Robust Optimization: Applications in Portfolio Selection Problems

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Outline

- Motivations
- Robust optimization: modeling and computational efficiency
- Applications
  - Mean-variance model
  - Sharpe ratio maximization

Motivations

Problem input data frequently suffers from estimation error, and sometimes even very small errors can render the "optimal" solution irrelevant (e.g. seriously infeasible).

e.g. Pilot 4 from NETLIB : constraint 372

$$\begin{aligned} a^T x = & -15.79081x_{826} - 8.598819x_{827} - 1.88789x_{828} - 1.362417x_{829} \\ & - 1.526049x_{830} - 0.031883x_{849} - 28.725555x_{850} - 10.792065x_{851} \\ & - 0.19004x_{852} - 2.757176x_{853} - 12.290832x_{854} + 717.562256x_{855} \\ & - 0.057865x_{856} - 3.785417x_{857} - 78.30661x_{858} - 122.163055x_{859} \\ & - 6.46609x_{860} - 0.48371x_{861} - 0.615264x_{862} - 1.353783x_{863} \\ & - 84.644257x_{864} - 122.459045x_{865} - 43.15593x_{866} - 1.712592x_{870} \\ & - 0.401597x_{871} + x_{880} - 0.946049x_{898} - 0.946049x_{916} \\ \geqslant & \; b = 23.387405 \end{aligned}$$

Motivations

Problem input data frequently suffers from estimation error, and because of this the "optimal" solution computed is far from optimal in reality.

e.g. Computation of efficient frontier from mean-variance model

- True asset returns μ and covariance matrix Q
- Estimate asset returns $\hat{\mu}$ and covariance matrix $\hat{Q}$

$$\max_x \ \mu^T x - \lambda x^T Q x \quad \text{s.t.} \quad \mathbb{1}^T x = 1 \quad (QP_\lambda)$$

Family of true optimal portfolio $\{x_\lambda : x_\lambda \text{ solves } (QP_\lambda),\ \lambda > 0\}$
True efficient frontier : $\{(\sqrt{x_\lambda^T Q x_\lambda^T}, \mu^T x_\lambda) : \lambda > 0\}$

(↑ What you would have obtained *if* you knew the true parameters μ and Q.)

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$$\max_x \ \hat{\mu}^T x - \lambda x^T \hat{Q} x \quad \text{s.t.} \quad \mathbb{1}^T x = 1 \quad (\hat{Q}P_\lambda)$$

Family of "optimal" portfolio $\{\hat{x}_\lambda : \hat{x}_\lambda \text{ solves } (\hat{Q}P_\lambda), \ \lambda > 0\}$

($\uparrow$ What you actually obtain based on the estimates $\hat{\mu}$ and $\hat{Q}$.)

Estimated efficient frontier : $\{(\sqrt{\hat{x}_\lambda^T \hat{Q} \hat{x}_\lambda^T}, \hat{\mu}^T \hat{x}_\lambda) : \lambda > 0\}$

(This efficient frontier is what you *think* you would get.)

Motivations

Problem input data frequently suffers from estimation error, and because of this the "optimal" solution computed is far from optimal in reality.

e.g. Computation of efficient frontier from mean-variance model

- True asset returns μ and covariance matrix Q
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$$\max_x \ \hat{\mu}^T x - \lambda x^T \hat{Q} x \quad \text{s.t.} \quad \mathbb{1}^T x = 1 \quad (\hat{QP}_\lambda)$$

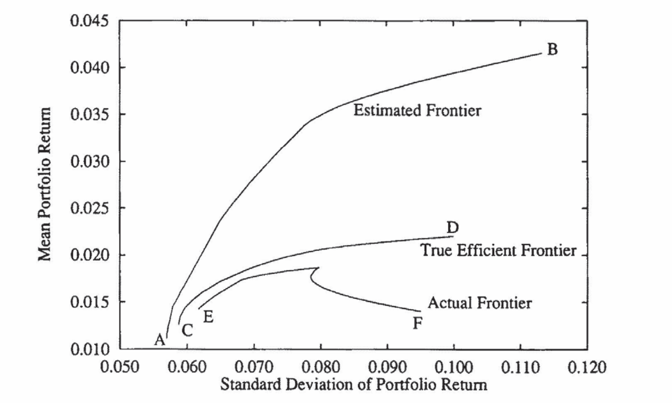
Family of "optimal" portfolio $\{\hat{x}_\lambda : \hat{x}_\lambda \text{ solves } (\hat{QP}_\lambda),\ \lambda > 0\}$

($\uparrow$ What you actually obtain based on the estimates $\hat{\mu}$ and $\hat{Q}$.)

Actual efficient frontier : $\{(\sqrt{\hat{x}_\lambda^T Q \hat{x}_\lambda^T}, \mu^T \hat{x}_\lambda) : \lambda > 0\}$

(This efficient frontier is what you *actually* get in reality!)

