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An Explicit Linear Solution for the Quadratic Dynamic Programming Problem

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Abstract. For a given vector x_0, the sequence $\{x_t\}$ which optimizes the sum of discounted rewards $r(x_t, x_{t+1})$, where r is a quadratic function, is shown to be generated by a linear decision rule $x_{t+1} = Sx_t + R$. Moreover, the coefficients R, S are given by explicit formulas in terms of the coefficients of the reward function r. A unique steady-state is shown to exist (except for a degenerate case), and its stability is discussed.

Key Words. Dynamic programming, discrete-time control theory, linear decision rules.

1. Introduction

Dynamic programming can be used to model a wide variety of discrete-time continuous-state optimization problems. Some examples include production theory (Ref. 1), control theory (Ref. 2), and economic planning (Ref. 3). These models are often solved, theoretically as well as numerically, by using the value improvement iterative method. However, for the case of a quadratic objective function, it has long been known (see Ref. 2 or Ref. 4) that the optimal solution, at least in special models, is characterized by a decision rule which is a linear function of the state variables. The purpose of this paper is to show that this result holds in general and that the optimal rule can be stated explicitly in terms of the coefficients of the quadratic objective function.

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The dynamic programming problem is defined, and the main result is presented in Section 2. Section 3 describes the sequence space for which the problem has finite values. This section also presents the necessary and sufficient conditions for optimality. Section 4 describes the kinds of generalized inverse matrix and square root matrix which are used in the statement of the main theorem. Section 5 then shows that there exists a linear decision rule which satisfies the sufficient conditions for optimality and thereby completes the proof. The existence and stability of a steady solution is also described.

2. Dynamic Programming Problem

The dynamic programming problem can be stated, in general terms, for the case of continuous state variables and discrete time, as follows: Given x_0, find a sequence $\tilde{x} = \{x_t\}$ of vectors which maximizes the value of the objective function

$$f(\tilde{x}) = \sum_{t=0}^{\infty} r_t(x_t, x_{t+1}). \tag{1}$$

The problem is said to be discounted if the reward function satisfies

$$r_t(x, y) = \delta^t r(x, y), \tag{2}$$

for all vectors x, y and all $t = 0, 1, 2, \ldots$, where $0 < \delta < 1$ is the discount factor. The nondiscounted problem corresponds to the case $\delta = 1$.

The reward function $r(x, y)$ is usually assumed to be a strictly concave function of the transitions (x, y). In this paper, we will require that the reward be a quadratic function,

$$r(x, y) = \tfrac{1}{2}x'Ax + x'By + \tfrac{1}{2}y'Cy + D'x + E'y, \tag{3}$$

where the prime superscript denotes the transpose of a matrix. A sufficient condition that r be strictly concave is that the block matrix of its associated quadratic form,

$$G = \tfrac{1}{2}\begin{bmatrix} A & B \\ B & C \end{bmatrix}, \tag{4}$$

be a symmetric, negative-definite matrix. Note that we assume that B is symmetric.

Since A and C are negative-definite matrices, then so is the matrix $\delta A+C$ for any $\delta>0$. Consequently, the matrices

$$M=(\delta A+C)^{-1}B, \tag{5}$$

$$N=-(\delta A+C)^{-1}(\delta D+E) \tag{6}$$

exist. These matrices are useful in concisely stating our results.

Next, a problem will be said to be degenerate if the matrix

$$H=(I+(1+\delta)M) \tag{7}$$

is singular, that is, if $\lambda=-1/(1+\delta)$ is an eigenvalue of M. The nondiscounted problem can easily be shown, using the quadratic form defined by G in Eq. (4), to be a nondegenerate problem. A discounted problem can always be made nondegenerate by a slight change in its discount factor.

The main result of the paper can now be stated. We consider only the discounted case in this paper.

Theorem 2.1. The quadratic dynamic programming problem has a unique optimal solution $\bar{x}$, for given x_0, defined by the linear decision rule

$$x_{t+1}=Sx_t+R, \tag{8}$$

for $t=0, 1, \ldots$, with coefficients

$$R=(I+\delta M(I+S))^{-1}N, \tag{9}$$

$$S=(1/2\delta)M^+(-I+(I-4\delta M^2)^{1/2}), \tag{10}$$

where the superscripts + and 1/2 denote the generalized matrix inverse and the positive matrix square root, respectively.

