Extending cover inequalities for the quadratic knapsack problem to relaxations in lifted space

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Abstract

We address the binary quadratic knapsack problem (**QKP**), where the variable $x \in \{0,1\}^n$ indicates whether an item is selected for the knapsack or not. We consider relaxations of the **QKP** in the symmetric matrix space determined by the lifting $X := xx^T$, and present valid inequalities for them on the matrix variable X, which are obtained by extending the well known cover inequalities for the knapsack problem.

Keywords: Quadratic knapsack problem; relaxation; cover inequalities; lifted matrix space.

1 Introduction

We address the binary *quadratic knapsack problem,* ***QKP***,

$$(\mathbf{QKP}) \qquad \begin{array}{rl} \max & x^T Q x \\ \text{s.t.} & w^T x \leq c \\ & x \in \{0,1\}^n, \end{array} \tag{1}$$

where $Q \in \mathbb{S}^n$ is a symmetric $n \times n$ nonnegative integer *profit matrix*, $w \in \mathbb{Z}^n_{++}$ is the vector of positive integer weights for the items, and $c \in \mathbb{Z}_{++}$ is the knapsack capacity with $c \geq w_i$, for all $i \in N := \{1, \ldots, n\}$. The binary variable x indicates whether an item is chosen for the knapsack or not, and the inequality in the model, known as a knapsack inequality, ensures that the selection of items does not exceed the knapsack capacity. We note that any linear costs in the objective can be included on the diagonal of Q by exploiting the $\{0,1\}$ constraints and, therefore, are not considered.

The **QKP** was introduced in [**?**] and was proved to be NP-Hard in the strong sense by reduction from the clique problem. The quadratic knapsack problem is a generalization of the *knapsack problem,* ***KP***. The **KP** can be solved in pseudo-polynomial time using dynamic programming approaches with complexity of $O(nc)$.

The **QKP** appears in a wide variety of fields, such as biology, logistics, capital budgeting, telecommunications and graph theory, and has received a lot of attention in the last decades. Several papers have proposed branch-and-bound algorithms for the **QKP** and the main difference between them is the method used to obtain upper bounds for the subproblems [**? ? ? ? ? ?**]. The well known trade-off between the strength of the bounds and the computational effort required to obtain them is intensively discussed in [**?**], where *semidefinite programming,* ***SDP***, relaxations proposed in [**?**] and [**?**] are presented as the strongest relaxations for the **QKP**. The *linear programming,* ***LP***, relaxation proposed in [**?**], on the other side, is presented as the most computationally inexpensive. Both the **SDP** relaxations and the **LP** relaxation have a common feature, they are defined in the symmetric matrix lifted space determined by the equation $X = xx^T$, and by the replacement of the quadratic objective function in (1) with the linear function in X, trace(QX). As the constraint $X = xx^T$ is nonconvex, it is relaxed by convex constraints on the relaxations. The well known McCormick inequalities [**?**], and also the semidefinite constraint $X - xx^T \succeq 0$, or equivalently, $Y \succeq 0$, where

$$Y := \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}, \tag{2}$$

have been extensively used to relax the nonconvex constraint $X = xx^T$, in relaxations of the **QKP**.

In this work we present valid inequalities to strengthen relaxations in lifted space for the **QKP**, which are derived from cover inequalities for the **KP**. Taking advantage of the lifting $X := xx^T$, we propose new valid inequalities, which can also be applied to more general relaxations of mixed-integer linear and nonlinear programs that use the same lifting. We also discuss how cuts for the relaxations can be obtained by separation algorithms for cover inequalities.

1.1 Preliminaries: knapsack polyhedron and cover inequalities

The *knapsack polytope* is the convex hull of the *feasible points,* ***KF*** $:= \{x \in \{0,1\}^n : w^T x \leq c\}$.

