E$ is called a *partial matrix*. Define now the projection map $\mathcal{P} : \mathcal{S}^n \to \mathbb{R}^E$ by setting

$$\mathcal{P}(A) = (A_{ij})_{ij \in E},$$

that is $\mathcal{P}(A)$ is the vector of all the entries of A indexed by E. The adjoint map $\mathcal{P}^* : \mathbb{R}^E \to \mathcal{S}^n$ is found by setting

$$(\mathcal{P}^*(y))_{ij} = \begin{cases} y_{ij}, & \text{if } ij \in E \\ 0, & \text{otherwise,} \end{cases}$$

for indices $i \le j$. Define also the *Laplacian operator* $\mathcal{L}\colon \mathbb{R}^E \to \mathcal{S}^n$ by setting

$$\mathcal{L}(a) := \frac{1}{2}(\mathcal{P} \circ \mathcal{K})^*(a) = \mathrm{Diag}(\mathcal{P}^*(a)e) - \mathcal{P}^*(a).$$

Consider a partial matrix $a \in \mathbb{R}^E$ whose components are all strictly positive. Classically then the Laplacian matrix $\mathcal{L}(a)$ is positive semi-definite and moreover the kernel of $\mathcal{L}(a)$ is only determined by the connectivity of the graph G; see for example [7], [16, Chapter 47]. Consequently all partial matrices with strictly positive weights define the same minimal face of the positive semi-definite cone. In particular, when G is connected, we have the equalities

$$\ker \mathcal{L}(a) = \mathrm{span}\{e\} \qquad \text{and} \qquad \mathrm{face}(\mathcal{L}(a), \mathcal{S}^n_+) = \mathcal{S}_c \cap \mathcal{S}^n_+. \tag{2.6}$$

A symmetric matrix $A \in \mathcal{S}^n$ is a *completion* of a partial matrix $a \in \mathbb{R}^E$ if it satisfies $\mathcal{P}(A) = a$. We say that a completion $A \in \mathcal{S}^n$ of a partial matrix $a \in \mathbb{R}^E$ is a

PSD completion if A is a PSD matrix. Thus the image $\mathcal{P}(\mathcal{S}^n_+)$ is the set of all partial matrices that are PSD completable. A partial matrix $a \in \mathbb{R}^E$ is a *partial PSD matrix* if all existing principal submatrices, defined by a, are PSD matrices. Finally we call G itself a *PSD completable graph* if every partial PSD matrix $a \in \mathbb{R}^E$ is completable to a PSD matrix. *PD completions*, *partial PD matrices*, and *PD completable graphs* are defined similarly.

We call a graph *chordal* if any cycle of four or more nodes has a chord, i.e., an edge exists joining any two nodes that are not adjacent in the cycle. Before we proceed, a few comments on completability are in order. In [11, Proposition 1], the authors claim that G is PSD completable (PD respectively) if and only if the graph induced on L by G is PSD completable (PD respectively). In light of this, the authors then reduce all of their arguments to this induced subgraph. It is easy to see that the statement above does not hold for PSD completability (see the example below), but is indeed valid for PD completability. Taking this into account, the correct statement of their main result [11, Theorem 7] is as follows.

Theorem 2.4 (PSD completable matrices & chordal graphs). *The following are true.*

1. *The graph G is PD completable if and only if the graph induced by G on L is chordal.*
2. *Supposing equality $L = V$ holds, the graph G is PSD completable if and only if G is chordal.*

Without the assumption $L = V$, the second part of the theorem does not hold. Consider for example the partial PSD matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & ? \end{bmatrix}$$

which is clearly not PSD completable. In Corollary 3.2, we get rid of this assumption and observe that PSD completable graphs are precisely the chordal graphs for which L is disconnected from L^c.

With regard to EDMs, we will always assume $L = \emptyset$ for the simple reason that the diagonal of an EDM is always fixed at zero. With this in mind, we say that a completion $A \in \mathcal{S}^n$ of a partial matrix $a \in \mathbb{R}^E$ is an *EDM completion* if A is an EDM. Thus the image $\mathcal{P}(\mathcal{E}^n)$ (or equivalently $\mathcal{L}^*(\mathcal{S}^n_+)$) is the set of all partial matrices that are EDM completable. We say that a partial matrix $a \in \mathbb{R}^E$ is a *partial EDM* if any existing principal submatrix, defined by a, is an EDM. Finally we say that G is an EDM completable graph if any partial EDM is completable to an EDM. The following theorem is analogous to Theorem 2.4. For a proof, see [4].

Theorem 2.5 (Euclidean distance completability & chordal graphs). *The graph G is EDM completable if and only if G is chordal.*

3. Closedness of the projected PSD and EDM cones.

We begin this section by characterizing when the projection of the PSD cone $\mathcal{S}^n_+$ onto some subentries is closed. To illustrate, consider the simplest setting $n = 2$, namely

$$\mathcal{S}^2_+ = \left\{ \begin{bmatrix} x & y \\ y & z \end{bmatrix} : x \geq 0, z \geq 0, xz \geq y^2 \right\}.$$

Abusing notation slightly, one can easily verify:

$$\mathcal{P}_z(\mathcal{S}^2_+) = \mathbb{R}_+, \qquad \mathcal{P}_y(\mathcal{S}^2_+) = \mathbb{R}, \qquad \mathcal{P}_{x,z}(\mathcal{S}^2_+) = \mathbb{R}^2_+.$$

Clearly all of these projected sets are closed. Projecting $\mathcal{S}^2_+$ onto a single row, on the other hand, yields a set that is not closed:

$$\mathcal{P}_{x,y}(\mathcal{S}^2_+) = \mathcal{P}_{z,y}(\mathcal{S}^2_+) = \{(0,0)\} \cup (\mathbb{R}_{++} \times \mathbb{R}).$$

In this case, the graph G has two vertices and two edges, and in particular, there is an edge joining L with L^c. The following theorem shows that this connectivity property is the only obstacle to $\mathcal{P}(\mathcal{S}^n_+)$ being closed.