The remainder of this paper mostly consists of a sequence of lemmas which lead up to the proof of this result for the discounted case $\delta<1$.

3. Sufficient Conditions for Optimality

The feasible solutions of the dynamic programming problem consist of all sequences $\bar{x}=\{x_t\}$ of vectors in n-space. Letting $\|x\|$ be the Euclidean norm, then we can define the sequence norm

$$|||\bar{x}|||=\left(\sum_{t=0}^{\infty}\delta^t\|x_t\|^2\right)^{1/2} \tag{11}$$

and the sequence space

$$X=\{\bar{x}: |||\bar{x}|||<\infty\}. \tag{12}$$

This space is the set of good sequences, in the sense that, as shown by the next lemma, the objective function f is finite only for sequences in X.

Lemma 3.1. Let f be defined by Eqs. (1), (2), (3). Then:
(i) $|f(\tilde{x})|<\infty$, for $\tilde{x}\in X$;
(ii) $|f(\tilde{x})|\to\infty$, as $|||\tilde{x}|||\to\infty$;
(iii) $f(\tilde{x})=-\infty$, if $\tilde{x}\notin X$;
(iv) f is strictly concave on X.

Proof. (i) Since r is a quadratic function, there exist constants J, K, L such that

$$|r(x,y)|\le J+K\|x\|^2+L\|y\|^2,$$

so that, for any sequence $\tilde{x}\in X$, we get

$$|f(\tilde{x})|\le\sum_{t=0}^{\infty}\delta^t|r(x_t,x_{t+1})|$$

$$\le J\sum_0^\infty\delta^t+K\sum_0^\infty\delta^t\|x_t\|^2+L\sum_0^\infty\delta^t\|x_{t+1}\|^2<\infty.$$

(ii) Since the negative-definite quadratic form

$$g(x,y)=(x',y')G\begin{pmatrix}x\\y\end{pmatrix}=\tfrac{1}{2}x'Ax+x'By+\tfrac{1}{2}y'Cy$$

dominates the linear terms in r and since

$$r(x,y)=g(x,y)+Dx+Ey,$$

we get that there exists some constant $\gamma>0$ such that

$$|g(x,y)|>2|Dx+Ey|,\qquad \text{whenever } \max\{\|x\|,\|y\|\}>\gamma.$$

Let $\tilde{x}\in X$, and define

$$N=\{t: \max\{\|x_t\|,\|x_{t+1}\|\}>\gamma\}.$$

Then, for some constant K, summing for $t\in N$, we have

$$|f(\tilde{x})|=\sum\delta^t|g(x_t,x_{t+1})+(Dx_t+Ex_{t+1})|+K$$

$$\ge\sum\delta^t\|g(x_t,x_{t+1})|-|Dx_t+Ex_{t+1}\|+K$$

$$>\tfrac{1}{2}\sum\delta^t|g(x_t,x_{t+1})|+K$$

$$\ge\tfrac{1}{2}|\lambda_1|\sum\delta^t\min\{\|x_t\|^2,\|x_{t+1}\|^2\}+K,$$

where $\lambda_1<0$ is the eigenvalue of G of smallest magnitude in absolute value. It now follows that

$$|f(\tilde{x})|\to\infty,\qquad \text{as } |||\tilde{x}|||\to\infty.$$

(iii) Since $r(x, y) \leq K$, for some constant K, then $f(\tilde{x}) < K/(1-\delta)$ holds for any sequence x. Hence, by (ii), it follows that

$$f(\tilde{x}) = -\infty, \qquad \text{if } \tilde{x} \notin X.$$

(iv) The strict concavity of f is a direct consequence of that for r. □

One of the main requirements for optimality of a sequence $\tilde{x}$ is that it satisfy a first-order condition, such as the derivative condition $\nabla f(\tilde{x}) = 0$. In the case of a discounted dynamic programming problem, this first-order condition reduces to a second-order difference equation. Consequently, a sequence $\tilde{x}$ is said to satisfy a first-order condition if

$$\frac{\partial r}{\partial y}(x_{t-1}, x_t) + \delta \frac{\partial r}{\partial x}(x_t, x_{t+1}) = 0 \tag{13}$$

holds for all $t = 1, 2, \ldots$.

