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Overcompressive shocks and compound shocks in 2D and 3D magnetohydrodynamic flows

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Abstract. It is shown by numerical simulation that overcompressive shocks and compound shocks arise naturally in two-dimensional and three-dimensional magnetohydrodynamic bow shock flows with strong upstream magnetic fields. The stability of these kinds of shocks has been debated for a long time, but the simulation results presented show clearly that they arise in realistic flows, confirming recent theoretical stability results.

1. Introduction

Shocks are formed in fluid flows described by hyperbolic systems of nonlinear conservation laws when one or more families of characteristics converge (Fig. 1) [16]. Consider a hyperbolic system with n equations for n conserved quantities, such that in every state n characteristics can be drawn in the x, t diagram (Fig. 1). For strictly hyperbolic systems — i.e. systems for which the wave speeds do not

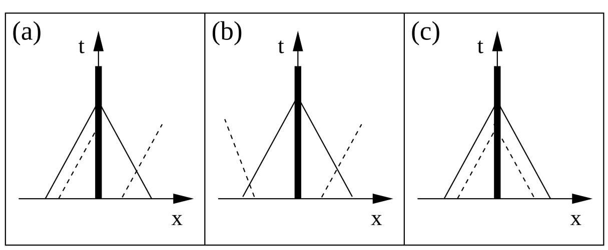

FIGURE 1. Possible shock types in a $n = 2$ hyperbolic system. The shock is thick solid, and the two families of characteristics are thin solid and thin dashed. (a) Lax shock. (b) Undercompressive shock. (c) Overcompressive shock.

coincide —, small-amplitude shocks necessarily have to be Lax shocks [12]. Lax shocks are defined by two conditions. First, exactly $n+1$ characteristics originating from the states on the left and the right of the shock converge into the shock, and second, of the converging characteristics there is exactly one pair belonging to the same characteristic family. For strictly hyperbolic systems these two conditions are equivalent. Undercompressive shocks are shocks in which less than $n+1$ characteristics converge, while more than $n+1$ characteristics converge in overcompressive shocks. In systems in which only the wave speeds of the linear modes may coincide, e.g. the Euler system, all shocks necessarily have to be Lax shocks as well. Non-strictly hyperbolic systems, of which the magnetohydrodynamic (MHD) system [15] is a well-known example, can exhibit small-amplitude undercompressive and overcompressive shocks [10]. It is generally found that Lax shocks are stable and can consequently arise in flow solutions described by the hyperbolic equations, but it turns out that undercompressive and overcompressive shocks are often rather unstable — in a sense to be explained below. The problem of the stability of undercompressive and overcompressive shocks has been widely studied during the last decades, both in the mathematics and the physics communities. In the present paper we give an overview of the occurrence of overcompressive shocks in two-dimensional (2D) and three-dimensional (3D) MHD bow shock flows that have been described in the physics literature in the last few years [3, 4, 6]. We explain how these simulation results confirm recent theoretical results that prove stability of overcompressive MHD shocks when dissipation mechanisms are taken into account [9]. At the same time these 2D and 3D simulation results confirm and generalize results of earlier numerical simulations that where restricted to flows in one spatial dimension (1D) with coplanarity of left and right states explicitly imposed (e.g., [19]). The present paper is to be seen as an extension of our previously reported work on overcompressive shocks in 2D MHD flows [5], and in fact puts into the right daylight some issues that were not very clearly described in our earlier papers on this subject [3, 4, 5], thus reflecting our increased understanding of this complex matter. Compound shocks, which in general can arise for wave modes of a hyperbolic system for which the flux function is not convex [16], occur in the same MHD bow shock flows. MHD compound shocks were observed in earlier 1D simulation results [2, 19], and again our results in 2D and 3D confirm and generalize these findings. Due to space constraints, we cannot discuss compound shocks in full detail in the present paper, but in stead we refer the reader to [3, 4, 5].