THEOREM 3.1 (Closedness of the projected PSD cone). *The projected set $\mathcal{P}(\mathcal{S}^n_+)$ is closed if, and only if, the vertices in L are disconnected from those in the complement L^c. Moreover, if the latter condition fails, then for any edge $i^*j^* \in E$ joining a vertex in L with a vertex in L^c, any partial matrix $a \in \mathbb{R}^E$ satisfying*

$$a_{i^*j^*} \neq 0 \quad \text{and} \quad a_{ij} = 0 \text{ for all } ij \in E \cap (L \times L),$$

lies in $\big(\operatorname{cl} \mathcal{P}(\mathcal{S}^n_+)\big) \setminus \mathcal{P}(\mathcal{S}^n_+)$.

Proof. First, whenever $L = \emptyset$ one can easily verify the equation $\mathcal{P}(\mathcal{S}^n_+) = \mathbb{R}^E$. Hence the theorem holds trivially in this case. Without loss of generality, we now permute the vertices V so that we have $L = \{1, \ldots, r\}$ for some integer $r \geq 1$. We will proceed by applying Theorem 2.1 with $\mathcal{M} := \mathcal{P}^*$ and $C := (\mathcal{S}^n_+)^* = \mathcal{S}^n_+$. To this end, observe the equality

$$\mathcal{S}^n_+ \cap \operatorname{rge} \mathcal{P}^* = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : A \in \mathcal{S}^r_+ \text{ and } A_{ij} = 0 \text{ when } ij \notin E \right\}.$$

Thus we obtain the inclusion

$$X := \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \in \operatorname{ri}(\mathcal{S}^n_+ \cap \operatorname{rge} \mathcal{P}^*).$$

Observe

$$\operatorname{face}(X, \mathcal{S}^n_+) = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : A \in \mathcal{S}^r_+ \right\}.$$

From [27, Lemma 3], we have the description

$$\operatorname{tcone}(X, \mathcal{S}^n_+) = \left\{ \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} : C \in \mathcal{S}^{n-r}_+ \right\},$$

while on the other hand

$$\operatorname{dir}(X, \mathcal{S}^n_+) = \left\{ \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} : C \in \mathcal{S}^{n-r}_+ \text{ and } \operatorname{rge} B^T \subseteq \operatorname{rge} C \right\}.$$

Thus if the intersection $E \cap \big(\{1, \ldots, r\} \times \{r+1, \ldots, n\}\big)$ is empty, then for any matrix

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in \operatorname{tcone}(X, \mathcal{S}^n_+) \cap \operatorname{rge} \mathcal{P}^*,$$

we have $B = 0$, and consequently this matrix lies in $\operatorname{dir}(X, \mathcal{S}^n_+)$. Using Theorem 2.1, we deduce that the image $\mathcal{P}(\mathcal{S}^n_+)$ is closed. Conversely, for any edge $i^*j^* \in E \cap \big(\{1, \ldots, r\} \times \{r+1, \ldots, n\}\big)$, we can define the matrix

$$V := e_{i^*} e^T_{j^*} + e_{j^*} e^T_{i^*} \in \big\{\operatorname{tcone}(X, \mathcal{S}^n_+) \setminus \operatorname{dir}(X, \mathcal{S}^n_+)\big\} \cap \operatorname{rge} \mathcal{P}^*,$$

where e_{i^*} and e_{j^*} denote the i^*'th and j^*'th unit vectors in $\mathbb{R}^n$. Theorem 2.1 immediately implies that the image $\mathcal{P}(\mathcal{S}^n_+)$ is not closed. Moreover, in this case, define $A \in \mathcal{S}^n$ to be any matrix satisfying $A_{i^*j^*} < 0$ and $A_{ij} = 0$ whenever $ij \in \{1,\ldots,r\} \times \{1,\ldots,r\}$. Then A lies in $(\text{face}(X, \mathcal{S}^n_+))^\perp$ and the inequality, $\langle A, V\rangle = 2A_{i^*j^*} < 0$, holds. Again appealing to Theorem 2.1, we deduce $\mathcal{P}(A) \in (\text{cl}\,\mathcal{P}(\mathcal{S}^n_+)) \setminus \mathcal{P}(\mathcal{S}^n_+)$, as we had to show. Replacing V by $-V$ shows that the same conclusion holds in the case $A_{i^*j^*} > 0$. This completes the proof. □

As a corollary, we obtain a characterization of PSD completable graphs — an immediate refinement of Theorem 2.4 and a correction of [11, Proposition 1].

