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The Arab Spring: A Simple Compartmental Model for the Dynamics of a Revolution

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Abstract

We introduce a simple compartmental model for the dynamics of a revolution in dictatorial regimes that employ censorship and police repression. A defining property of the model is the use of visibility and policing terms that feature rapid transitions as a function of the size of the revolution, for which we provide conceptual and network-based mathematical justifications. The complete mathematical classification of the dynamical behaviour of the model leads to a division in parameter space that is interpreted naturally in terms of stability of the regime (stable police state, meta-stable police state, unstable police state, and failed state). We show that these dynamical properties of the model are generic for a broad class of visibility and policing functions that feature rapid transitions. We investigate how the model can be applied to the peaceful revolutions of the Arab Spring in Tunisia and Egypt, taking into account the influence of the Internet and new media on the visibility of the revolution and the ensuing reduced effectivity of censorship. Within the model this leads to significant, discontinuous changes in regime stability, which greatly increase the probability of realized revolutions. These properties of the model inform possible answers to questions on causes and timing of the Arab Spring revolutions, and the role of the Internet and new media. The broader relevance of the model classification is also investigated by applying it to the current political situation in some other countries with regimes that employ censorship and police repression.

Keywords: revolution, modelling, dynamical systems, Arab Spring, new media

1. Introduction and Motivation

> "After decades of political stagnation... new winds of hope were felt in the Middle East, accompanied by a new catchword making the rounds in the American media, 'Arab Spring'... The age of the old patriarchs, it appeared, was nearing its end. And the new media - satellite television, mobile phones, the Internet - were often regarded as having precipitated this development by undermining governments' hegemonic control over the flow of information."

When A. Hofheinz wrote these words about the Arab Spring he was referring to modest advancements being made in democracy and political liberalization in a handful of Middle Eastern countries in 2005 [1]. He did not foresee the events sparked by Mohamed Bouazizi's self-immolation in a small Tunisian city on December 17, 2010 that ultimately led to the 2010-2011 Arab Spring revolutions[1]. Nevertheless, his analysis of new media and their impact on Arab society is eerily prescient, especially considering that in 2005 social media was either in its infancy or completely non-existent. Indeed, Facebook was launched in 2004 and was still an invitation-only service in 2005, Youtube was founded in early 2005, and Twitter was not founded until the spring of 2006 [2].

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[1] Henceforth, when we use the term Arab Spring we are referring to the 2010-2011 Arab Spring.

The predominant (but not universal [3]) view today is that the Internet and social media played a critical role in the Arab Spring [4, 5, 6, 7, 8, 9, 10, 11, 12]. Although some rigorous work has been done attempting to determine a link between social media and protests using, for example, Twitter data [4, 5], most of these opinions are based on anectodal evidence and *ad hoc* reasoning. Thus, the goal of this paper is twofold. First, we formulate a new mathematical model for revolutionary events that aims to capture essential features of the dynamics of peaceful revolutions in regimes that employ censorship and police repression. We provide conceptual and network-based mathematical justifications for the model. Analysis of the mathematical properties of the model leads to a division in parameter space that is interpreted naturally in terms of stability of the regime. It also identifies discontinuous changes in regime stability for certain changes in the balance between the visibility of the revolution and the policing power of the regime, which may greatly increase the probability of realized revolutions. Second, we investigate how the model can be applied to the Arab Spring revolutions in Tunisia and Egypt, taking into account the influence of the Internet and new media on the visibility of the revolution and the ensuing reduced effectivity of censorship.

It should be noted that models of opinion/norm formation [13], conflict [14, 15, 16, 17], and revolution [18, 19] already exist. However, these models either do not apply specifically to the peaceful political revolutions of the Arab Spring or are highly complex. Although complex models may in principle be able to offer a more complete description, they also have limitations such as requiring the calibration of a large number of parameters. This makes complex models analytically intractable, difficult to interpret, and computationally expensive to simulate. In Epstein's categorization of reasons to model [20], ours is to illustrate the core dynamics of the balance between, on one hand, the growth of a revolutionary movement and the influence of censorship and information flow on this growth, and on the other hand, the suppression of the revolution by police force. As a simple conceptual model, our model follows the tradition of well-known differential equation models in conflict analysis like the Richardson arms race model [14]. Thus, we emphasize that the main goal of this paper is to develop a simple model that is nevertheless able to capture essential features of political revolutions in dictatorial regimes that employ censorship and police repression.

