Some Necessary and Some Sufficient Trace Inequalities for Euclidean Distance Matrices

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Abstract

In this paper, we use known bounds on the smallest eigenvalue of a symmetric matrix and Schoenberg's Theorem to provide both necessary as well as sufficient trace inequalities that guarantee a matrix D is a Euclidean distance matrix, ***EDM***. We also provide necessary and sufficient trace inequalities that guarantee a matrix D is an ***EDM*** generated by a regular figure.

1 Introduction

A real, $n \times n$, symmetric matrix $D = (d_{ij})$ is called a *predistance matrix* if it is nonnegative elementwise with zero diagonal. If, in addition, there exist points $p^1, \dots, p^n$ in some Euclidean space $\Re^r$ such that

$$d_{ij} = ||p^i - p^j||^2 \text{ for all } i, j = 1, \dots, n,$$

then D is called a *Euclidean distance matrix*, ***EDM***, and the dimension of the smallest space containing the points $p^1, \dots, p^n$ is called the *embedding dimension* of D. A well-known theorem of Schoenberg [7] states that a predistance matrix D is ***EDM*** if and only if D is negative semidefinite on the subspace $M := e^\perp = \{x \in \Re^n : e^T x = 0\}$, where e is the vector of all ones. This provides a relationship between the convex cone of ***EDMs*** and the convex cone of positive semidefinite matrices.

It is well known that a real symmetric $n \times n$ matrix X is positive semidefinite if and only if all the eigenvalues are nonnegative. Therefore, bounds on the smallest nonzero eigenvalue can be

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used to provide both necessary as well as sufficient conditions for positive semidefiniteness. In this paper we use known relationships between ***EDMs*** and positive semidefinite matrices and known eigenvalue bounds, to get necessary as well as sufficient inequalities that guarantee a matrix is ***EDM*** .

In this paper, we let e denote the vector of all ones of appropriate dimension; $\mathcal{S}^n$ denotes the space of real, symmetric, $n \times n$ matrices; $D \in \mathcal{S}^n$ denotes a *nonzero* predistance matrix; and for $X \in \mathcal{S}^n$, we use $X \succeq 0$ to denote that X is positive semidefinite.

1.1 Known Eigenvalue Bounds

Bounds for eigenvalues of matrices are well known in the literature. A survey of bounds is given in e.g. [5, 4]. The following upper and lower bounds on the smallest nonzero eigenvalue of a symmetric matrix follow from the results in [9]. We ouline a proof for completeness.

Theorem 1.1 *[9] Suppose that A is an $n \times n$, real symmetric matrix of rank at most r, $r \geq 2$. Let*

$$m := \frac{\operatorname{trace} A}{r}, \qquad s^2 := \frac{\operatorname{trace} A^2}{r} - \left(\frac{\operatorname{trace} A}{r}\right)^2. \tag{1.1}$$

Then the smallest nonzero eigenvalue of A, denoted $\lambda_1(A)$, satisfies

$$m - \sqrt{r-1}\; s \leq \lambda_1(A) \leq m - \frac{1}{\sqrt{r-1}}\; s. \tag{1.2}$$

Proof. We outline a proof for the lower bound for λ_1. The proof of the upper bound is similar but more involved. Let A be a symmetric matrix of rank r and let $\lambda = (\lambda_i)$ be the vector of the nonzero eigenvalues of A. We know that $e^T\lambda = \sum_{j=1}^r \lambda_j = \operatorname{trace} A$ and $\sum_{j=1}^r \lambda_j^2 = \operatorname{trace} A^2$. Moreover, the Cauchy-Schwartz inequality implies that $r\left(\frac{\operatorname{trace} A}{r}\right)^2 = \frac{1}{r}(e^T\lambda)^2 \leq \frac{1}{r}\|e\|^2\|\lambda\|^2 = \operatorname{trace} A^2$, with equality if and only if all the eigenvalues are equal to $\frac{1}{r}\operatorname{trace} A$, in which case the lower bound is trivially true. Therefore, we can assume that strict inequality holds, $r\left(\frac{\operatorname{trace} A}{r}\right)^2 < \operatorname{trace} A^2$. Consider the convex program

$$\begin{array}{rl} \min & \lambda_1 \\ \text{subject to} & \sum_{j=1}^r \lambda_j = \operatorname{trace} A \\ & \sum_{j=1}^r \lambda_j^2 \leq \operatorname{trace} A^2. \end{array}$$

By the strict inequality assumption, the generalized Slater constraint qualification holds for this convex program. Therefore, we can apply the (necessary and sufficient) optimality conditions (Karush-Kuhn-Tucker conditions), with Lagrange multipliers α, β:

$$\begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} + \alpha e + 2\beta\lambda = 0, \quad \beta\,(\sum_{j=1}^r \lambda_j^2 - \operatorname{trace} B^2) = 0, \quad \beta \geq 0.$$

The optimality conditions are satisfied by $\beta > 0$ and $m - \sqrt{r-1}\; s = \lambda_1 < \lambda_2 = \cdots = \lambda_r = m + \frac{1}{\sqrt{r-1}}\; s$. ■

Note that the bounds get tighter if r can be chosen smaller.