Corollary 3.2 (PSD completability, chordal graphs, and connectivity). *The graph G is PSD completable if and only if the graph induced by G on L is chordal and L is disconnected from L^c.*

Proof. Permuting the vertices, we may assume $L = \{1,\ldots,r\}$. Suppose first that G is PSD completable. Then the projection $\mathcal{P}(\mathcal{S}^n_+)$ coincides with the set of all partial PSD matrices, which is clearly a closed set. Theorem 3.1 then immediately implies that L is disconnected from L^c. Now denote by $G_L = (L, E_L)$ the graph induced by G on L, and suppose that G_L is not chordal. Then by Theorem 2.4 there exists a partial matrix $a \in \mathbb{R}^{E_L}$ that is not PSD completable to a matrix in $\mathcal{S}^r_+$. Extending a to $\mathbb{R}^E$ by setting it to be zero elsewhere, we obtain a partial PSD matrix that is not PSD completable, a contradiction. Thus the graph induced by G on L is chordal.

To see the converse, suppose that the graph induced by G on L is chordal and L is disconnected from L^c. Then given a partial PSD matrix $a \in \mathbb{R}^E$, consider its restriction to the graph induced on L, denoted by a_L. By Theorem 2.4, there exists a PSD completion $A_L \in \mathcal{S}^r_+$ of a_L. Since the diagonal elements indexed by L^c are free, we can set them to a sufficiently large value and obtain a PSD completion $A_{L^c} \in \mathcal{S}^{n-r}_+$ of a_{L^c}. Consequently, the matrix $\begin{bmatrix} A_L & 0 \\ 0 & A_{L^c}\end{bmatrix}$ is a PSD completion of a. We conclude that G is PSD completable. □

In contrast to Theorem 3.1, we now show that the projected image of the EDM cone $\mathcal{E}^n$ is <u>always</u> closed.

Theorem 3.3 (Closedness of the projected EDM cone). *The projected image $\mathcal{P}(\mathcal{E}^n)$ is always closed.*

Proof. First, we claim that we can assume without loss of generality that the graph G is connected. To see this, let $G_i = (V_i, E_i)$ for $i = 1,\ldots,l$ be the connected components of G. Then one can easily verify that $\mathcal{P}(\mathcal{E}^n)$ coincides with the Cartesian product $P_{E_1}(\mathcal{E}^{|V_1|}) \times \ldots \times P_{E_l}(\mathcal{E}^{|V_l|})$. Thus if each image $\mathcal{P}_{E_i}(\mathcal{E}^{|V_i|})$ is closed, then so is the product $\mathcal{P}(\mathcal{E}^n)$. We may therefore assume that G is connected.

The proof proceeds by applying Corollary 2.3. To this end, in the notation of that corollary, we set $C := \mathcal{S}^n_+$ and $\mathcal{M} = \mathcal{L} = \frac{1}{2}\mathcal{K}^* \circ \mathcal{P}^*$. Clearly then we have the equality $\mathcal{M}^*C^* = \mathcal{P}(\mathcal{E}^n)$.

Define now the partial matrix $x \in \mathbb{R}^E$ with $x_{ij} = 1$ for all $ij \in E$, and set $X := \mathcal{L}(x)$. We now claim that the inclusion

$$X \in \text{ri}\left(\mathcal{S}^n_+ \cap \text{rge}\,\mathcal{L}\right) \qquad \text{holds.} \tag{3.1}$$

To see this, observe that X lies in the intersection $\mathcal{S}^n_+ \cap \text{rge}\,\mathcal{L}$, since X is a positively weighed Laplacian. Now let $Y \in \mathcal{S}^n_+ \cap \text{rge}\,\mathcal{L}$ be arbitrary, then $Y = \mathcal{L}(y)$ for some partial matrix $y \in \mathbb{R}^E$. Consider the matrices

$$X \pm \epsilon(X - Y) \;=\; \mathcal{L}(x \pm \epsilon(x - y)).$$

If $\epsilon > 0$ is small, then $x \pm \epsilon(x-y)$ has all positive components, and so $X \pm \epsilon(X-Y)$ is a positively weighed Laplacian, hence positive semidefinite. This proves (3.1). Now define $F = \mathrm{face}(X, \mathcal{S}^n_+)$. We claim that equation $\mathrm{span}\, F = \mathcal{S}_c$ holds. To see this, recall that the nullspace of X is one-dimensional, being generated by e. Consequently F has dimension $\frac{n(n-1)}{2}$. On the other hand F is clearly contained in $\mathcal{S}_c$, a linear subspace of dimension $\frac{n(n-1)}{2}$. We deduce $\mathrm{span}\, F = \mathcal{S}_c$, as claimed. The closure now follows from Corollary 2.3. ☐

4. Boundaries of projected sets & facial reduction.

To motivate the discussion, consider the conic system

$$F := \{X \in C : \mathcal{M}(X) = b\}, \tag{4.1}$$

where C is a closed convex cone in an Euclidean space $\mathbb{E}$ and $\mathcal{M} \colon \mathbb{E} \to \mathbb{R}^m$ is a linear operator onto $\mathbb{R}^m$. Classically we say that the Slater condition holds for this problem whenever there exists X in the interior of C satisfying the system $\mathcal{M}(X) = b$. Since $\mathcal{M}$ is surjective, and hence an open mapping, this amounts to requiring b to lie in the interior of the image $\mathcal{M}(C)$. Thus, recognizing that b lies on the boundary of $\mathcal{M}(C)$ certifies that the Slater condition has failed. On the other hand, much more is true, as the following theorem shows: if a vector v exposes $\mathrm{face}(b, \mathcal{M}(C))$, then $\mathcal{M}^*v$ exposes $\mathrm{face}(F, C)$.