Definition 1.1 (zero-one knapsack polytope)**.**

$$\boldsymbol{KPol} := \text{conv}(\boldsymbol{KF}) = \text{conv}\{x \in \{0,1\}^n : w^T x \leq c\}.$$

Proposition 1.2. *The dimension*

$$\dim(\boldsymbol{KPol}) = n,$$

and ***KPol*** *is an independence system, i.e.,*

$$x \in \boldsymbol{KPol}, y \in \{0,1\}^n, y \leq x \implies y \in \boldsymbol{KPol}.$$

Cover inequalities were originally presented in [**? ?**]. These inequalities can be used not only for knapsack problems, **KP**, but also for more general mixed-integer linear programs.

Definition 1.3 (*cover inequality, **CI***)**.** *The subset $C \subseteq N$ is a* cover *if it satisfies*

$$\sum_{j \in C} w_j > c.$$

*The (valid) **CI** is*

$$\sum_{j \in C} x_j \leq |C| - 1.$$

The cover inequality is minimal *if no proper subset of C is also a cover.*

Definition 1.4 (*extended **CI**, **ECI***)**.** *Let $w^* := \max_{j \in C} w_j$ and define the extension of C as*

$$E(C) := C \cup \{j \in N \backslash C : w_j \geq w^*\}.$$

*The **ECI** is*

$$\sum_{j \in E(C)} x_j \leq |C| - 1.$$

Definition 1.5 (*lifted **CI**, **LCI***)**.** *Given any minimal cover C, there exists at least one facet-defining* lifted **CI**, **LCI** *of the form*

$$\sum_{j \in C} x_j + \sum_{j \in N \backslash C} \alpha_j x_j \leq |C| - 1, \tag{3}$$

*where $\alpha_j \geq 0, \forall j \in N \backslash C$. Moreover, each such **LCI** dominates the extended **CI**.*

Details about the computational complexity of **LCI** are presented in [**?**]. Algorithm 1.1, from [**?**], shows how to derive a facet-defining **LCI** from a given minimal cover C.

Algorithm 1.1 Procedure to find **LCI**

Sort the elements in ascending w_i order $i \in N \setminus C$, defining $\{i_1, i_2, \ldots, i_r\}$
For: t=1 **to** r

$$\begin{array}{rl} \zeta_t = \max & \sum_{j=1}^{t-1} \alpha_{i_j} x_{i_j} + \sum_{i \in C} x_i \\ st & \sum_{j=1}^{t-1} w_{i_j} x_{i_j} + \sum_{i \in C} w_i x_i \leq c - w_{i_t} \\ & x \in \{0,1\}^{|C|+t-1}. \end{array} \tag{4}$$

Set $\alpha_{i_t} = |C| - 1 - \zeta_t$.
End For

2 Valid inequalities

We are now interested in finding valid inequalities to strengthen relaxations of the **QKP** in the lifted space determined by the lifting $X := xx^T$. Let us denote by **ConvRel**, any convex relaxation of the **QKP** in the lifted space, where the equation $X = xx^T$ was relaxed somehow, by convex constraints. Let Y be defined as in (2). We initially note that if the inequality

$$\tau^T x \leq \beta \tag{5}$$

is valid for the **QKP**, where $\tau \in \mathbb{Z}_+^n$ and $\beta \in \mathbb{Z}_+$, then, as x is nonnegative and $X := xx^T$,

$$Y \begin{pmatrix} -\beta \\ \tau \end{pmatrix} \leq 0 \tag{6}$$

is a valid inequality for **ConvRel**.

For simplicity, we will say that the solution $\bar{Y}$ of **ConvRel** satisfies (5) if it satisfies the corresponding valid inequality (6).