2 Some Necessary and Some Sufficent Trace Inequalities for EDMs

As stated above, it is well known [7] that a predistance matrix D is $\boldsymbol{EDM}$ if and only if D is negative semidefinite on M. Let V be the $n \times n-1$ matrix whose columns form an orthonormal basis for M. Then it immediately follows that a predistance matrix D is $\boldsymbol{EDM}$ if and only if $-V^TDV$ is positive semidefinite. Note also that $J := VV^T = I - \frac{1}{n}ee^T$ is the orthogonal projection onto M. Now, by applying Theorem 1.1 to the matrix $X = -V^TDV$ we obtain the following theorem.

Theorem 2.1 *Let $D \neq 0$ be an $n \times n$, $n \geq 3$, predistance matrix. Then*

1. *The following is a sufficient condition for D to be an EDM*

$$\frac{2}{n}e^TD^2e - \frac{(n-3)}{n^2(n-2)}(e^TDe)^2 \geq \operatorname{trace} D^2. \tag{2.3}$$

2. *If D is an EDM then D satisfies*

$$\frac{2}{n}\, e^TD^2e \geq \operatorname{trace} D^2. \tag{2.4}$$

Proof. It is clear that D is $\boldsymbol{EDM}$ if and only if the smallest nonzero eigenvalue of the $(n-1)\times(n-1)$ matrix $X = -V^TDV$ is nonnegative. But rank $X \leq n-1$. Let

$$m = \frac{\operatorname{trace} X}{n-1} = -\frac{1}{n-1}\operatorname{trace} DVV^T = -\frac{1}{n-1}\operatorname{trace} D(I - \frac{1}{n}ee^T) = \frac{e^TDe}{n(n-1)}$$

and

$$\begin{aligned} s^2 &= \frac{1}{n-1}\operatorname{trace} X^2 - m^2 \\ &= \frac{1}{n-1}\operatorname{trace} D^2 - \frac{2}{n(n-1)}e^TD^2e + \frac{(n-2)}{(n-1)^2n^2}(e^TDe)^2. \end{aligned}$$

Then, Theorem 1.1 and the fact that $m \geq 0$ imply that the smallest nonzero eigenvalue of X is nonnegative if $m^2 \geq (n-2)s^2$. Note that

$$(n-1)(m^2 - (n-2)s^2) = -(n-2)\operatorname{trace} D^2 + \frac{2(n-2)}{n}e^TD^2e - \frac{(n-3)}{n^2}(e^TDe)^2.$$

Therefore, Condition 1 holds.

The second condition follows from the upper bound on the smallest eigenvalue, i.e. if $m \geq 0$ and $m^2 - s^2/(n-2) < 0$, then D is not $\boldsymbol{EDM}$. Therefore, We get the required necessary condition in (2.4) since

$$(n-1)(m^2 - s^2/(n-2)) = -\frac{1}{n-2}\operatorname{trace} D^2 + \frac{2}{n(n-2)}\, e^TD^2e.$$

∎

The following is an immediate corollary of Theorem 2.1

Corollary 2.1 *Let D be 3×3 predistance matrix. Then D is **EDM** if and only if*

$$\frac{2}{3}\, e^T D^2 e \geq \operatorname{trace} D^2. \tag{2.5}$$

The results in Theorem 2.1 can be strengthened by weakening the sufficient condition (2.3), if the rank of D is known. Note that the necessary condition in Theorem 2.1 is independent of rank of D. We get the following result.

Theorem 2.2 *Let $D \neq 0$ be an $n \times n$, $n \geq 3$, predistance matrix and assume that rank $D = k \leq n-1$. Then the following is a sufficient condition for D to be an EDM*

$$\frac{2}{n} e^T D^2 e - \frac{(k-2)}{n^2(k-1)} (e^T De)^2 \geq \operatorname{trace} D^2. \tag{2.6}$$