THEOREM 4.1 (Facial reduction). *Consider a linear operator $\mathcal{M} \colon \mathbb{E} \to \mathbb{Y}$, between two Euclidean spaces $\mathbb{E}$ and $\mathbb{Y}$, and let $C \subset \mathbb{E}$ be a closed convex cone. Define the feasible set*

$$F := \{X \in C : \mathcal{M}(X) = b\}$$

for some point $b \in \mathbb{Y}$. Then for any vector v exposing $\mathrm{face}(b, \mathcal{M}(C))$*, the vector $\mathcal{M}^*v$ exposes* $\mathrm{face}(F, C)$.

Proof. For notational convenience, define $N := \mathrm{face}(b, \mathcal{M}(C))$. Then we have

$$N = v^\perp \cap \mathcal{M}(C), \qquad b \in \mathrm{ri}\, N, \qquad v \in N^\triangle = b^\perp \cap (\mathcal{M}(C))^*.$$

Observe now

$$\langle \mathcal{M}^*v, X\rangle = \langle v, \mathcal{M}(X)\rangle \geq 0, \qquad \text{for any } X \in C,$$

and hence the inclusion $\mathcal{M}^*v \in C^*$ holds. Thus $C \cap (\mathcal{M}^*v)^\perp$ is indeed an exposed face of C. Moreover for any $X \in F$, we have $\langle \mathcal{M}^*v, X\rangle = \langle v, b\rangle = 0$, and therefore F is contained in $C \cap (\mathcal{M}^*v)^\perp$. It is standard now to verify the equality

$$\mathcal{M}(C \cap (\mathcal{M}^*v)^\perp) = \mathcal{M}(C) \cap v^\perp = N.$$

Combining this with [30, Theorem 6.6], we deduce

$$\mathrm{ri}(N) = \mathcal{M}(\mathrm{ri}(C \cap (\mathcal{M}^*v)^\perp)).$$

Thus b can be written as $\mathcal{M}(X)$ for some $X \in \mathrm{ri}(C \cap (\mathcal{M}^*v)^\perp)$. We deduce that the intersection $F \cap \mathrm{ri}(C \cap (\mathcal{M}^*v)^\perp)$ is nonempty. Appealing to [25, Proposition 2.2(ii)], we conclude that $C \cap (\mathcal{M}^*v)^\perp$ is the minimal face of C containing F. ☐

In light of this theorem, we may hope to then restrict the system (4.1) to the linear span of $\mathrm{face}(F, C)$, i.e., replace C by $\mathrm{face}(F, C)$. The obvious advantage of this is a reduction in dimension, and a Slater condition now holding in this linear span. (We note that $\text{face}(F, C)$ is equivalently defined as the smallest face of C that contains some feasible $\bar{X} \in \text{ri}(F)$; and if $C = \mathcal{S}^n_+$, then such an $\bar{X}$ is a maximum *rank* PSD matrix in the affine subspace $\mathcal{M}(X) = b$.)

This is essentially the philosophy of the facial reduction algorithm of Borwein and Wolkowicz [5, 6]. The difficulty in implementing this strategy is that $\mathcal{M}(C)$ is usually not a well-understood set: systematically recognizing points on its boundary is hopeless and exposing vectors are out of reach. The authors of [5, 6] propose to rectify this problem by solving a sequence of auxiliary semidefinite programs. Another approach is through Ramana's extended dual [29] or the close variants [23, 28, 35]. All these strategies either increase the size of the problem or require one to solve a (potentially long) sequence of auxiliary problems.

For those problems with highly structured constraints one can hope to do better. The idea is extremely simple: fix a subset $I \subset \{1, \ldots, m\}$ and let $\mathcal{M}_I(X)$ and b_I, respectively, denote restrictions of $\mathcal{M}(X)$ and b to coordinates indexed by I. Consider then the relaxation:

$$F_I := \{X \in C : \mathcal{M}_I(X) = b_I\}.$$

If the index set I is chosen so that the image $\mathcal{M}_I(C)$ is "simple", then we may find the minimal face $\text{face}(F_I, C)$, as discussed above. Intersecting such faces for varying index sets I may yield a drastic dimensional decrease. Moreover, observe that this preprocessing step is entirely parallelizable.

Interpreting this technique in the context of matrix completion problems, we recover the Krislock-Wolkowicz algorithm [17]. Namely note that when $\mathcal{M}$ is simply the projection $\mathcal{P}$ and we set $C = \mathcal{S}^n_+$ or $C = \mathcal{E}^n$, we obtain the PSD and EDM completion problems,

$$F := \{X \in C : \mathcal{P}(X) = a\} = \{X \in C : X_{ij} = a_{ij} \text{ for all } ij \in E\},$$

where $a \in \mathbb{R}^E$ is a partial matrix. It is then natural to consider indices $I \subset E$ describing clique edges in the graph since then the images $\mathcal{P}_I(C)$ are the smaller dimensional PSD and EDM cones, respectively — sets that are well understood. This algorithmic strategy becomes increasingly effective when the rank (for the PSD case) or the embedding dimension (for the EDM case) of the specified principal minors are all small. Moreover, we show that under a chordality assumption, the minimal face of C containing the feasible region is guaranteed to be discovered if all the cliques were to be considered; see Theorems 4.5 and 4.9. This, in part, explains why the EDM completion algorithm of [17] works so well. Understanding the geometry of $\mathcal{P}_I(C)$ for a wider class of index sets I would yield an even better preprocessing strategy. We defer to [17] for extensive numerical results and implementation issues showing that the discussed algorithmic idea is extremely effective for EDM completions.