The next lemma shows for quadratic problems that the first-order condition is equivalent to solving a pair of matrix equations.

Lemma 3.2. A sequence $\tilde{x}$ satisfies the first-order condition if it is generated by a linear decision rule (8), where R, S satisfy the equations

$$((\delta A + C) + \delta B(I + S))R = -(\delta D + E), \tag{14}$$

$$\delta B S^2 + (\delta A + C)S + B = 0. \tag{15}$$

Proof. Equation (13), in the case of a quadratic r given by (3), reduces to

$$(Bx_{t-1} + Cx_t + E) + \delta(Ax_t + Bx_{t+1} + D) = 0. \tag{16}$$

This difference equation is linear with constant coefficients. Using Eq. (8) to eliminate the variables x_{t+1} and x_t, respectively, this equation can be rewritten in the form

$$\begin{aligned}&(B + CS + \delta AS + \delta BS^2)x_{t-1}\\ &+ (CR + E + \delta AR + \delta BSR + \delta BR + \delta D) = 0.\end{aligned} \tag{17}$$

With the first-order condition expressed in this form, it is clear that it holds, for all t and all x_0, if and only if both the constant term and the coefficient of x_{t-1} in (17) are equal to zero. These conditions, slightly rearranged, are Eqs. (14) and (15). □

Using Eqs. (5), (6), the system of matrix equations (14), (15) can be expressed more simply as

$$(I + \delta M(I + S))R = N, \tag{18}$$

$$\delta M S^2 + S + M = 0. \tag{19}$$

The next lemma adds a second-order condition, concavity, and gives an inequality which will be used to establish the optimality of a sequence.

Lemma 3.3. Let $\tilde{x}^* \in X$ satisfy the first-order condition (13), and let $\tilde{x} \in X$ be any other sequence from x_0^*. Then

$$f(\tilde{x}) - f(\tilde{x}^*) \leq \lim_{T \to \infty} \delta^T \frac{\partial r}{\partial x}(x_T^*, x_{T+1}^*)(x_T^* - x_T). \tag{20}$$

Proof. Since $r(x, y)$ is concave and differentiable, then

$$r(x_t, x_{t+1}) - r(x_t^*, x_{t+1}^*) \leq \frac{\partial r}{\partial x}(x_t^*, x_{t+1}^*)(x_t - x_t^*)$$

$$+ \frac{\partial r}{\partial y}(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*).$$

If these inequalities are multiplied by δ^t and summed from $t = 0$ to $t = T - 1$, then

$$\sum_{t=0}^{T-1} \delta^t (r(x_t, x_{t+1}) - r(x_t^*, x_{t+1}^*))$$

$$\leq \frac{\partial r}{\partial x}(x_0^*, x_1^*)(x_0 - x_0^*)$$

$$+ \sum_{t=0}^{T-1} \delta^{t-1} \left(\frac{\partial r}{\partial y}(x_{t-1}^*, x_t^*) + \delta \frac{\partial r}{\partial x}(x_t^*, x_{t+1}^*) \right)(x_t - x_t^*)$$

$$+ \delta^{T-1} \frac{\partial r}{\partial y}(x_{T-1}^*, x_T^*)(x_T - x_T^*).$$

But the first and second terms are zero; and, using (13), the third term can be written as

$$-\delta^T \frac{\partial r}{\partial x}(x_T^*, x_{T+1}^*)(x_T - x_T^*),$$

proving the inequality. □

By Lemma 3.3, a sufficient condition for the optimality of a sequence $\tilde{x}^* \in X$ is that it satisfy both a first-order condition and a transversality condition, namely,

$$\lim_{T \to \infty} \delta^T \frac{\partial r}{\partial x}(x_T^*, x_{T+1}^*)(x_T - x_T^*) = 0, \tag{21}$$

for all $\tilde{x} \in X$.

4. Generalized Inverses and Square Roots

The matrix M, defined in Eq. (5), plays a central role in this paper. In the definition of the matrix S in Eq. (10), the following generalized inverse is needed.