Proof. If rank $D = k$ then rank $X = -V^T DV \leq k$. Note that $k \geq 2$ since $D \neq 0$ and $\operatorname{trace} D = 0$. Therefore, in this case

$$m = \frac{\operatorname{trace} X}{k} = -\frac{1}{k} \operatorname{trace} DVV^T = -\frac{1}{k} \operatorname{trace} D(I - \frac{1}{n} ee^T) = \frac{e^T De}{kn}.$$

and

$$\begin{aligned} s^2 &= \frac{1}{k} \operatorname{trace} X^2 - m^2 \\ &= \frac{1}{k} \operatorname{trace} D^2 - \frac{2}{kn} e^T D^2 e + \frac{(k-1)}{k^2 n^2} (e^T De)^2. \end{aligned}$$

The result now follows from a similar argument to that in the proof of Theorem 2.1. ■

A recent, different sufficient condition for a predistance matrix to be an ***EDM*** is derived by Bénasséni [2]. This is in the form of a variance inequality equivalent to $\frac{(e^T De)^2}{n^2 - n - 1} > \operatorname{trace} D^2$. The condition is derived using a continuity argument on the ***EDM*** corresponding to the standard simplex.

The following is an immediate corollary of Theorem 2.2

Corollary 2.2 *Let D be an $n \times n$ predistance matrix of rank 2. Then D is an EDM if and only if*

$$\frac{2}{n}\, e^T D^2 e \geq \operatorname{trace} D^2. \tag{2.7}$$

Theorem 2.3 *Let $D \neq 0$ be an $n \times n$ EDM. Then D satisfies inequality 2.4 in Theorem 2.1 as an equality if and only if the embedding dimension of D is 1.*

Proof. Let $D \neq 0$ be an $n \times n$ EDM and let $B = -\frac{1}{2} JDJ$, where $J = VV^T$ is the orthogonal projection on the subspace $M = e^\perp$. Then $B \succeq 0$ and the embedding dimension of D is well known to be equal to the rank of B. Furthermore, D can be written in terms of B as

$$D = \operatorname{diag} B e^T + e(\operatorname{diag} B)^T - 2B, \tag{2.8}$$

where $\operatorname{diag} B$ denotes the vector consisting of the diagonal elements of B.

Using (2.8), it is easy to show that $\frac{2}{n} e^T D^2 e \geq \operatorname{trace} D^2$ is equivalent to $(\operatorname{trace} B)^2 \geq \operatorname{trace} B^2$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of B. Therefore, D satisfies inequality 2.4 in Theorem 2.1 as an equality if and only if $(\operatorname{trace} B)^2 = \operatorname{trace} B^2$ if and only if $(\sum_{i=1}^n \lambda_i)^2 = \sum_{i=1}^n \lambda_i^2$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$ and $\lambda_n > 0$ since $B \succeq 0$. ■

3 Spherical EDMs

An ***EDM*** D is said to be a *spherical* ***EDM*** if the points that generate D lie on a hypersphere. If, in addition, this hypersphere is centered at the origin, then, following [3], we say that D is generated by a regular figure.[1] The following result is known.

Lemma 3.1 ([6]) *Let D be a spherical EDM and let the points that generate D lie on a hypersphere of radius R. Then $\lambda^* = 2R^2$ is the minimum value of λ such that $\lambda ee^T - D \succeq 0$.*

Proof. (For completeness we include a proof of this lemma based on a recent characetrization of the rangespace and the nullspace of spherical EDMs [1].) Let D be a spherical EDM of embedding dimension r and let $B = -\frac{1}{2}JDJ$. Let B be factorized as $B = PP^T$, where P is $n \times r$ of rank r. Furthermore, let Z be a Gale matrix corresponding to D. Z is defined to satisfy

$$\text{Range } Z := \text{ Nullspace } \begin{bmatrix} P^T \\ e^T \end{bmatrix}, \quad Z \text{ full rank.}$$

Then it was shown in [1] that Range $D =$ Range $[\ P\ e\ ]$ and Nullspace $D =$ Range Z.

Define the nonsingular matrix $Q = [\ P\ e\ Z\ ]$. Then $\lambda ee^T - D \succeq 0$ if and only if $Q^T(\lambda ee^T - D)Q \succeq 0$. But

$$Q^T(\lambda ee^T - D)Q = \begin{pmatrix} -P^TDP & -P^TDe & 0 \\ -e^TDP & \lambda n^2 - e^TDe & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $\lambda ee^T - D \succeq 0$ if and only if $\begin{pmatrix} 2(P^TP)^2 & -P^TDe \\ -e^TDP & \lambda n^2 - e^TDe \end{pmatrix} \succeq 0$, if and only if $\lambda n^2 - e^TDe - \frac{1}{2}e^TDP(P^TP)^{-2}P^TDe \geq 0$. This implies that

$$\begin{aligned} \lambda^* &= \frac{e^TDe}{n^2} - \frac{e^TDP(P^TP)^{-2}P^TDe}{2n^2} \\ &= \frac{e^TDe}{n^2} - \frac{e^TDB^\dagger De}{2n^2}, \end{aligned} \tag{3.9}$$

where $B^\dagger$ denotes the Moore-Penrose inverse of B. But the center of the hypersphere containing the points that generate D is given by $a = (P^TP)^{-1}P^TDe/2n$. Hence, $\lambda^* = e^TDe/n^2 + 2a^Ta = 2R^2$.