In what follows, by the term "clique χ in G" we will mean a collection of k pairwise connected vertices of G. The symbol $|\chi|$ will indicate the cardinality of χ (i.e. the number of vertices) while $E(\chi)$ will denote the edge set in the subgraph induced by G on χ. For a partial matrix $a \in \mathbb{R}^E$, the symbol a_χ will mean the restriction of a to $E(\chi)$, whereas $\mathcal{P}_\chi$ will be the projection of $\mathcal{S}^n$ onto $E(\chi)$. The symbol $\mathcal{S}^\chi$ will indicate the set of $|\chi| \times |\chi|$ symmetric matrices whose rows and columns are indexed by χ. Similar notation will be reserved for $\mathcal{S}^\chi_+$. If χ is contained in L, then we may equivalently think of a_χ as a vector lying in $\mathbb{R}^{E(\chi)}$ or as a matrix lying in $\mathcal{S}^\chi$. Thus the adjoint $\mathcal{P}^*_\chi$ assigns to a partial matrix $a_\chi \in \mathcal{S}^\chi$ an $n \times n$ matrix whose principal submatrix indexed by χ coincides with a_χ and whose all other entries are zero.

THEOREM 4.2 (Clique facial reduction for PSD completions). *Let $\chi \subseteq L$ be any k-clique in the graph G. Let $a \in \mathbb{R}^E$ be a partial PSD matrix and define*

$$F_\chi := \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij} \text{ for all } ij \in E(\chi)\}$$

Then for any matrix v_χ exposing $\operatorname{face}(a_\chi, \mathcal{S}^\chi_+)$, *the matrix*

$$\mathcal{P}^*_\chi v_\chi \quad \textit{exposes} \quad \operatorname{face}(F_\chi, \mathcal{S}^n_+).$$

Proof. Simply apply Theorem 4.1 with $C = \mathcal{S}^n_+$, $\mathcal{M} = \mathcal{P}_\chi$, and $b = a_\chi$. □

Theorem 4.2 is transparent and easy. Consequently it is natural to ask whether the minimal face of $\mathcal{S}^n_+$ containing the feasible region of a PSD completion problem can be found using solely faces arising from cliques, that is those faces described in Theorem 4.2. The answer is no in general: the following example exhibits a PSD completion problem that fails the Slater condition but for which all specified principal submatrices are definite, and hence all faces arising from Theorem 4.2 are trivial.

EXAMPLE 4.3 (Slater condition & nonchordal graphs).
Let $G = (V, E)$ be a cycle on four vertices with each vertex attached to a loop, that is $V = \{1, 2, 3, 4\}$ and $E = \{12, 23, 34, 14\} \cup \{11, 22, 33, 44\}$. Define the following PSD completion problems $C(\epsilon)$, parametrized by $\epsilon \geq 0$:

$$C(\epsilon): \qquad \begin{bmatrix} 1+\epsilon & 1 & ? & -1 \\ 1 & 1+\epsilon & 1 & ? \\ ? & 1 & 1+\epsilon & 1 \\ -1 & ? & 1 & 1+\epsilon \end{bmatrix}.$$

Let $a(\epsilon) \in \mathbb{R}^E$ denote the corresponding partial matrices. According to [11, Lemma 6] there is a unique positive semidefinite matrix A satisfying $A_{ij} = 1, \forall |i-j| \leq 1$, namely the matrix of all 1's. Hence the PSD completion problem $C(0)$ is infeasible, that is $a(0)$ lies outside of $\mathcal{P}(\mathcal{S}^4_+)$. On the other hand, for all sufficiently large ϵ, the partial matrices $a(\epsilon)$ do lie in $\mathcal{P}(\mathcal{S}^4_+)$ due to the diagonal dominance. Taking into account that $\mathcal{P}(\mathcal{S}^4_+)$ is closed (by Theorem 3.1), we deduce that there exists $\hat{\epsilon} > 0$, so that $a(\hat{\epsilon})$ lies on the boundary of $\mathcal{P}(\mathcal{S}^4_+)$, that is the Slater condition fails for the completion problem $C(\hat{\epsilon})$. On the other hand for all $\epsilon > 0$, the partial matrices $a(\epsilon)$ are clearly positive definite, and hence $a(\hat{\epsilon})$ is a partial PD matrix. In fact, we can prove $\hat{\epsilon} = \sqrt{2} - 1$, by solving the semi-definite program:

$$\text{(4.2)} \qquad \begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \begin{bmatrix} 1+\epsilon & 1 & \alpha & -1 \\ 1 & 1+\epsilon & 1 & \beta \\ \alpha & 1 & 1+\epsilon & 1 \\ -1 & \beta & 1 & 1+\epsilon \end{bmatrix} \succeq 0 \end{array}$$

Doing so, we deduce that $\hat{\epsilon} = \sqrt{2} - 1, \hat{\alpha} = \hat{\beta} = 0$ is optimal. Formally, we can verify this by finding the dual of (4.2) and checking feasibility and complementary slackness for the primal-dual optimal pair $\widehat{X}$ and $\widehat{Z}$

$$\widehat{X} = \begin{bmatrix} \sqrt{2} & 1 & 0 & -1 \\ 1 & \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} & 1 \\ -1 & 0 & 1 & \sqrt{2} \end{bmatrix}, \qquad \widehat{Z} = \frac{1}{4}\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}.$$

Despite this pathological example, we now show that at least for chordal graphs, the minimal face of the PSD completion problem can be found solely from faces corresponding to cliques in the graph. We begin with the following simple lemma.