2.1 Adding cuts to the relaxation

Our first idea is to iteratively obtain a **CI**, formulated as $\alpha^T x \leq e^T \alpha - 1$, where $\alpha \in \{0,1\}^n$ and e denotes the ones vector, which is most violated by the current solution $\bar{Y}$ of **ConvRel**. More specifically, we search for the **CI** that maximizes the maximum violation among the inequalities in $\bar{Y}\mathbf{cut}(\alpha) \leq 0$, where

$$\mathbf{cut}(\alpha) = \begin{pmatrix} -e^T\alpha + 1 \\ \alpha \end{pmatrix}.$$

To obtain such **CI**, we solve the following linear knapsack problems, for all $i = 1, \ldots, n+1$, where e_i denotes the unit vector.

$$v_i^* := \max_{\alpha} \{ e_i^T \bar{Y} \mathbf{cut}(\alpha) : w^T \alpha \geq c + 1, \, \alpha \in \{0,1\}^n \}. \tag{7}$$

If $v^* := \max_{i \in \{1,\ldots,n+1\}} \{v_i^*\} > 0$, the **CI** given by the corresponding solution of (7) is violated by $\bar{Y}$. In this case, we apply Algorithm 1.1 to the **CI** obtained and lift it to a **LCI**, which is formulated as (5). Finally, we add the corresponding valid inequality (6) to **ConvRel**.

Remark 2.1. *It is worth noting that if $Y \geq 0$ in **ConvRel**, then if Y satisfies the constraint (6) derived from a **LCI**, Y also satisfies the constraints derived from a **CI** that can be lifted to the **LCI**. Therefore, the dominance relation between **LCI** and **CI** is maintained on the corresponding constraints on the variable Y.*

2.2 New valid inequalities in the lifted space

As previously discussed, after finding any valid inequality in the form of (5) for the **QKP**, we may add the constraint (6) to **ConvRel**. We note that besides (6), we can also generate stronger valid inequalities in the lifted space by taking advantage of the lifting $X := xx^T$. In the following, we show how the idea can be applied to cover inequalities.

Let

$$\sum_{j \in C_l} x_j \leq \beta. \tag{8}$$

be a valid inequality for **KPol**.

The inequality (8) can be either a cover inequality, **CI**, or an extended cover inequality, **ECI**, or a particular lifted cover inequality, **LCI**, where $\alpha_j \in \{0,1\}, \forall j \in N \backslash C$ in (3). Furthermore, given a general **LCI**, where $\alpha_j \in \mathbb{Z}_+$, for all $j \in N \backslash C$, a valid inequality of type (8) can be constructed by replacing each α_j with $\min\{\alpha_j, 1\}$ in the **LCI**.

Definition 2.2 (*Cover inequality in the lifted space,* ***CILS***)**.** *Considering* (8)*, we conclude that at most* $\binom{\beta}{2}$ *products of variables* $x_i x_j$*, where* $i,j \in C_l$*, can be equal to 1. Therefore, we introduce the following inequality on the lifted variable* X*, which we denote by* ***CILS****.*

$$\sum_{i,j \in C_l, i<j} X_{ij} \leq \binom{\beta}{2}. \tag{9}$$

Remark 2.3. *When* $\beta = 1$*, the inequality* (8) *is well known as a* clique cut*, widely used to model decision problems, and frequently used as a cut in branch-and-cut algorithms. We note that, if* $\beta = 1$*, the inequality* (9) *makes it possible to fix*

$$X_{ij} = 0, \text{ for all } i,j \in C_l, i < j.$$

Remark 2.4. *A similar observation to Remark 2.1, is that, if* $Y \geq 0$ *in* ***ConvRel****, then if* Y *satisfies a* ***CILS*** *derived from a* ***LCI****,* Y *also satisfies any* ***CILS*** *derived from a* ***CI*** *that can be lifted to the* ***LCI****. Therefore, such* ***CILS*** *derived from a* ***LCI****, dominates the* ***CILS*** *derived from the* ***CI****.*

Besides defining one cover inequality in the lifted space considering all possible pairs of indexes in C_l, we can also define a set of cover inequalities in the lifted space, considering in each inequality, a partition of the indexes in C_l into subsets of cardinality 2. In this case, the right hand side of the inequalities is never bigger than $\beta/2$. The idea is better specified below.