The remainder of this paper is divided into five sections and an appendix. We specify and justify our model in Section 2, we provide a mathematical analysis of our model in Section 3, and we investigate how our model can be applied to the Arab Spring revolutions in Tunisia and Egypt, and the current situation in some other countries with regimes that employ censorship and police repression, in Section 4. Appendix A provides a network-based justification of our model, and Appendix B specifies and analyzes an extension to our model that relaxes some of the simplifying assumptions made in the basic form of the model, confirming that the dynamical properties of the basic model are generic for a broader class of models. Finally, conclusions are given in Section 5.

2. Model Specification

2.1. Basic Model

We begin by specifying a simple model describing the process by which citizens engage in revolution in regimes that employ censorship and police repression. Let $r(t)$ be the fraction of *protesters* or *revolutionaries* in the population at time t. The model is given by a single differential equation for $r(t)$,

$$\dot{r}(t) = \frac{dr}{dt} = \underbrace{c_1 \; v(r;\alpha) \; (1-r)}_{g(r)} - \underbrace{c_2 \; p(r;\beta) \; r}_{d(r)}, \quad (1)$$

where $\alpha, \beta \in (0,1)$ and $c_1, c_2 > 0$ are parameters, and where the functions $g, d : [0,1] \to \mathbb{R}^+$ are called the *growth* and *decay* terms, respectively, since they model the growth and decay of the fraction of protesters.

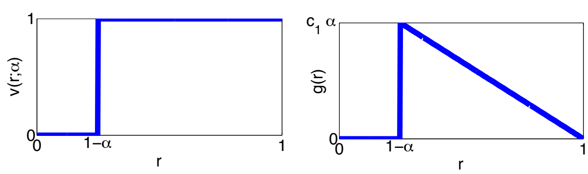

Figure 1: (Left) Visibility term $v(r;\alpha)$. (Right) Growth term $g(r)$.

In our model the fraction of protesters can only grow when the protest movement is sufficiently large to be visible to the general population, defying attempts at censorship by the regime. We call the proportionality constant c_1 and the parameter α the *enthusiasm* and *visibility* of the protesters, respectively. The *visibility term* $v(r;\alpha)$ is modelled as a step function which shuts off the growth term when the fraction of protesters is below the *visibility threshold* $1-\alpha$: $v(r;\alpha) = 0$ if $r \leq 1-\alpha$, and $v(r;\alpha) = 1$ otherwise (see Figure 1).

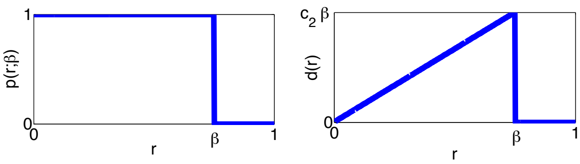

Figure 2: (Left) Policing term $p(r;\beta)$. (Right) Decay term $d(r)$.

The *policing term*, $p(r;\beta)$, is also modelled as a step function, which shuts down the decay term when the fraction of protesters is above the police capacity threshold β: $p(r;\beta) = 0$ if $r \geq \beta$, and $p(r;\beta) = 1$ otherwise (see Figure 2). We call the proportionality constant c_2 and the threshold parameter β the *policing efficiency* and *police capacity*, respectively. Finally, we observe from Equation (1) that if $r = 0$ or $r = 1$ then $\dot{r} = 0$. Thus, we call $r = 0$ and $r = 1$ the equilibria of *total state control* and of the *realized revolution*, respectively.

We provide a conceptual justification for the model developed above in the following section, and a network-based mathematical justification for the step function form of the visibility term in Section Appendix A.

2.2. Conceptual Justification of Basic model

In order to arrive at the model given in (1) some simplifying assumptions were made. First, our model is developed for describing rapid revolutionary transitions on a short time scale (on the order of months), and neglects demographic and other long-term effects. Second, we assume that the regime is very unpopular and that all individuals would privately like to see the regime changed. While this does not apply to all cases of political revolution (e.g., the 2011 revolutions in Lybia or the current civil war in Syria), it is nevertheless a reasonable modelling approximation in many cases (e.g. the Arab Spring revolutions in Tunisia or Egypt). These two assumptions imply a constant population that can be divided into two compartments: the population participating $(r(t))$ and not participating $(1-r(t))$ in the revolution. We note that, by the second assumption, the fraction of the population potentially willing to join the revolution at time t is $1-r(t)$.

Dictatorial regimes are known to keep tight control on the flow of political information through state control of the media and through censorship [1, 7, 8, 13, 18, 21], for obvious reasons: if political protests are kept hidden from the general population, protest movements have little chance of growing. We model this effect through the visibility term $v(r;\alpha)$, which we assume undergoes a rapid transition from 0 to 1 that, for simplicity, we model as a step function. As soon as the fraction of protesters reaches the visibility threshold $1-\alpha$ and is large enough to be visible to the general population, the revolution is assumed to grow proportional to $1-r$, where the time scale of growth is determined by the protesters' enthusiasm c_1.