LEMMA 4.4 (Maximal rank completions). *Suppose without loss of generality* $L = \{1, \ldots, r\}$ *and let* $G_L := (L, E_L)$ *be the graph induced on* L *by* G*. Let* $a \in \mathbb{R}^E$ *be a partial matrix and* a_{E_L} *the restriction of* a *to* E_L*. Suppose that* $X_L \in \mathcal{S}^r_+$ *is a maximum rank PSD completion of* a_{E_L}*, and*

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

is an arbitrary PSD completion of a*. Then*

$$X_\mu := \begin{bmatrix} X_L & B \\ B^T & C + \mu I \end{bmatrix}$$

is a maximal rank PSD completion of $a \in \mathbb{R}^E$ *for all sufficiently large* μ*.*

Proof. We *construct* the maximal rank PSD completion from the arbitrary PSD completion X by moving from A to X_L and from C to $C + \mu I$ while staying in the same minimal face for the completions. To this end, define the sets

$$\begin{aligned} F &= \left\{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij}, \text{ for all } ij \in E\right\}, \\ F_L &= \left\{X \in \mathcal{S}^r_+ : X_{ij} = a_{ij}, \text{ for all } ij \in E_L\right\}, \\ \widehat{F} &= \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij}, \text{ for all } ij \in E_L\}. \end{aligned}$$

Then X_L is a maximum rank PSD matrix in F_L. Observe that the rank of any PSD matrix $\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}$ is bounded by $\operatorname{rank} P + \operatorname{rank} R$. Consequently the rank of any PSD matrix in F and also in $\widehat{F}$ is bounded by $\operatorname{rank} X_L + (n - r)$, and the matrix

$$\bar{X} = \begin{bmatrix} X_L & 0 \\ 0 & I \end{bmatrix}$$

has maximal rank in $\widehat{F}$, i.e.,

$$\bar{X} \in \operatorname{ri}(\widehat{F}). \tag{4.3}$$

Let U be a matrix of eigenvectors of X_L, with eigenvectors corresponding to 0 eigenvalues coming first. Then

$$U^T X_L U = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix},$$

where $0 \prec \Lambda \in \mathcal{S}^k_+$ is a diagonal matrix with all positive diagonal elements.

Define

$$Q = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}.$$

Let X be as in the statement of the lemma; then clearly $X \in \widehat{F}$ and we deduce using (4.3) that

$$\bar{X} \pm \epsilon(\bar{X} - X) \in \mathcal{S}^n_+ \Leftrightarrow Q^T \bar{X} Q \pm \epsilon Q^T (\bar{X} - X) Q \in \mathcal{S}^n_+, \tag{4.4}$$

for some small $\epsilon > 0$. We now have

$$\begin{aligned} Q^T \bar{X} Q &= \begin{bmatrix} U^T X_L U & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & I \end{bmatrix}, \\ Q^T X Q &= \begin{bmatrix} U^T A U & U^T B \\ B^T U & C \end{bmatrix} &= \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}^T & V_{22} & V_{23} \\ V_{13}^T & V_{23}^T & V_{33} \end{bmatrix}, \end{aligned}$$

where $V_{11} \in \mathcal{S}^{r-k}$, $V_{22} \in \mathcal{S}^k$, $V_{33} \in \mathcal{S}^{n-r}$. From (4.4) we deduce $V_{11} = 0$, $V_{12} = 0$, $V_{13} = 0$. Therefore

$$Q^T X_\mu Q = \begin{bmatrix} U^T X_L U & U^T B \\ B^T U & \mu I + C \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda & V_{23} \\ 0 & V_{23}^T & \mu I + C \end{bmatrix}.$$

By the Schur complement condition for positive semidefiniteness we have that for sufficiently large μ the matrix X_μ is PSD, and $\operatorname{rank} X_\mu = \operatorname{rank} X_L + (n - r)$; hence it is a maximal rank PSD matrix in F. ☐

Theorem 4.5 (Finding the minimal face on chordal graphs). *Suppose that the graph induced by G on L is chordal. Consider a partial PSD matrix $a \in \mathbb{R}^E$ and the region*

$$F = \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij} \text{ for all } ij \in E\}.$$

Then the equality

$$\operatorname{face}(F, \mathcal{S}^n_+) = \bigcap_{\chi \in \Theta} \operatorname{face}(F_\chi, \mathcal{S}^n_+) \qquad \textit{holds},$$

where Θ denotes the set of all cliques in the restriction of G to L, and for each $\chi \in \Theta$ we define the relaxation

$$F_\chi := \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij} \text{ for all } ij \in E(\chi)\}.$$

Proof. For brevity, set

$$H = \bigcap_{\chi \in \Theta} \operatorname{face}(F_\chi, \mathcal{S}^n_+).$$

We first prove the theorem under the assumption that L is disconnected from L^c. To this end, for each clique $\chi \in \Theta$, let $v_\chi \in \mathcal{S}^\chi_+$ denote the exposing vector of face(a_χ, $\mathcal{S}^\chi_+$). Then by Theorem 4.2, we have

$$\operatorname{face}(F_\chi, \mathcal{S}^n_+) = \mathcal{S}^n_+ \cap (\mathcal{P}^*_\chi v_\chi)^\perp.$$