Definition 2.5 (*Set of cover inequalities in the lifted space,* ***SCILS***)**.** *Let* C_l *be the cover in the valid inequality* (8) *for our* ***QKP****. Let*

- $C_{ls} := \{(i_1, j_1), \ldots, (i_p, j_p)\}$ *be a partition of* C_l*, if* $|C_l|$ *is even.*
- $C_{ls} := \{(i_1, j_1), \ldots, (i_p, j_p)\}$ *be a partition of* $C_l \setminus \{i_0\}$ *for each* $i_0 \in C_l$*, if* $|C_l|$ *is odd and* β *is odd.*
- $C_{ls} := \{(i_0, i_0), (i_1, j_1), \ldots, (i_p, j_p)\}$*, where* $\{(i_1, j_1), \ldots, (i_p, j_p)\}$ *is a partition of* $C_l \setminus \{i_0\}$*, for each* $i_0 \in C_l$*, if* $|C_l|$ *is odd and* β *is even.*

In all cases, $i_k < j_k$ *for all* $k = 1, \ldots, p$*.*

The inequalities in the ***SCILS*** *corresponding to* (8) *are given by*

$$\sum_{(i,j) \in C_{ls}} X_{ij} \leq \left\lfloor \frac{\beta}{2} \right\rfloor,$$

for all partitions C_{ls} *defined as above.*

Finally, we extend the ideas presented above to the more general case of knapsack inequalities. We note that the following discussion applies to a general **LCI**, where $\alpha_j \in \mathbb{Z}_+, \forall j \in N \backslash C$.

Let

$$\sum_{j \in N} \alpha_j x_j \leq \beta. \tag{10}$$

be a valid knapsack inequality for **KPol**, with $\alpha_j, \beta \in \mathbb{Z}_+, \beta \geq \alpha_j, \forall j \in N$.

Definition 2.6 (*Set of knapsack inequalities in the lifted space,* ***SKILS***)**.** *Let* $\{C_1, \ldots, C_q\}$ *be the partition of* N*, such that for every* $j_k \in C_k$*,* α_{j_k} *assumes the same value* $\tilde{\alpha}_k$ *in* (10)*, for all* $k = 1, \ldots, q$*, i.e.,* $\alpha_{j_r} = \alpha_{j_s}$ *if* $j_r, j_s \in C_k$*, for some* k*, and* $\alpha_{j_r} \neq \alpha_{j_s}$*, otherwise. The knapsack inequality* (10) *can then be rewritten as*

$$\sum_{k=1}^{q} \left(\tilde{\alpha}_k \sum_{j \in C_k} x_j \right) \leq \beta. \tag{11}$$

Now, for $k = 1, \ldots, q$*, let* $C_{l_k} := \{(i_{k_1}, j_{k_1}), \ldots, (i_{k_{p_k}}, j_{k_{p_k}})\}$*, where* $i < j$ *for all* $(i,j) \in C_{l_k}$*, and*

- C_{l_k} *is a partition of* C_k*, if* $|C_k|$ *is even.*
- C_{l_k} *is a partition of* $C_k \setminus \{i_{k_0}\}$*, where* $i_{k_0} \in C_k$*, if* $|C_k|$ *is odd.*

The inequalities in the ***SKILS*** *corresponding to* (10) *are given by*

$$\sum_{k=1}^{q} \left(\tilde{\alpha}_k X_{i_{k_0} i_{k_0}} + 2\tilde{\alpha}_k \sum_{(i,j) \in C_{l_k}} X_{ij} \right) \leq \beta, \tag{12}$$

for all partitions C_{l_k}*,* $k = 1, \ldots, q$*, defined as above, and for* $i_{k_0} \in C_k \setminus C_{l_k}$*. (If* $|C_k|$ *is even,* $C_k \setminus C_{l_k} = \emptyset$*, and the term in the variable* $X_{i_{k_0} i_{k_0}}$ *does not exist.)*

Proposition 2.7.

- *If inequality* (8) *is valid for* ***QKP****, then* ***CILS*** *and the inequalities in* ***SCILS*** *are valid for* ***ConvRel****.*
- *If inequality* (10) *is valid for* ***QKP****, then inequalities in* ***SKILS*** *are valid for* ***ConvRel****.*

3 Example

Let us now illustrate the application of the cuts proposed in this work, to two instances of the **QKP**, with 6 candidate items to be selected ($N = \{1, \ldots, 6\}$). The instances were constructed with the purpose of showing situations where our proposed cuts **CILS** and **SCILS** are both effective if used separately, and are even stronger if used together. Furthermore, we aim to show through the instances that, when the cuts are used separately, each one of them may be stronger than the other.