Corollary 3.1 *Let D be an $n \times n$ predistance matrix. Then D is a spherical EDM if and only if $\lambda^* ee^T - D \succeq 0$, where λ^* is given in (3.9).*

Corollary 3.2 ([3]) *Let D be an $n \times n$ predistance matrix. Then D is a spherical EDM generated by a regular figure if and only if $\lambda^* ee^T - D \succeq 0$, where*

$$\lambda^* = \frac{e^TDe}{n^2}. \tag{3.10}$$

[1]Some authors refer to these as ***EDMs*** of strength one, [6]

4 Sufficent and Necessary Trace Inequalities for EDMs Generated by Regular Figures

Since λ^* given by (3.10) is easy to compute , in the section we present sufficient and necessary trace inequalities for a predistance matrix to be an EDM generated by a regular figure.

Theorem 4.1 *Let D be an $n \times n, n \geq 3$ predistance matrix. Then*

1. *The following is a sufficient condition for D to be an EDM generated by a regular figure.*

$$\frac{n-1}{n-2}\,\frac{(e^T De)^2}{n^2} \geq \operatorname{trace} D^2 \tag{4.11}$$

2. *If D is an EDM generated by a regular figure then*

$$2\,\frac{(e^T De)^2}{n^2} \geq \operatorname{trace} D^2. \tag{4.12}$$

Proof. Let $A = \lambda^* ee^T - D$ then rank $A \leq n-1$. Let

$$m = \frac{\operatorname{trace} A}{n-1} = \frac{n}{n-1}\lambda^*.$$

and

$$\begin{aligned} s^2 &= \frac{1}{n-1}\operatorname{trace} A^2 - m^2 \\ &= \frac{1}{n-1}\operatorname{trace} D^2 - \frac{2\lambda^*}{n-1} e^T De + \frac{n^2\,(n-2)}{(n-1)^2}\lambda^{*2}, \\ &= \frac{1}{n-1}\operatorname{trace} D^2 - \frac{1}{n\,(n-1)^2}(e^T De)^2. \end{aligned}$$

Then, Theorem 1.1 implies that the smallest eigenvalue of A is nonnegative if $m^2 \geq (n-2)s^2$. But

$$(n-1)(m^2 - (n-2)\,s^2) = -(n-2)\operatorname{trace} D^2 + \frac{(n-1)}{n^2}(e^T De)^2.$$

Therefore, Condition 1 holds.

Condition 2 follows from the upper bound on the smallest eigenvalue of A, i.e. if D is an EDM generated by a regular figure then $m^2 - s^2/(n-2) \geq 0$. We get

$$(n-1)(m^2 - s^2/(n-2)) = -\frac{1}{(n-2)}\operatorname{trace} D^2 + \frac{2}{n^2\,(n-2)}(e^T De)^2.$$

■

As was the case in Theorem 2.2, the sufficient condition in Theorem 4.1 can be weakened if the rank of D is known. Hence, we have the following theorem

Theorem 4.2 *Let $D \neq 0$ be an $n \times n, n \geq 3$ predistance matrix of rank $k \leq n-1$. Then the following is a sufficient condition for D to be an EDM generated by a regular figure.*

$$\frac{k}{k-1}\,\frac{(e^T De)^2}{n^2} \geq \operatorname{trace} D^2 \tag{4.13}$$

Proof. let D be an EDM generated by a regular figure of rank $k \leq n-1$, $k \geq 2$ since $D \neq 0$ and trace $D = 0$. then rank $V^T DV \leq k$. Consequently, rank $A = \lambda^* ee^T - D \leq k$. Let

$$m = \frac{\text{trace}\, A}{k} = \frac{n}{k}\lambda^*.$$

and

$$\begin{aligned} s^2 &= \frac{1}{k}\,\text{trace}\, A^2 - m^2 \\ &= \frac{1}{k}\,\text{trace}\, D^2 - \frac{2\lambda^*}{k} e^T De + \frac{n^2\,(k-1)}{k^2}\lambda^{*2}, \\ &= \frac{1}{k}\,\text{trace}\, D^2 - \frac{(k+1)}{k^2\, n^2}(e^T De)^2, \end{aligned}$$

and the result follows by a similar argument as in the proof of Theorem 4.1. ■

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