It is straightforward to see that $\mathcal{P}_\chi^* v_\chi$ is simply the $n \times n$ matrix whose principal submatrix indexed by χ coincides with v_χ and whose all other entries are zero. Letting $Y[\chi]$ denote the principal submatrix indexed by χ of any matrix $Y \in \mathcal{S}_+^n$, we successively deduce

$$\begin{aligned}\mathcal{P}(H) &= \mathcal{P}\Big(\{Y \succeq 0 : Y[\chi] \in v_\chi^\perp \quad \forall \chi \in \Theta\}\Big)\\ &= \mathcal{P}(\mathcal{S}_+^n) \cap \{b \in \mathbb{R}^E : b_\chi \in v_\chi^\perp \quad \forall \chi \in \Theta\}.\end{aligned}$$

On the other hand, since the restriction of G to L is chordal and L is disconnected from L^c, Corollary 3.2 implies that G is PSD completable. Hence we have the representation $\mathcal{P}(\mathcal{S}_+^n) = \{b \in \mathbb{R}^E : b_\chi \in \mathcal{S}_+^\chi \quad \forall \chi \in \Theta\}$. Combining this with the equations above, we obtain

$$\begin{aligned}\mathcal{P}(H) &= \{b \in \mathbb{R}^E : b_\chi \in \mathcal{S}_+^\chi \cap v_\chi^\perp \quad \forall \chi \in \Theta\}\\ &= \{b \in \mathbb{R}^E : b_\chi \in \operatorname{face}(a_\chi, \mathcal{S}_+^\chi) \quad \forall \chi \in \Theta\}\\ &= \bigcap_{\chi \in \Theta} \{b \in \mathbb{R}^E : b_\chi \in \operatorname{face}(a_\chi, \mathcal{S}_+^\chi)\},\end{aligned}$$

Clearly a lies in the relative interior of each set $\{b \in \mathbb{R}^E : b_\chi \in \operatorname{face}(a_\chi, \mathcal{S}_+^\chi)\}$. Using [30, Theorems 6.5,6.6], we deduce

$$a \in \operatorname{ri} \mathcal{P}(H) = \mathcal{P}(\operatorname{ri} H).$$

Thus the intersection $F \cap \operatorname{ri} H$ is nonempty. Taking into account that F is contained in H, and appealing to [25, Proposition 2.2(ii)], we conclude that H is the minimal face of $\mathcal{S}_+^n$ containing F, as claimed.

We now prove the theorem in full generality, that is when there may exist an edge joining L and L^c. To this end, let $\widehat{G}_L = (V, E_L)$ be the graph obtained from G by deleting all edges adjacent to L^c. Clearly, L and L^c are disconnected in $\widehat{G}_L$. Applying the special case of the theorem that we have just proved, we deduce that in terms of the set

$$\widehat{F} \quad = \quad \{X \in \mathcal{S}_+^n : X_{ij} = a_{ij} \text{ for all } ij \in E_L\},$$

we have

$$\operatorname{face}(\widehat{F}, \mathcal{S}_+^n) = H.$$

The X_μ matrix of Lemma 4.4 is a maximum rank PSD matrix in F, and also in $\widehat{F}$. Since $F \subseteq \widehat{F}$, we deduce $\operatorname{face}(F, \mathcal{S}_+^n) = \operatorname{face}(\widehat{F}, \mathcal{S}_+^n)$, and this completes the proof. $\square$

Example 4.6 (Finding the minimal face on chordal graphs). Let Ω consist of all matrices $X \in \mathcal{S}_+^4$ solving the PSD completion problem

$$\begin{bmatrix} 1 & 1 & ? & ? \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & -1 \\ ? & ? & -1 & 2 \end{bmatrix}.$$

There are three nontrivial cliques in the graph. Observe that the minimal face of $\mathcal{S}_+^2$ containing the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is exposed by

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Classically, an intersection of two faces is exposed by the sum of the exposing vectors. Using Theorem 4.5, we deduce that the minimal face of $\mathcal{S}^4_+$ containing Ω is the one exposed by the sum

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Diagonalizing this matrix, we obtain

$$\text{face}(\Omega, \mathcal{S}^4_+) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} \mathcal{S}^2_+ \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}^T.$$

We now turn to an analogous development for the EDM completion problem. To this end, recall from (2.5) that the the mapping $\mathcal{K} \colon \mathcal{S}^n \to \mathcal{S}^n$ restricts to an isomorphism $\mathcal{K} \colon \mathcal{S}_c \to \mathcal{S}_H$ carrying $\mathcal{S}_c \cap \mathcal{S}^n_+$ onto $\mathcal{E}^n$. Moreover, it turns out that the Moore-Penrose pseudoinverse $\mathcal{K}^\dagger$ restricts to the inverse of this isomorphism $\mathcal{K}^\dagger \colon \mathcal{S}_H \to \mathcal{S}_c$. As a result, it is convenient to study the faces of $\mathcal{E}^n$ using the faces of $\mathcal{S}_c \cap \mathcal{S}^n_+$. This is elucidated by the following standard result.