Motivations



Source: Computing Efficient Frontiers using Estimated Parameters, M. Broadie, 1993, Annals of Operations Research, Vol. 45, 21-58.

Motivations

- Problem input data frequently suffers from estimation error.
- Solving an optimization problem based on nominal data alone could produce some "optimal solution" that is irrelevant in reality.
- Even if feasibility is not an issue, the "optimal" solution obtained is very likely far from optimal.

Robust optimization: modeling

- Nominal LP problem:

$$\min_x \ \hat{c}^T x + \hat{d} \quad \text{s.t.} \quad \hat{A}x \leqslant \hat{b}$$

- Uncertain data $(\hat{A}, \hat{b})$ takes value in $\mathcal{U}_{A,b}$ (and $(\hat{c}, \hat{d})$ in $\mathcal{U}_{c,d}$)
- Robust counterpart:

$$\min_x \ \tilde{c}(x) \quad \text{s.t.} \quad Ax \leqslant b \quad \text{for all } (A, b) \in \mathcal{U}_{A,\,b}$$

where $\tilde{c}(x) := \sup_{(c,\,d) \in \mathcal{U}_{c,\,d}} c^T x + d$.

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where $\tilde{c}(x) := \sup_{(c,\,d) \in \mathcal{U}_{c,\,d}} c^T x + d$.

Robust optimization: modeling

- Nominal problem:

$$\min_x f(x;\hat{\eta}) \quad \text{s.t.} \quad g(x;\hat{\xi}) \leqslant 0$$

- Uncertain data $\hat{\eta}$ and $\hat{\xi}$ vary within $\mathcal{U}_\eta$ and $\mathcal{U}_\xi$ resp.
- Robust counterpart:

$$\min_x \tilde{f}(x) \quad \text{s.t.} \quad g(x;\xi) \leqslant 0 \text{ for all } \xi \in \mathcal{U}_\xi$$

where $\tilde{f}(x) := \sup_{\eta \in \mathcal{U}_\eta} f(x;\eta)$.

Robust optimization: modeling

- But what are the uncertainty sets?
- In finance, the uncertainty structure corresponds to the confidence region.
- So this problem is deterministic, but is a semi-infinite (and possibly non-smooth) programming problem!
- Infinite number of constraints poses computational difficulty.
- But constraints such as the last one can be dealt with efficiently.

Robust optimization: efficiency

An illustration in LP case:

- Consider an uncertain constraint

$$\hat{a}^T x \leqslant \hat{b} \quad \text{where } [\hat{a};\hat{b}] = \left\{ [a^0; b^0] + \sum_{l=1}^{L} \zeta_l [a^l; b^l] : \|\zeta\|_2 \leqslant 1 \right\}$$

- Robust counterpart:

$$(a^0)^T x + \sum_{l=1}^{L} \zeta_l [a^l]^T x \leqslant b^0 + \sum_{l=1}^{L} \zeta_l b^l \quad \forall \|\zeta\|_2 \leqslant 1$$

Robust optimization: efficiency

An illustration in LP case:

- Consider an uncertain constraint

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- Robust counterpart:

$$\left\| ([a^1]^T x - b^1, \ldots, [a^L]^T x - b^L)^T \right\|_2 \leqslant b^0 - (a^0)^T x$$

which is not a “hard” constraint to deal with.

Robust optimization: efficiency

Robust counterparts of some classes of non-linear programming problems and LP with different uncertainty sets may have a finite representation, and can be solved efficiently.

Mean-variance model

(λ-)parametric QP

$$\max_x \ \mu^T x - \lambda x^T Q x \quad \text{s.t.} \quad \mathbb{1}^T x = 1$$

where		(assuming n stocks are available to choose)
$x \in \mathbb{R}^n$	:	proportion of investment on the available assets
$\mu \in \mathbb{R}^n$	:	expected return of the available assets
$Q \in \mathbb{R}^{n\times n}$	:	covariance matrix of the available assets
$\lambda > 0$	:	risk aversion parameter

Mean-variance model

(μ_0-)parametric QP

$$\min_x \; x^T Q x \quad \text{s.t.} \quad \mu^T x \geqslant \mu_0 \,,\; \mathbb{1}^T x = 1$$

where		(assuming n stocks are available to choose)
$x \in \mathbb{R}^n$	:	proportion of investment on the available assets
$\mu \in \mathbb{R}^n$	:	expected return of the available assets
$Q \in \mathbb{R}^{n\times n}$	:	covariance matrix of the available assets
$\mu_0 \in \mathbb{R}$	:	minimum expected return guarantee

Mean-variance model

Uncertain (μ_0-)parametric QP

$$\min_x \; x^T \hat{Q} x \quad \text{s.t.} \quad \hat{\mu}^T x \geqslant \mu_0 \, , \; \mathbb{1}^T x = 1$$

Robust counterpart

$$\min_x \left\{ \max_{Q \in \mathcal{U}_Q} \; x^T Q x \; : \; \mu^T x \geqslant \mu_0 \quad \forall \, \mu \in \mathcal{U}_\mu \, , \; \mathbb{1}^T x = 1 \right\}$$