Definition 4.1. Let M be a square matrix. The generalized inverse of M, denoted by M^+, is the matrix which satisfies the following conditions:

(i) $MM^+M = M,$ (22)

(ii) $M^+MM^+ = M^+,$ (23)

(iii) $MM^+ = M^+M.$ (24)

This is known as the group inverse (see Ref. 5, page 162), which exists if and only if the ranks of M and M^2 are equal. If the group inverse exists, it is unique. In the case of a unitarily diagonalizable matrix, the group inverse is also the Moore-Penrose generalized inverse. The next lemma shows how to construct M^+ for such matrices.

Lemma 4.1. Let M be a diagonalizable square matrix, i.e., $M = PQP^{-1}$, for some invertible matrix P and some diagonal matrix Q. Then, the group inverse exists and is defined by $M^+ = PQ^+P^{-1}$, where Q^+ is the diagonal matrix with entries

$$q_{ii}^+ = q_{ii}^{-1}, \qquad \text{if } q_{ii} \neq 0,$$

$$q_{ii}^+ = 0, \qquad \text{otherwise.}$$

Proof. The matrix M^+ clearly exists and it is a straightforward matter to verify that it satisfies the three identities (22), (23), (24). □

The definition and construction of a matrix square root is quite similar. Suppose that M is a diagonalizable matrix with nonnegative real eigenvalues, i.e.,

$$M = PQP^{-1}, \qquad \text{with } q_{ii} \geq 0.$$

Then, the positive square root of M is defined by

$$M^{1/2} = PQ^{1/2}P^{-1}, \tag{25}$$

where $Q^{1/2}$ is the diagonal matrix with entries $q_{ii}^{1/2}$. Notice that, by reversing the signs of some of these entries, there may be up to 2^n distinct square roots of a matrix, and an infinite number in the case of a repeated eigenvalue.

Lemma 4.2. The matrix $M=(\delta A+C)^{-1}B$ is diagonalizable.

Proof. Since M can be taken to be the product of a positive-definite matrix $U=-(\delta A+C)^{-1}$ and a symmetric real matrix $V=-B$, then M is similar to a Hermitian matrix, i.e.,

$$U^{-1/2}MU^{1/2}=-U^{1/2}VU^{1/2},$$

where $U^{1/2}$ is the positive square root of U. Hence, M is diagonalizable. □

Lemma 4.3. The spectral radius $\rho(M)$, for $M=(\delta A+C)^{-1}B$, satisfies the inequality

$$2\delta^{1/2}\rho(M)<1.$$

Proof. Let α be a real number such that $0\ll|\alpha|<\delta^{1/2}$, and let U be the matrix

$$U=\begin{bmatrix}\alpha I & 0\\ I & I\end{bmatrix}.$$

Then, as the matrix G in (4) is negative definite, so is the matrix

$$U'GU=\tfrac{1}{2}\begin{bmatrix}\alpha^2AC+2\alpha B & \alpha B+C\\ \alpha B+C & C\end{bmatrix}.$$

This implies that $\alpha^2AC+2\alpha\beta$ is negative definite, and consequently so is $A+\delta C+2\alpha B$. But then, $\lambda=-1/(2\alpha)$ cannot be an eigenvalue of M for $0<|\alpha|<\delta^{1/2}$, which places a bound on the size of $\rho(M)$. □

5. Optimal Solution

The results of the previous two sections can now be combined to prove the main theorem. We begin with the existence of the matrix S.

Lemma 5.1. The matrix S defined in (10) exists and solves (15).

Proof. By Lemma 4.2, the matrix M is diagonalizable, so by Lemma 4.1 its group inverse M^+ exists. By Lemma 4.3, the matrix $I-4\delta M^2$ will be diagonalizable with nonnegative eigenvalues, so that its positive square root exists. So, S exists. The fact that it satisfies (15), or equivalently (19), can be verified by direct substitution. The three properties for the group inverse are used extensively, with the passing of M^+ through the square root being an essential step in the simplification. □

The existence of the vector R follows next.

Lemma 5.2. The vector R defined in (9) exists and solves (14).