2. MHD shocks

The equations of single-fluid MHD [15] in conservative form are given by

$$\frac{\partial}{\partial t}\left[\begin{array}{c}\rho\\ \rho\,\vec{v}\\ \vec{B}\\ e\end{array}\right]+\nabla\cdot\left[\begin{array}{c}\rho\,\vec{v}\\ \rho\,\vec{v}\vec{v}+\mathsf{I}\left(p+\vec{B}\cdot\vec{B}\,/2\right)-\vec{B}\vec{B}\\ \vec{v}\vec{B}-\vec{B}\vec{v}\\ \left(e+p+\vec{B}\cdot\vec{B}\,/2\right)\vec{v}-\left(\vec{v}\cdot\vec{B}\right)\vec{B}\end{array}\right]=\mathsf{D}, \qquad (1)$$

supplemented with the divergence free condition $\nabla\cdot\vec{B}=0$ as an initial condition. In this $n=8$ system ρ and p are the plasma density and pressure respectively, $\vec{v}$ is the plasma velocity, $\vec{B}$ the magnetic field, and $e=p/(\gamma-1)+\rho\,\vec{v}\cdot\vec{v}/2+\vec{B}\cdot\vec{B}/2$ is the total energy density of the plasma. I is the unity matrix. The right hand side D contains dissipative terms [15, 19, 9], which may include resistivity, viscosity and heat conduction. The magnetic permeability $\mu=1$ in our units. We take $\gamma=5/3$ for the adiabatic index. MHD allows for three different anisotropic wave modes superposed on the plasma advection, namely the fast, the Alfvén and the slow wave. The phase speeds of the three MHD waves in propagation direction x are denoted by c_{fx}, c_{Ax} and c_{sx}, and the respective Mach numbers are defined by $M_{fx}=|v_x|/c_{fx}$, $M_{Ax}=|v_x|/c_{Ax}$ and $M_{sx}=|v_x|/c_{sx}$. The eigenvalues of the MHD flux Jacobian are given by v_x-c_f, v_x-c_A, v_x-c_s, v_x, v_x+c_s, v_x+c_A, and v_x+c_f, and determine the slopes of the characteristics in every state in the x,t diagram. Corresponding to the three types of waves, the MHD equations allow for three different types of shocks, namely the fast, intermediate and slow shocks. All MHD shocks have the property of coplanarity, which means that the downstream magnetic field lies in the plane defined by the upstream magnetic field and the shock normal. The three types of shocks connect plasma states which are labeled from 1 to 4, with state 1 super-fast ($v_n>c_{fn}$ in the shockframe, with n the direction of the shock normal), state 2 sub-fast but super-Alfvénic, state 3 sub-Alfvénic but super-slow, and state 4 sub-slow. Here we follow the notation that is customary in the physics literature, while mathematicians usually use a numbering from 0 to 3. The fast 1–2 shock refracts the magnetic field away from the shock normal. A limiting case of the fast shock is the 1–2=3 switch-on shock — where 2=3 means that $v_n=c_{An}$ downstream —, for which the upstream magnetic field is parallel to the shock normal, while the magnetic field makes a finite angle with the shock normal in the downstream state. The tangential component of the magnetic field is thus switched on. Intermediate shocks (1–3, 1–4, 2–3 and 2–4) bring a super-Alfvénic upstream plasma to a sub-Alfvénic downstream state, while the magnetic field is flipped over the shock normal — the tangential component of the magnetic field changes sign —, and the Alfvénic Mach number jumps from an upstream value above unity to a downstream value below one. The slow 3–4 shock refracts the magnetic field towards the shock normal.

An important distinction has to be made between the properties of *full* MHD with three vector components in 1D, 2D or 3D space on one hand, and the reduced