As further motivation for the rapid transition from 0 to 1 in $v(r;\alpha)$ consider that, given the policing limitations of the regime, the decision of individuals whether or not to act is a collective action problem [18]. If individuals protest individually then the state is capable of severe retaliation, however, if individuals protest in sufficient numbers then the state loses its ability to punish. Thus, the case can be made that the most important factor for individuals deciding to join a revolution is the *perceived size* of the revolution. A network-based mathematical justification of how this would lead to rapid transitions from 0 to 1 in the visibility term of our model is provided in Appendix A.

We assume that the regime is capable of arresting/dispersing protesters at a rate proportional to the size of the revolution $r(t)$, provided that the number of protesters does not exceed the regime's finite police capacity β. Provided that no new protesters join the revolution $(v = 0)$ and that the number of protesters does not exceed the regime's police capacity $(p = 1)$, this corresponds to exponential decay in the number of protesters with the time scale determined by the policing efficiency c_2. We make the further simplifying assumption that the regime loses all ability to punish once the number of protesters exceeds the regime's police capacity. Thus, $p(r;\beta)$ takes the form of a step function in our basic model.

We note that the growth term in our model (with parameters c_1 and α) can be related to aspects of *grievance* [15], the utility of protest, and the overall emotional state of the population [22]. In contrast, parameters c_2 and β can be related to aspects of *political opportunity* [15] and *state capacity* [23].

In Appendix B we relax the step function transitions of $v(r;\alpha)$ and $p(r;\beta)$ to broader classes of sigmoid-type rapid transitions between 0 and 1, and show that the resulting more general models exhibit essentially the same dynamics as our basic model with step functions. This shows that the discontinuities in the step function transitions between 0 and 1 are not essential for obtaining the dynamics of the model and do not introduce artefacts in the dynamics, and that the dynamics are indeed generic for a broader class of models with rapid transitions between 0 and 1 in visibility and policing terms. We choose step functions in our basic model because they require the smallest number of parameters in describing rapid transitions between 0 and 1, and yet maintain the essential dynamics of more complicated models with more generic rapid transitions between 0 and 1.

Finally, due to the simplicity of our model, it is unable to capture singular events such as the self-immolation of Mohamed Bouazizi on December 17, 2010. Although this type of event could be modelled stochastically, to keep our model as simple as possible we introduce the concept of *shocks*: events external to our model which affect the fraction of revolutionaries $r(t)$ either directly, or indirectly via a change in the parameters α, β, c_1, or c_2.

3. Model Analysis

3.1. Classification of Parameter Regimes

The mathematical classification of the different types of dynamical behaviour that may occur in model (1) proceeds case-wise by considering parameter regions $\alpha+\beta=1$, $\alpha+\beta<1$, and $\alpha+\beta>1$, which we call Regions I, II, and III, respectively, see Figure 3. See Figure 4 for the phase portraits of the different regions.

Before considering Regions I, II, and III separately we begin by considering the equilibria $r=0$ and $r=1$. When $r<\min\{1-\alpha,\beta\}$ we have $v(r;\alpha)=0$ and $p(r;\beta)=1$, which implies $\dot{r}\le 0$ and $\dot{r}=0 \iff r=0$. Similarly, when $r>\max\{1-\alpha,\beta\}$ we have $v(r;\alpha)=1$ and $p(r;\beta)=0$, which implies $\dot{r}\ge 0$ and $\dot{r}=0 \iff r=1$. It follows that $r=0$ and $r=1$ are locally asymptotically stable equilibria whose basins of attraction contain the intervals $[0,\min\{1-\alpha,\beta\})$ and $(\max\{1-\alpha,\beta\},1]$.

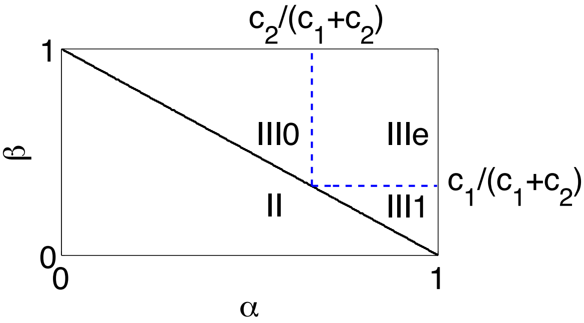

Figure 3: Division of $\alpha-\beta$ parameter space of model (1) into Regions I, II, IIIe, III0, and III1. Region I is given by the solid line $\alpha+\beta=1$.

