[15] 1. (Errors in numerical computation.)

(a) Given \( p_0 = 1/3 \) and \( p_1 = 1/5 \), a recurrence relation for calculating \( p_n \) is given by

\[
p_n = \frac{5}{6}p_{n-1} - \frac{1}{6}p_{n-2} \quad (n \geq 2).
\]

Assume that rounding errors occur in assigning the floating point values \( \hat{p}_0 \) and \( \hat{p}_1 \), but that no further errors occur in the subsequent calculations. Analyse the propagation of the absolute error \( \Delta p_n = p_n - \hat{p}_n \). Is the recurrence relation stable with respect to the absolute error?

(b) On a base – 2 machine, the distance between 7 and the next larger floating point number is \( 2^{-12} \). What is the (exact) distance between 71 and the next largest floating point number?

[15] 2. (Interpolation.)

(a) Let \( f(x) = x^6 - 3x^5 + 7x^4 - 3x^3 + 2x^2 + x - 1 \).

Let \( p_5(x) \) be the polynomial of degree 5 that interpolates \( f(x) \) in the points \( x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4 \) and \( x_5 = 5 \). What is the leading coefficient of \( p_5(x) \)?

(Hint: Consider the error formula \( E_5(x) = f(x) - p_5(x) \). You should be able to compute this precisely without constructing \( p_5(x) \).)

(b) With \( l_i(x), i = 0, \cdots, n \) being the Lagrange polynomials, for which values of \( m \) is it true that \( \sum_{k=0}^{n} x_k^m l_k(x) = x^m \). Explain why.
3. (Differentiation.)  

(a) It is known that \( \hat{y}'_0(h) = \frac{y_h - y_0}{h} \) is an approximation for \( y'_0 \) with truncation error \( T(h) = ch + O(h^2) \). Find a formula that approximates the truncation error \( T(h) \), given two approximations \( \hat{y}'_0(h) \) and \( \hat{y}'_0(2h) \) of \( y'_0 \). (Hint: assume that \( T(h) = ch \).)

(Integration.)

(b) Let \( f(x) = \frac{1}{x} \). What is the fewest number of intervals and function evaluations necessary to approximate \( I = \int_1^2 f(x)dx \) to within an error that is guaranteed to be less than \( \frac{24}{2880} \cdot 10^{-4} \), using the Composite Simpson rule? (Hint: recall that the global truncation error for the Composite Simpson rule is bounded as follows: \( |T_{\text{global}}| \leq \frac{(b-a)}{2880} h^4 \max_{\zeta \in [a,b]} |f^{(4)}(\zeta)| \), with \( h = \frac{b-a}{n} \)).

4. (Fourier Methods.)

(a) Calculate the DFT of \( (f[n]) = (3, 2, 1) \).

(b) The convolution of two time domain functions \( h(t) \) and \( u(t) \) is given by \( y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \). Show that the Fourier Transforms \( Y(q), H(q) \) and \( U(q) \) satisfy the relation \( Y(q) = H(q) \cdot U(q) \).

5. (Fourier Methods.)

Let the DFT of \( f[n] \) be given by \( F[k] \).

Find the DFT \( G[k] \) of time series \( g[n] = f[n] \cdot (-1)^n \), in terms of \( F[k] \). (Hint: \( G[k] \) is related to \( F[k] \) by a shift in the frequency domain.) (Assume that the length \( N \) of time series \( f[n] \) is even.)

6. (Linear Algebra.)

Find a permutation matrix \( P \), and unit - lower and upper triangular matrices \( L \) and \( U \), such that \( P \cdot A = L \cdot U \), for

\[
A = \begin{bmatrix}
0 & 1 & 2 \\
1 & 2 & 0 \\
0 & 2 & 0
\end{bmatrix}.
\]

7. (Linear Algebra.)

(a) Show that if \( A \) is orthonormal, then the two-norm condition number \( \kappa_2(A) = 1 \).

(b) Assume that you are given the decomposition \( A = L \cdot U \). Describe how you can use the decomposition to solve the system \( A^2 \bar{x} = \bar{b} \) in an efficient way. What is the computational complexity in terms of floating point operations? (Specify the coefficient of the highest-order term precisely.)