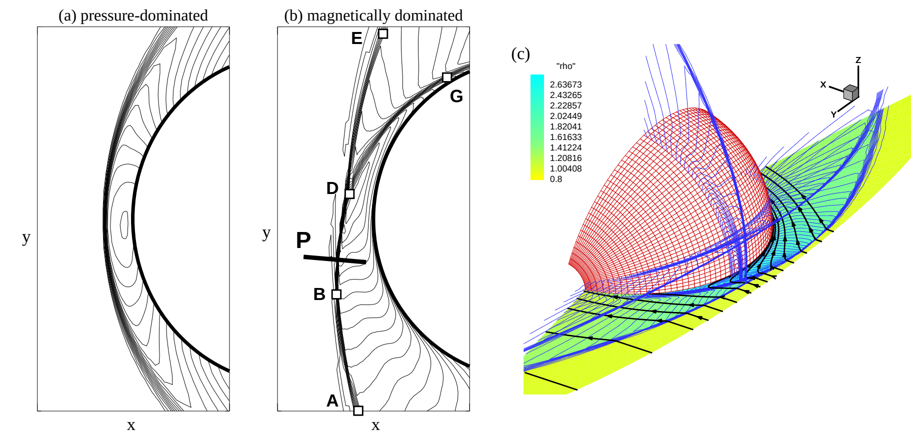

FIGURE 2. (a–b) 3D bow shock flows over a sphere (thick solid). The flow comes in from the left. Density contours (thin solid) are shown in the xy symmetry plane that is parallel to the uniform upstream magnetic and velocity fields. The incoming magnetic field is aligned with the x-axis. (a) Pressure-dominated flow: $M_{Ax} = 3.985$, $\beta = 0.4$, $\theta_{vB} = 5°$. (b) Magnetically dominated flow: $M_{Ax} = 1.5$, $\beta = 0.4$, $\theta_{vB} = 3.8°$. (c) 3D visualization of a magnetically dominated flow.

system of *planar* MHD with two vector components in 1D or 2D space, which is obtained by putting $B_z \equiv 0$ and $v_z \equiv 0$ in a Cartesian coordinate system. In the latter $n = 6$ system the Alfvén waves drop out. It is interesting to note that the intermediate shocks, however, still arise in planar MHD. This shows that the intermediate shocks are in fact to be associated as much with the fast and slow wave modes as with the Alfvén mode [19].

3. MHD bow shock flows

The overcompressive MHD shocks we report on in this paper were found in simulation results of stationary MHD bow shock flows around perfectly conducting spheres in 3D [6] and around perfectly conducting cylinders in 2D [3, 4, 5]. The 2D problem, for which the magnetic field necessarily has to be aligned to the plasma flow, has two free parameters. We choose the upstream plasma $\beta = 2p/B^2$, and the Alfvénic Mach number M_{Ax} along the upstream magnetic field lines. The 3D problem has one additional free parameter, for which the angle θ_{vB} between the upstream velocity field and magnetic field is chosen. We evolve the time-dependent ideal MHD equations until a steady state solution is obtained using a second order accurate conservative finite volume shock capturing scheme [3]. Numerical dissipation plays a role analogous to a small physical dissipation. Future simulations with explicit control of the dissipation [19, 8] are necessary to confirm that the bow shock topologies to be presented in this paper occur generally for large ranges of the dissipation coefficients, even though we have obtained our present results using various grid resolutions and numerical schemes — i.e. various effective dissipations.

For the 3D problem Fig. 2a shows that for an upstream flow with a *weak* magnetic field — to be made more specific below — a bow shock flow is obtained with a single shock front. This is the classical bow shock topology which is well-known from hydrodynamic bow shocks, and which until recently was believed to arise for all fast MHD bow shock flows as well. Figs. 2b and 2c, however, show that for an upstream flow with a *strong* magnetic field the leading bow shock front is followed by a secondary shock front. In the following Section it is shown that this new bow shock topology contains overcompressive shock segments. This previously unknown complex bow shock topology arises for bow shock flows when the uniform upstream plasma state satisfies the conditions that

$$B^2 > \gamma p \quad \text{and} \quad \rho v_x^2 > B^2 > \rho v_x^2 \frac{\gamma - 1}{\gamma(1-\beta)+1}, \tag{2}$$

with x the direction along the upstream magnetic field. These conditions are satisfied when the upstream magnetic field is strong and thermal and dynamical pressure effects are dominated by magnetic effects. We call a state satisfying these conditions magnetically dominated, as opposed to pressure-dominated.

