1. (Errors in numerical computation.)

With $x = 101$ and $y = 100$, you are asked to calculate $z = x^2 - y^2$.

a) Calculate $z = x^2 - y^2$ in the floating point number system $F_1(b = 10, m = 5, e = 3)$ (assume chopping).

b) Calculate $z = x^2 - y^2$ in the floating point number system $F_2(b = 10, m = 3, e = 3)$ (assume chopping). Show your work and explain why the result is not the same as in a).

c) Devise a different, more stable method to calculate $z = x^2 - y^2$, and show that using this method you can obtain the exact result for $z$ when calculating in floating point system $F_2(b = 10, m = 3, e = 3)$.

2. (Interpolation.)

The piecewise polynomial function $y(x)$ is defined as

$$y(x) = \begin{cases} 
ax^3 + bx^2 - 9/2x + c & \text{for } x \in [-3,-1] \\
x^3 & x \in [-1,0] \\
dx^3 + ex^2 & x \in [0,1] \\
-x^3 + 6x^2 - 6x + f & x \in [1,2]
\end{cases}$$

Can the coefficients $a, b, c, d, e$ and $f$ be determined such that $y(x)$ is a natural cubic spline? If so, determine the coefficients $a, b, c, d, e$ and $f$. 
3. (Interpolation. Integration.)

a) Given \( f_0, f_1, f_1' \) in points \( x_0 = 0 \) and \( x_1 = h \), determine the coefficients \( a, b \) and \( c \) of the interpolating polynomial \( y(x) = a(x - h) + b(x - h)^2 + c \) that satisfies \( y(x_0) = f_0, y(x_1) = f_1 \) and \( y'(x_1) = f_1' \).

b) Derive an integration rule for \( I = \int_0^h f(x)dx \) by integrating the interpolating polynomial \( y(x) \). Express your result as \( \hat{I} = w_0 f_0 + w_1 f_1 + w_2 f_1' \) (determine \( w_0, w_1, w_2 \)).

c) Using the integration rule \( \hat{I} \), derive a composite integration rule \( \hat{I}_{\text{composite}} \) for \( I = \int_a^b f(x)dx \), using \( n+1 \) equidistant points \( x_i = a + i(b - a)/n \) (\( i = 0, \cdots, n \)), with \( h = (b - a)/n \). Find the weights of \( f_i(i = 0, \cdots, n) \), and \( f_i'(i = 1, \cdots, n) \) in \( \hat{I}_{\text{composite}} \).

4. (Differentiation.)

a) Given \( f_0 \) and \( f_1 \) in points \( x_0 \) and \( x_1 \), determine \( a \) and \( b \) in \( \hat{f}_0' = af_0 + bf_1 \) such that \( \hat{f}_0' \) approximates \( f_0' \) as accurately as possible. Determine the leading order term of the truncation error.

b) Given \( f_0, f_1, \) and \( f_1' \) in point \( x_0 \) and \( x_1 \), determine \( a, b \) and \( c \) in \( \hat{f}_0' = af_0 + bf_1 + cf_1' \) such that \( \hat{f}_0' \) approximates \( f_0' \) as accurately as possible. Determine the leading order term of the truncation error.

5. (Integration.)

Given \( \{ (x_i, f_i = f(x_i)) \}_{i=0}^n \) with \( x_i \neq x_j \) when \( i \neq j \), derive an integration rule \( \hat{I} = \sum_{i=0}^n w_i f_i \) for approximating \( I = \int_a^b f(x)dx \) with degree of precision \( n \). (This means that the functions \( f(x) = 1, f(x) = x, f(x) = x^2, \cdots, f(x) = x^n \) are integrated exactly). Show that the weights \( w_i \) can be found by solving a matrix equation \( A \cdot w = r \), with \( w \) the vector containing the weights \( w_i \). Find expressions for the matrix \( A \) and the right hand side \( r \). Discuss existence and uniqueness of the solution \( w \).