Proof. Since M is diagonalizable, then by Lemma 4.2 the coefficient of R in Eq. (18) can be rewritten, using (10), as follows:

$$\begin{aligned} &I+\delta M(I+S)\\ &=I+\delta M(I+(1/2\delta)M^+(-I+(I-4\delta M^2)^{1/2}))\\ &=I+\delta M+\tfrac{1}{2}MM^+(-I+(I-4\delta M^2)^{1/2})\\ &=I+\delta M-\tfrac{1}{2}MM^++\tfrac{1}{2}MM^+(I-4\delta M^2)^{1/2}\\ &=P[I+\delta Q-\tfrac{1}{2}QQ^++\tfrac{1}{2}QQ^+(I-4\delta Q^2)^{1/2}]P^{-1}. \end{aligned} \tag{26}$$

But the middle matrix in (26) is a diagonal matrix with entries equal to 1, if $q_{ii}=0$, and otherwise of the form

$$(\tfrac{1}{2}+\delta q_{ii})+\tfrac{1}{2}(1-4\delta q_{ii}^2)^{1/2}. \tag{27}$$

Now, by Lemma 4.3,

$$2\delta^{1/2}|q_{ii}|<1,$$

so that all the terms in expression (27) are positive. Hence, this matrix is positive definite and invertible. □

The spectral properties of the matrix S are described next.

Lemma 5.3. Let S be defined by (10). Then, the spectral radius $\rho(S)$ satisfies the inequality $\delta^{1/2}\rho(S)<1$. Moreover, $\rho(S)<1$ holds if and only if $\rho(M)<1/(1+\delta)$.

Proof. Since $M=PQP^{-1}$, for Q diagonal, we see by (10) that

$$\begin{aligned} \rho(S)&=\rho((1/2\delta)M^+(-I+(I-4\delta M^2)^{1/2}))\\ &=\rho((1/2\delta)PQ^+(-I+(I-4\delta Q^2)^{1/2})P^{-1})\\ &=(1/2\delta)\rho(Q^+)\rho(-I+(I-4\delta Q^2)^{1/2}). \end{aligned} \tag{28}$$

Now, if μ is a nonzero eigenvalue of S, then

$$|\mu|=(1/2\delta)|\lambda^{-1}|\,|-1+(1-4\delta\lambda^2)^{1/2}|, \tag{29}$$

for some nonzero eigenvalue λ of Q. But, as $\rho(Q)=\rho(M)$, then $2\delta^{1/2}|\lambda|<1$ holds by Lemma 4.3, which implies that

$$1-2\delta^{1/2}|\lambda|<(1-4\delta\lambda^2)^{1/2}.$$

Using this inequality in Eq. (29) proves the inequality

$$|\mu| < 1/\delta^{1/2}.$$

The second inequality for $\rho(S)$ is also based on Eq. (28). Assuming that $\rho(S) \neq 0$, then $\rho(S) < 1$ if and only if

$$1-(1-4\delta\rho(Q^2))^{1/2} < 2\delta\rho(Q).$$

But this inequality is equivalent to

$$(1-(1+\delta)\rho(Q))\rho(Q) > 0,$$

which proves the result, since $\rho(Q) = \rho(M)$ and $\rho(M) > 0$. □

Lemma 5.4. If R, S are defined by (9) and (10), then the sequences $\tilde{x}$ generated by the decision rule (8) belong to the space X.

Proof. Let $\alpha = \|S\|$. By Lemma 5.3, we can assume that $\alpha < \delta^{-1/2}$. Moreover, there exists some constant $\beta > 0$ such that $\alpha + \beta > 1$ and $\alpha + \beta < \delta^{-1/2}$, since $\delta < 1$. Next, let

$$T = \min\{t: \|x_t\| \geq \|R\|/\beta\},$$

noting that, whenever this inequality holds, then by (8),

$$\|x_{t+1}\| \leq \|S\| \, \|x_t\| + \|R\| < (\alpha+\beta)\|x_t\|.$$

On the other hand, if this inequality fails to hold, for some $t > T$, then

$$\begin{aligned}\|x_{t+1}\| &\leq \|S\| \, \|x_t\| + \|R\| \\ &\leq \|S\| \, \|x_T\| + \|R\|, \qquad \text{as } \|x_t\| < \|x_T\|, \\ &\leq (\alpha+\beta)\|x_T\| \\ &\leq (\alpha+\beta)^{(t-T+1)}\|x_T\|, \qquad \text{as } (\alpha+\beta) > 1.\end{aligned}$$

Upon combining these cases, we have

$$\begin{aligned}|||\tilde{x}|||^2 &= \sum_{t<T} \delta^t \|x_t\|^2 + \sum_{t \geq T} \delta^t \|x_t\|^2 \\ &\leq \sum_{t<T} \delta^t \|x_t\|^2 + \sum_{t \geq T} \delta^t ((\alpha+\beta)^2)^{(t-T)} \|x_T\|^2,\end{aligned}$$

which is finite, since $\delta(\alpha+\beta)^2 < 1$. □

Finally, we are now in a position to prove the main theorem.