The corresponding bow shock problem in planar 2D MHD is the flow around a conducting cylinder. Fig. 3 shows that for this planar flow similar phenomena occur. When the upstream magnetic field is weak such that conditions (2) are not satisfied, a regular single-front bow shock is obtained (Fig. 3a). When the magnetic field is strong enough for the conditions (2) to be satisfied, a complicated multiple-front bow shock flow develops (Fig. 3b). The topology of this flow is sketched in Fig. 3c. The horizontal line through the center of the cylinder is now a line of top-bottom symmetry because $\theta_{vB} \equiv 0$ in the steady 2D case. Both above and below this line of symmetry the topology of the 3D solution (Fig. 2b) can be observed, and near the line of symmetry additional secondary discontinuities have developed. It has to be noted that numerical experiments indicate that this 2D flow solution around a cylinder is only marginally stable, because any non-zero angle θ_{vB} imposed at the boundary makes the flow evolve into a non-symmetrical time-dependent pattern in which the secondary discontinuities are swept upwards or downwards — depending on the sign of θ_{vB} —, such that only one secondary shock is retained in a configuration resembling the 3D topology of Fig. 2b. The symmetrical solution of Fig. 3b can numerically only be obtained when the top-bottom symmetry at the horizontal line through the center of the cylinder is explicitly imposed in the computations.

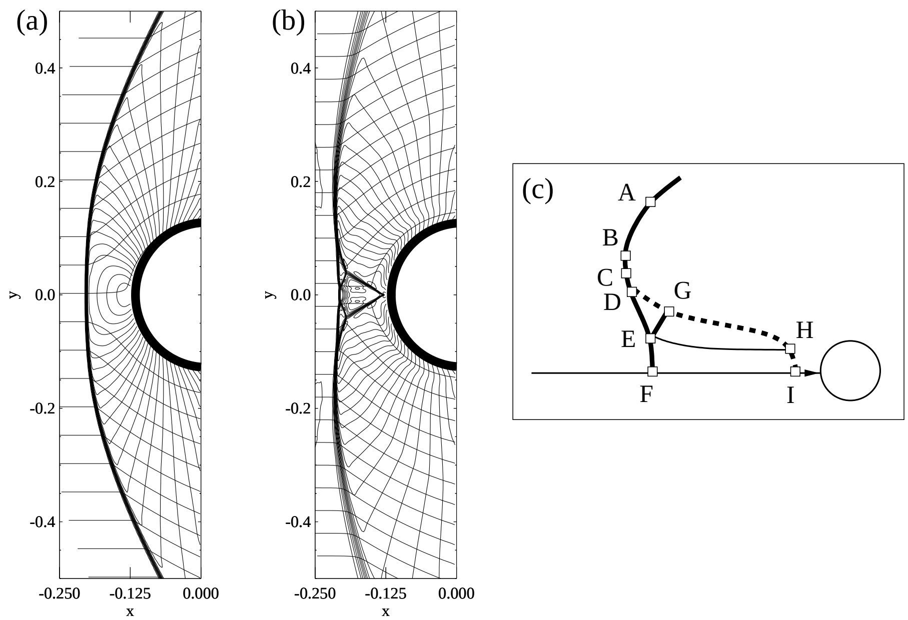

FIGURE 3. 2D planar bow shock flows around a cylinder (thick solid). The flow comes in from the left. Density contours and magnetic field lines (thin solid) are shown. (a) Pressure-dominated flow: $M_{Ax} = 2$, $\beta = 0.4$. (b) Magnetically dominated flow: $M_{Ax} = 1.5$, $\beta = 0.4$. (c) Topology of the magnetically dominated flow above the horizontal line of symmetry.

4. Overcompressive shocks

In the dissipationless case overcompressive shocks have since long been known to be unstable as they break up instantaneously upon arbitrary small perturbations [1, 15]. This can be understood intuitively by conservation considerations [13, 14], and corresponds to the fact that the perturbation problem of the shock using linearized equations is overdetermined [12]. Shocks for which the linearized perturbation problem is overdetermined (or underdetermined) were initially termed 'non-evolutionary' [1, 15], and they were later also called 'not linearized stable (not LS)' (e.g., [10]). Recent study has shown that there exist several very distinct types of overcompressive, not LS shocks. One type of overcompressive shocks, which we call shocks of type A [13], split up into two shocks that move with the *same* speed as the original shock, and remain thus stable in the L1 norm, as smaller perturbations lead to smaller separation between the two newly formed shocks and the unperturbed shock. Another type of overcompressive shocks, which we call type

B shocks [14], behaves very differently. Upon perturbation type B shocks split up into two shocks that move with *different* speeds. Type B shocks are thus not L1 stable, as even very small perturbations make the shocks separate over increasing distances for increasing time. It has been proven that type B shocks generically arise in non-strictly hyperbolic systems with rotationally equivariant flux function [8]. The full 3D MHD system has such a rotationally equivariant flux function with the Alfvén wave acting as the rotational wave. Overcompressive shocks of type B do thus arise in full MHD. The planar MHD system does not have an Alfvén mode: the flux is not rotationally equivariant, and overcompressive shocks of type B do not arise in planar MHD.

