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A Note on Maximizing the Permanent of a Positive Definite Hermitian Matrix, Given the Eigenvalues*

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As a step toward understanding the unsolved problem of determining how large the permanent of a positive semi-definite matrix can be, given the eigenvalues, we note that a necessary condition for A to be a permanent maximizing matrix is that A commute with its permanental adjoint.

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For an n-by-n matrix $A = (a_{ij})$ the *permanent* is defined by

$$\operatorname{per} A = \sum_{\tau \in S_n} \prod_{i=1}^{n} a_{i\tau(i)}$$

in which S_n denotes the full symmetric group on n objects. It is well known that if A is positive semi-definite Hermitian, then

$$\det A \leqslant \operatorname{per} A.$$

For given $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, $\lambda_1 \geqslant 0$, the following question has long been open [1, 2]:

$$\textit{maximize } \operatorname{per} A \qquad (*)$$

$$\textit{subject to: } A \text{ Hermitian}; \ \sigma(A) = \{\lambda_1, \ldots, \lambda_n\}.$$

The minimum of per A subject to the same conditions is clearly $\det A = \lambda_1 \cdots \lambda_n$. The problem (*) has a finite solution, as an equivalent formulation is

$$\textit{maximize } \operatorname{per} U^*\Lambda U \qquad (**)$$

$$\textit{subject to: } U \text{ is } n\text{-by-}n \text{ unitary}$$

in which $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Since the permanent is a continuous function of the entries and the unitary group is compact, a finite maximum must be achieved for at least one U. We do not solve the problem (*) here, but simply wish to note an interesting necessary condition for A to be an optimizing matrix.

For an n-by-n matrix $A = (a_{ij})$ the *permanental adjoint* of A is defined to be the n-by-n matrix padj A whose i, j entry is the permanent of the $(n-1)$-by-$(n-1)$ submatrix of A resulting from deletion of row i and column j. Our observation is the

THEOREM *If the matrix A is a solution to* (*), *then A and* padj A *commute.*

Proof Let $M = C^{n\times n}$ be the space of $n \times n$ complex matrices, $k(U) = U^*\Lambda U$, $g(U) = U^*U - I$, and $f(U) = \operatorname{per}(U^*\Lambda U)$. We let $\langle A, B\rangle = \operatorname{tr} A^*B$ denote the inner product in M. The Frechet differentials of the above functions acting on the increment $h \in M$ are:

$$dk(U; h) = k'(U)h = U^*\Lambda h + h^*\Lambda U,$$

$$dg(U; h) = g'(U)h = U^*h + h^*U,$$

$$d\,\mathrm{per}(V;h) = \langle \mathrm{padj}\, V, h\rangle = \mathrm{tr}(\mathrm{padj}\, V)^*h,$$

$$\begin{aligned} df(U;h) &= d\,\mathrm{per}(V; dk(U;h)) \\ &= \mathrm{tr}(\mathrm{padj}\, V)^*(U^*\Lambda h + h^*\Lambda U), \qquad V = U^*\Lambda U. \end{aligned}$$

The above derivatives can be verified directly from the definition of a Frechet differential. Let us now write (*) as

$$\max\{f(U)\colon g(U) = 0,\ U \in M\},$$

and try to solve this problem using Lagrange multipliers. The Lagrangian is

$$L(U,\sigma) = f(U) + \langle \sigma, g(U)\rangle,$$

where the Lagrange multiplier $\sigma = \sigma^* \in M$. Note that we can assume σ Hermitian since $g(U)$ is Hermitian for each U. Now suppose that U solves (*) and $A = U^*\Lambda U$. Note that the Frechet derivative $g'(U)$, for U unitary, is a linear operator from M onto the Hermitians in M, i.e. for $K = K^* \in M$, set $h = \frac{1}{2}UK$. Thus the standard constraint qualification holds for (**) and U is a stationary point of L for some σ, i.e. for all h in M, we have

$$\begin{aligned} 0 = \frac{\partial L(U,\sigma)}{\partial U} h &= df(U;h) + \langle \sigma, dg(U;h)\rangle \\ &= \mathrm{tr}[(\mathrm{padj}\, A)(U^*\Lambda h + h^*\Lambda U) + \sigma(U^*h + h^*U)] \\ &= \langle \Lambda U(\mathrm{padj}\, A) + U\sigma, h\rangle + \langle h, \Lambda U(\mathrm{padj}\, A) + U\sigma\rangle \end{aligned}$$

since $\mathrm{tr}\, BC = \mathrm{tr}\, CB$ and $\sigma = \sigma^*$.

This implies

$$\Lambda U(\mathrm{padj}\, A) = -U\sigma$$

or equivalently

$$A(\mathrm{padj}\, A) = -\sigma.$$

Since σ, padj A and A are Hermitian, the above yields the theorem. ∎

We note that the commutativity condition $A(\mathrm{padj}\, A) = (\mathrm{padj}\, A)A$ is also necessary for minimization of the permanent when $\lambda_1 > 0$. Then the minimum occurs if and only if per $A = \det A$, or if and only if A is diagonal. If A is diagonal, padj A is diagonal and they commute. In general commutativity with padj A does seem to place a significant restriction upon A, in contrast to the case of the determinantal adjoint ($A(\operatorname{adj} A) = (\operatorname{adj} A)A$, always).

The theorem does permit a simple solution of (*) when $n = 2$.

COROLLARY *If $n = 2$, the solution to (*) is $(\lambda_1^2 + \lambda_2^2)/2$ and a maximizing matrix is characterized by equal diagonal entries.*

Proof If

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

then a calculation reveals that A and padj A commute if and only if either $a = c$ or $b = 0$. In the latter case the permanent is minimized. Thus,

$$A = \begin{bmatrix} \frac{1}{2}(\lambda_1 + \lambda_2) & \pm\sqrt{\left(\frac{\lambda_1 + \lambda_2}{2}\right)^2 - \lambda_1\lambda_2} \\ \pm\sqrt{\left(\frac{\lambda_1 + \lambda_2}{2}\right)^2 - \lambda_1\lambda_2} & \frac{1}{2}(\lambda_1 + \lambda_2) \end{bmatrix}$$

for maximization, and a further calculation reveals that per $A = \frac{1}{2}(\lambda_1^2 + \lambda_2^2)$. ■

If $n = 3$, application of the commutativity condition is already complex. Let

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Then $A(\operatorname{padj} A) = (\operatorname{padj} A)A$ if and only if

$$(a - d)(bf + ce) = b(e^2 - c^2),$$
$$(a - f)(be + cd) = c(e^2 - b^2),$$
$$(d - f)(ae + bc) = e(c^2 - b^2).$$

This permits a number of solutions. A diagonal matrix again corresponds to minimization. The matrix $A = \alpha I + \beta J$, J the matrix of 1's, generalizes the 2-by-2 maximum (equal diagonal entries and equal off-diagonal entries) and gives the maximum when the smallest eigenvalue has multiplicity two. But there are other cases, such as matrices of the form

$$A = \begin{bmatrix} a & 0 & c \\ 0 & a & c \\ c & c & \dfrac{a^2 - c^2}{a} \end{bmatrix},$$

among others. It is not immediately clear what solution corresponds to the maximum in the general case of three distinct eigenvalues.

The first part of the above corollary is a special case of the Marcus–Minc inequality, see [2, p. 113],

$$\text{per } A \leqslant \frac{1}{n} \sum_i \lambda_i^n. \qquad (***)$$

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References

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