For the first instance (Instance 1) we consider $Q_{jj} = 1$, for $j \in N$, $Q_{12} = Q_{13} = Q_{56} = Q_{21} = Q_{31} = Q_{65} = 100$, and all other elements in the matrix Q equal to 2. We also consider $w_j = 1$, for all $j \in N$, and $c = 3$. For the second instance (Instance 2), we only change the following elements of the matrix Q: $Q_{14} = Q_{15} = Q_{41} = Q_{51} = 60$. The remaining data is kept as it is for Instance 1. An optimal solution for both instances is obtained by the selection of items $1, 2, 3$, with value $z^* = 407$.

We consider the simple initial convex relaxation of the **QKP** as the following linear program in the lifted space:

$$(\textbf{ConvRel}) \qquad \begin{array}{rl} \max & \text{trace}(QX) \\ \text{s.t.} & w^T x \leq c \\ & X - xx^T \succeq 0 \\ & x \in [0,1]^n \\ & X \in [0,1]^{n\times n} \end{array} \tag{13}$$

The optimal solution of **ConvRel** is given by the sum of all elements in Q, as X can be taken as a matrix of all ones and x is a vector of zeros. Now we consider, for both instances, the lifted cover inequality, **LCI** :

$$\sum_{j\in N} x_j \leq 3,$$

the corresponding **CILS** :

$$\sum_{i,j\in N, i<j} X_{ij} \leq 3, \tag{14}$$

and the corresponding 15 inequalities in the **SCILS** , corresponding to each partition of N:

$$\begin{array}{c} X_{12} + X_{34} + X_{56} \leq 1, \\ \vdots \\ X_{16} + X_{25} + X_{34} \leq 1. \end{array} \tag{15}$$

In Table 1, we show the impact on the objective function of the convex relaxation of the **QKP**, given by the addition of the proposed valid inequalities derived from an **LCI** .

Inst	z^*	Cuts added to **ConvRel**			
		None	**CILS**	**SCILS**	**CILS** +**SCILS**
1	407	654	606	418	410
2	407	886	606	650	526

Table 1: Upper bound given by **ConvRel** with addition of cuts

Finally, we present a similar analysis, but considering a stronger initial convex relaxation **ConvRel**$^+$, where we add to **ConvRel** the valid inequalities $X_{jj} = x_j$, for all $j \in N$, and $w^T X \leq cx^T$. The results are presented in Table 2

Inst	z^*	Cuts added to **ConvRel**$^+$			
		None	**CILS**	**SCILS**	**CILS** +**SCILS**
1	407	603	603	**407**	**407**
2	407	603	603	431	431

Table 2: Upper bound given by **ConvRel**$^+$ with addition of cuts

We see from the results in Table 2 that, although the cuts **CILS** cannot tight the bound given by the stronger relaxation **ConvRel**$^+$, for these small instances, the set of inequalities **SCILS** is very effective.

4 Conclusions

We present in this work new valid inequalities for convex relaxations of the binary quadratic knapsack problem, **QKP**, defined in the lifted space determined by the equation $X = xx^T$, where x is a binary vector. The inequalities can be also applied to more general binary quadratic programs, or even more general mixed binary programs that are relaxed in this lifted space. Two different ideas are proposed to construct valid inequalities from a cover inequality for the knapsack problem. The first idea leads to a single valid inequality, denoted in this work by cover inequality in the lifted space, **CILS**. The second idea leads to a set of valid inequalities denoted by set of cover inequalities in the lifted space, **SCILS**. Through two small instances of the problem, we illustrate the application of these inequalities and show that neither of them dominates the other, and that when used together, they may be stronger than when used separately. Finally, we show how the idea in **SCILS** can be generalized to construct valid inequalities from knapsack constraints.

References