Recently is has been shown in convincing numerical examples [19] — including 1D simulations — and by rigorous mathematical proof [14, 9, 18, 10] that overcompressive MHD shocks can be stable when dissipation is included. The overdeterminacy of the perturbation problem can be removed by taking into account the degrees of freedom offered by the dissipative shock profile. It had in fact been realized a long time ago that dissipation could stabilize overcompressive shocks in the sense that stable dissipative profiles could be formed (e.g., [11]), but the significance of these effects for the possible occurrence of overcompressive shocks in realistic flows had not been realized before the more recent studies referred to above. In particular, it has been shown that in $n = 2$ model systems overcompressive shocks of type A are stable against perturbations with arbitrary mass — the mass of a perturbation is defined as its amplitude integrated in space — for suitable forms of the dissipation [13]. Type B shocks, however, are only conditionally stable for a diagonal dissipation matrix: if the mass of the rotational component of the perturbation exceeds a critical value, the shock splits up [14]. This critical value is typically quite small. Even more peculiar is that this critical value vanishes for vanishing dissipation strength. For more complicated forms of the dissipation matrix and for $n > 2$ systems dissipative profiles do not necessarily exist for all the values of the dissipative parameters, and bifurcations separate the parameter ranges for which the overcompressive shocks are stable from unstable parameter ranges. The full MHD system is characterized by such a complex behavior [9], with the overcompressive MHD shocks being the 1–3, 1–4 and 2–4 intermediate shocks. Although much understanding has been gained in recent years, it is realized that the precise influence of dissipation mechanisms and magnitudes on the stability of intermediate MHD shocks is very complicated, and the analysis of MHD shock stability remains incomplete. Especially the vanishing viscosity limit, and the stability of the 2–3 intermediate shock — which is undercompressive —, remain points of consideration. What is clear, however, is that the classical linearized stability analysis alone is not sufficient to investigate the full picture of the stability behavior of overcompressive shocks. Although after many years of debate this has been acknowledged by most researchers, attempts are still being made to explain overcompressive shock stability solely based on linear ‘evolutionarity’ analysis [17], and unfortunately such approaches keep on sending out confusing signals.

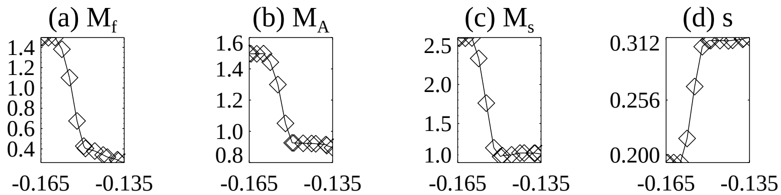

FIGURE 4. Normal Mach numbers and entropy along cut P in Fig. 2b.

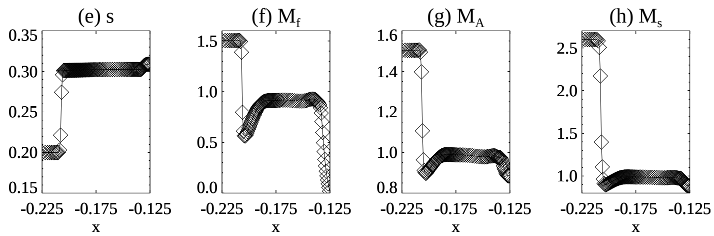

FIGURE 5. Plots along the horizontal line of symmetry for the 2D planar flow of Fig. 3b.