Lemma 4.7 (Faces under isomorphism). *Consider a linear isomorphism $\mathcal{M} \colon \mathbb{E} \to \mathbb{Y}$ between linear spaces $\mathbb{E}$ and $\mathbb{Y}$, and let $C \subset \mathbb{E}$ be a closed convex cone. Then the following are true*

1. $F \trianglelefteq C \iff \mathcal{M}F \trianglelefteq \mathcal{M}C$.
2. $(\mathcal{M}C)^* = (\mathcal{M}^{-1})^* C^*$.
3. *For any face $F \trianglelefteq C$, we have $(\mathcal{M}F)^\triangle = (\mathcal{M}^{-1})^* F^\triangle$.*

In turn, it is easy to see that $\mathcal{S}_c \cap \mathcal{S}^n_+$ is a face of $\mathcal{S}^n_+$ isomorphic to $\mathcal{S}^{n-1}_+$. More specifically for any $n \times n$ orthogonal matrix $\begin{bmatrix} \frac{1}{\sqrt{n}}e & U \end{bmatrix}$, we have the representation

$$\mathcal{S}_c \cap \mathcal{S}^n_+ = U \mathcal{S}^{n-1}_+ U$$

Consequently, with respect to the ambient space $\mathcal{S}_c$, the cone $\mathcal{S}_c \cap \mathcal{S}^n_+$ is self-dual and for any face $F \trianglelefteq \mathcal{S}^{n-1}_+$ we have

$$UFU^T \trianglelefteq \mathcal{S}_c \cap \mathcal{S}^n_+ \qquad \text{and} \qquad (UFU^T)^\triangle = UF^\triangle U^T.$$

As a result of these observations, we make the following important convention: the *ambient spaces* of $\mathcal{S}_c \cap \mathcal{S}^n_+$ and of $\mathcal{E}^n$ will always be taken as $\mathcal{S}_c$ and $\mathcal{S}_H$, respectively. Thus the facial conjugacy operations of these two cones will always be taken with respect to these ambient spaces and *not* with respect to the entire $\mathcal{S}^n$.

Given a clique χ in G, we let $\mathcal{E}^\chi$ denote the set of $|\chi| \times |\chi|$ Euclidean distance matrices indexed by χ. In what follows, given a partial matrix $a \in \mathbb{R}^E$, the restriction

a_χ can then be though of either as a vector in $\mathbb{R}^{E(\chi)}$ or as a hollow matrix in $\mathcal{S}^\chi$. We will also use the symbol $\mathcal{K}_\chi \colon \mathcal{S}^\chi \to \mathcal{S}^\chi$ to indicate the mapping $\mathcal{K}$ acting on $\mathcal{S}^\chi$.

THEOREM 4.8 (Clique facial reduction for EDM completions). *Let χ be any k-clique in the graph G. Let $a \in \mathbb{R}^E$ be a partial Euclidean distance matrix and define*

$$F_\chi \quad := \quad \{X \in \mathcal{S}^n_+ \cap \mathcal{S}_c : [\mathcal{K}(X)]_{ij} = a_{ij} \text{ for all } ij \in E(\chi)\}$$

Then for any matrix v_χ exposing $\operatorname{face}\left(\mathcal{K}^\dagger(a_\chi), \mathcal{S}^\chi_+ \cap \mathcal{S}_c\right)$, *the matrix*

$$\mathcal{P}^*_\chi v_\chi \qquad \text{exposes} \qquad \operatorname{face}(F, \mathcal{S}^n_+ \cap \mathcal{S}_c).$$

Proof. The proof proceeds by applying Theorem 4.1 with

$$C := \mathcal{S}^n_+ \cap \mathcal{S}_c, \quad \mathcal{M} := P_\chi \circ \mathcal{K}, \quad b := a_\chi.$$

To this end, first observe $\mathcal{M}(C) = (P_\chi \circ \mathcal{K})(\mathcal{S}^n_+ \cap \mathcal{S}_c) = \mathcal{E}^\chi$. By Lemma 4.7, the matrix $\mathcal{K}^{\dagger *}_\chi(v_\chi)$ exposes $\operatorname{face}(a_\chi, \mathcal{E}^\chi)$. Thus the minimal face of $\mathcal{S}^n_+ \cap \mathcal{S}_c$ containing F is the one exposed by the matrix

$$(P_\chi \circ \mathcal{K})^*(\mathcal{K}^{\dagger *}_\chi(v_\chi)) = \mathcal{K}^* P^*_\chi \mathcal{K}^{\dagger *}_\chi(v_\chi) = P^*_\chi \mathcal{K}^*_\chi \mathcal{K}^{\dagger *}_\chi(v_\chi) = P^*_\chi v_\chi.$$

The result follows. □

THEOREM 4.9 (Clique facial reduction for EDM is sufficient). *Suppose that G is chordal, and consider a partial Euclidean distance matrix $a \in \mathbb{R}^E$ and the region*

$$F := \{X \in \mathcal{S}_c \cap \mathcal{S}^n_+ : [\mathcal{K}(X)]_{ij} = a_{ij} \text{ for all } ij \in E\}.$$

Let Θ denote the set of all cliques in G, and for each $\chi \in \Theta$ define

$$F_\chi := \{X \in \mathcal{S}_c \cap \mathcal{S}^n_+ : [\mathcal{K}(X)]_{ij} = a_{ij} \text{ for all } ij \in E(\chi)\}.$$

Then the equality

$$\operatorname{face}(F, \mathcal{S}_c \cap \mathcal{S}^n_+) = \bigcap_{\chi \in \Theta} \operatorname{face}(F_\chi, \mathcal{S}_c \cap \mathcal{S}^n_+) \qquad \text{holds.}$$

Proof. The proof follows entirely along the same lines as the first part of the proof of Theorem 4.5. We omit the details for the sake of brevity. □

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