$\mathcal{U}_Q$	:	uncertainty set for Q
$\mathcal{U}_\mu$	:	uncertainty set for μ

Mean-variance model

Robust counterpart:

$$\min_{x} \left\{ \max_{Q \in \mathcal{U}_Q} x^T Q x \ : \ \mu^T x \geqslant \mu_0 \quad \forall \mu \in \mathcal{U}_\mu \, , \ \mathbb{1}^T x = 1 \right\}$$

Some special cases:

- Q is certain :

$$\mathcal{U}_\mu = \{\mu : \hat{\mu} - \delta \leqslant \mu \leqslant \hat{\mu} + \delta\}$$

$$\Longrightarrow \min_{x} \ x^T Q x \quad \text{s.t.} \quad \mu^T x - \delta^T |x| \geqslant \mu_0 \, , \ \mathbb{1}^T x = 1$$

Mean-variance model

Robust counterpart:

$$\min_{x} \left\{ \max_{Q \in \mathcal{U}_Q} x^T Q x \; : \; \mu^T x \geqslant \mu_0 \quad \forall \mu \in \mathcal{U}_\mu \, , \; \mathbb{1}^T x = 1 \right\}$$

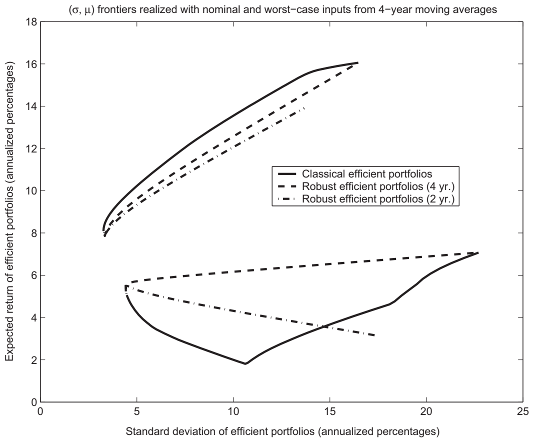
Some special cases:

- Q is certain :

$$\mathcal{U}_\mu = \{\mu : (\mu - \hat{\mu})^T Q^{-1} (\mu - \hat{\mu}) \leqslant \chi^2\}$$
$$\Longrightarrow \min_{x} \; x^T Q x \quad \text{s.t.} \quad \mu^T x \geqslant \mu_0' \; , \; \mathbb{1}^T x = 1$$

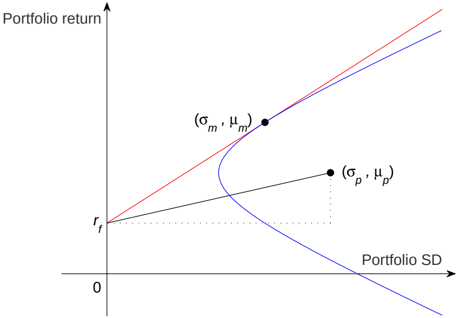
for some $\mu_0' \leqslant \mu_0$.

Mean-variance model



Source: Robust Asset Allocation, R.H. Tütüncü and M. Koenig. Annals of Operations Research, 132, 2004.

Sharpe ratio maximization



$$\max_x \frac{\mu^T x - r_f}{\sqrt{x^T Q x}} \quad \text{s.t.} \quad \mathbb{1}^T x = 1$$

(r_f = risk free rate)

Sharpe ratio maximization

Nominal problem:

$$\max_x \frac{\hat{\mu}^T x - r_f}{\sqrt{x^T \hat{Q} x}} \quad \text{s.t.} \quad \mathbb{1}^T x = 1 \quad , \quad x \geqslant 0$$

Robust counterpart:

$$\max_x \left\{ \min_{\mu,\, Q} \frac{\mu^T x - r_f}{\sqrt{x^T Q x}} \quad \text{s.t.} \quad \mu \in \mathcal{U}_\mu \,,\; Q \in \mathcal{U}_Q \right\} \quad \text{s.t.}\; \mathbb{1}^T x = 1 \quad , \quad x \geqslant 0$$

It can be shown that the above max-min problem can be reduced to a SOCP (with $2n + 6$ variables and 8 constraints).

Sharpe ratio maximization

Advantages of robustification:

- Lower turnover [Ceira]
  - At 95% confidence level, turnover drops by 4%.
  - At 99% confidence level, turnover drops by 7%.
- This indicates a lower aggregate transaction cost.
- Higher terminal wealth [Goldfarb and Iyengar]
  - At 95% confidence level, final wealth is 40% higher.
  - At 99% confidence level, final wealth is 50% higher.

Summary

- Robust optimization can give uncertainty-immune solutions
- Many robust optimization problems can be solved efficiently
- Robust optimization can producing “stable” solutions
  $\Longrightarrow$ lower turnover rate
  $\Longrightarrow$ suitable for long term planning