In full 3D MHD, the 1–3, 2–4 and 1–4 intermediate MHD shocks are overcompressive. For example, both a fast and an Alfvén characteristic family converge into 1–3 shocks, or, equivalently, both the fast and the Alfvénic Mach number jump from above one to below one over a 1–3 shock. Alfvén and slow characteristics converge into a 2–4 intermediate shock. The 1–4 intermediate shock has three converging characteristic families, one fast, one Alfvén, and one slow. Each of these three types of overcompressive shocks is of the rotational type B [14, 8]. Detailed analysis of the 3D magnetically dominated bow shock flows shows that many of the MHD shock types can be identified in the simulation result of Fig. 2b [6]. Shock fronts AB and DE are 1–2 fast, and BD is 1–3 intermediate. The Mach number profiles in Fig. 4 show convincingly that BD is indeed an overcompressive 1–3 intermediate shock. The secondary shock segment DG is 2–4 intermediate, evolving into 3–4 slow along the front [6]. The substantial difference between pressure-dominated and magnetically dominated MHD bow shock topologies can be explained in terms of the geometrical properties of MHD shocks [3, 5, 6].

In planar 2D MHD, the Alfvén waves are not present. Therefore the 1–3 shock is not overcompressive in this case, as only a family of fast characteristics converges into the shock, which is thus a regular fast Lax shock. The 2–4 shock is a regular slow Lax shock. The 1–4 shock, however, remains overcompressive: both a fast and a slow family of characteristics converge into the shock. The 1–4 shock is an overcompressive shock of type A for the planar system, as the rotational Alfvén wave is not present. Detailed analysis of the planar 2D magnetically dominated bow shock flow simulation results shows that many of the planar MHD shock types can be identified in the simulation result of Fig. 3b [3, 4, 5]. In the leading front, shock AB is a 1–2 fast shock, BCD is 1–3 intermediate, DE is 1–2 fast, and EF is 1–4 intermediate. The second front DGHI is 2–4 intermediate (evolving into 3–4 slow along the front). EG is a 1=2–3=4 intermediate shock. It is a double steady compound shock [18, 4]. For the full 3D bow shock problem of Fig. 2b the double compound structure also arises, but only as an intermediate feature during the time-dependent evolution towards the steady state. EH is a tangential discontinuity, and tangential discontinuities also stretch out from points D, G, and H, along the streamlines towards infinity. The 1–3 and 2–4 shocks are not overcompressive, but the 1–4 shock is overcompressive (Fig. 5).

5. Conclusion

It has been shown by numerical simulation that overcompressive shocks and compound shocks arise naturally in planar 2D and full 3D MHD bow shock flows [3, 4, 5, 6]. These types of shocks had been found before in 1D simulations of full MHD with three vector components [19]. However, 1D simulations are of limited generality because coplanarity of left and right states has to be imposed explicitly in order to obtain persistent intermediate shocks. It was not immediately clear that such coplanar left and right states would occur naturally in real flows at many locations and for long enough times for the overcompressive shocks to be formed. In our 3D bow shock simulations coplanarity is not explicitly imposed, but intermediate shocks *naturally* and *necessarily* arise in this flow because of topological reasons [6, 3, 5]. Our 3D results are thus not only the first clear confirmation of the occurrence of overcompressive MHD shocks in 3D, but also show that their occurrence is natural in realistic flows, and that configurations in which coplanar intermediate shocks arise are not exceptional. It can thus be expected that overcompressive MHD shocks may be found in real physical plasma flows as well. Traces of intermediate shocks have been identified in observations of space plasmas [6]. It has also to be noted that in most physical plasmas small wave perturbations possibly of a turbulent nature are omnipresent [6, 7, 17]. Those wave perturbations may inhibit the formation of the conditionally stable intermediate MHD shocks. Simulations have shown that if such perturbations destroy the intermediate shock segments in 3D bow shock flows, the shock segments of intermediate type are subsequently dynamically *reformed* in this driven problem, for the same topological reasons that caused their initial formation [7]. This reformation would not occur in the 1D settings that were considered in earlier simulations [19], but requires a driven 3D context. The fact that intermediate shocks are unstable against large enough perturbations does thus not necessarily preclude their presence ‘at large times’ in plasma flows with perturbations [7]. Our 2D and 3D simulation results are in full agreement with the theoretical results that prove (conditional) stability of overcompressive MHD shocks when dissipation mechanisms are taken into account [9, 10]. This is a nice confirmation of that theory, and also indicates that the sometimes abstract 1D setting of the theoretical stability problem bears direct relevance to realistic 2D and 3D flows.

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