Proof of Theorem 2.1. The existence of R, S such that the sequence $\tilde{x}^*$ generated by (8) satisfies the first-order condition and belongs to the space X is given by Lemma 5.1, 5.2, and 5.4. By the comment which follows Lemma 3.3, the optimality of $\tilde{x}^*$ is assured if the transversality condition (13) also holds, that is, if

$$\lim_{t\to\infty} \delta^t \frac{\partial r}{\partial x}(x_t^*, x_{t+1}^*)x_t = 0$$

holds for all sequences $\tilde{x} \in X$, including $\tilde{x}^*$. But this follows from the inequality

$$\left\|\delta^t \frac{\partial r}{\partial x}(x_t^*, x_{t+1}^*)x_t\right\| \leq \|\delta^{t/2}(Ax_t^* + Bx_{t+1}^* + D)\| \, \|\delta^{t/2}x_t\|,$$

where each term on the right tends to zero, since each of the sequences $\tilde{x}^*$ and $\tilde{x}$ are in X.

The uniqueness of the optimal sequence follows by part (iv) of Lemma 3.1. □

Thus far, we have established the existence of a unique optimal sequence, for every x_0, in the case of the discounted problems. This sequence can, by iterating (8) and using (9) to eliminate R, be expressed in the following form:

$$\begin{aligned} x_t &= (I+S+\cdots+S^{t-1})R + S^t x_0 \\ &= (I+S+\cdots+S^{t-1})(I+\delta M(I+S))^{-1}N + S^t x_0. \end{aligned} \tag{30}$$

The next lemma shows how this formula can be simplified for most problems.

Lemma 5.5. If M is nondegenerate, then Eq. (9) becomes

$$R = (I-S)(I+(1+\delta)M)^{-1}N.$$

Proof. The nondegeneracy assumption ensures that the matrix $(I+(1+\delta)M)$ is invertible. So it is sufficient to show that the vector R, as given above, satisfies Eq. (18). This is shown below, using the fact that the matrix S satisfies (19):

$$\begin{aligned} (I+\delta M(I+S))R &= (I+\delta M(I+S))(I-S)(I+(1+\delta)M)^{-1}N \\ &= (I+\delta M - S - \delta MS^2)(I+(1+\delta)M)^{-1}N \\ &= ((I+(1+\delta)M) - (\delta MS^2 + M + S)) \\ &\quad \times (I+(1+\delta)M)^{-1}N \\ &= N. \end{aligned}$$

□

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So, for nondegenerate problems, Eq. (30) can be simplified,

$$\begin{aligned} x_t &= (I+S+\cdots+S^{t-1})(I-S)(I+(1+\delta)M)^{-1}N+S^t x_0 \\ &= (I-S^t)(I+(1+\delta)M)^{-1}N+S^t x_0 \\ &= (I-S^t)x^*+S^t x_0. \end{aligned}$$

Thus,

$$x_t = x^*+S^t(x_0-x^*), \tag{31}$$

where the vector

$$x^* = (I+(1+\delta)M)^{-1}N \tag{32}$$

is variously known as the equilibrium, turnpike, or steady state.

Theorem 5.1. Let $\bar{x}^*$ be the optimal sequence, from some arbitrary x_0 in a nondegenerate problem. Then, x_t^* converges to the equilibrium state x^* if the spectral radius satisfies $\rho(M)<1/(1+\delta)$.

Proof. By Lemma 5.5, it follows that the equilibrium state x^* exists and that any optimal sequence can be expressed in the form of Eq. (31). Thus, x_t^* converges to x^*, for an arbitrarily given x_0, if $\rho(S)<1$. However, by Lemma 5.3, this is equivalent to the condition that $\rho(M)<1/(1+\delta)$. □

Note that, if $x_0=x^*$ and $\delta(M)<1/(1+\delta)$, then (31) implies that $x^*=Sx^*+R$ and that $\bar{x}^*=(x_t^*)=(x^*)$ is an optimal sequence.

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