3.1.1. Region I: $\alpha+\beta=1$

When $r=\beta=1-\alpha$ we have $v(r;\alpha)=p(r;\beta)=0$, which implies $\dot{r}=0$. It follows that $r=\beta=1-\alpha$ is an unstable equilibrium, and that $r=0$ and $r=1$ are locally asymptotically stable equilibria with basins of attraction $[0,\beta)$ and $(1-\alpha,1]$, respectively. We note that Region I, because it is one-dimensional, is unlikely to manifest itself and so we mostly disregard it in what follows.

3.1.2. Region II: $\alpha+\beta<1$

As in Section 3.1.1, $r=0$ and $r=1$ are locally asymptotically stable equilibria with basins of attraction $[0,\beta)$ and $(1-\alpha,1]$, respectively. When $r\in[\beta,1-\alpha]$ we have $v(r;\alpha)=p(r;\beta)=0$, which implies $\dot{r}=0$. It follows that all $r\in(\beta,1-\alpha)$ are locally stable equilibria and $r\in\{\beta,1-\alpha\}$ are unstable equilibria.

3.1.3. Region III: $\alpha+\beta>1$

When $r=1-\alpha$ we have $v(r;\alpha)=0$ and $p(r;\beta)=1$, which implies $\dot{r}<0$. Analogously, when $r=\beta$ we have $v(r;\alpha)=1$ and $p(r;\beta)=0$, which implies $\dot{r}>0$. Thus, the locally asymptotically stable equilibria $r=0$ and $r=1$ have basins of attraction containing $[0,1-\alpha]$ and $[\beta,1]$, respectively. Restricting our attention to the interval $r\in(1-\alpha,\beta)$ and solving the algebraic equation $dr/dt=0$ gives $r=c_1/(c_1+c_2)$. We define $c^*=c_1/(c_1+c_2)$ and observe that our analysis breaks down into a further three subcases:

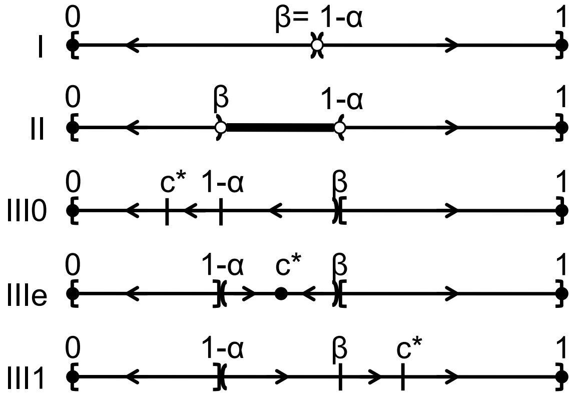

Figure 4: Equilibria, stability and basins of attraction on the r-axis ($r \in [0, 1]$) for parameters α, β, c_1 and c_2 of model (1) lying in Regions I, II, III0, IIIe and III1. Closed (open) circles represent locally asymptotically stable (unstable) equilibria. Left (right) arrows indicate regions where $\dot{r} < 0$ ($\dot{r} > 0$). The thick horizontal line indicates locally stable equilibria. Square ("[" and "]") or curved ("(" and ")") brackets indicate boundaries of basins of attraction that contain or do not contain their boundary points, respectively. Short vertical lines indicate special values of r that are not equilibria or boundaries of basins of attraction.

- Region III0 ($c^* \leq 1-\alpha < \beta$): If $c^* \in [0, 1-\alpha]$ then $\dot{r} = c_1(1-r) - c_2 r < 0$ for $r < \beta$, which implies that the interval $(1-\alpha, \beta)$ lies in the basin of attraction of $r = 0$.

- Region III1 ($1-\alpha < \beta \leq c^*$): If $c^* \in [\beta, 1]$ then $\dot{r} = c_1(1-r) - c_2 r > 0$ for $r > 1-\alpha$, which implies that the region $(1-\alpha, \beta)$ lies in the basin of attraction of $r = 1$.

- Region IIIe ($1-\alpha < c^* < \beta$): If $c^* \in (1-\alpha, \beta)$ then

$$\dot{r} = c_1(1-r) - c_2 r \begin{cases} > 0 & \text{if } r \in (1-\alpha, c^*) \\ = 0 & \text{if } r = c^* \\ < 0 & \text{if } r \in (c^*, \beta) \end{cases},$$

  which implies that there exists a third locally asymptotically stable equilibrium $r = c^*$ with basin of attraction $(1-\alpha, \beta)$.

We conclude the analysis of our model with a discussion of the discontinuities in the vector field of Equation (1) at $r = 1-\alpha$ or $r = \beta$, which result from choosing the visibility and policing terms to be step functions. Specifically, choosing $v(1-\alpha; \alpha) = p(\beta; \beta) = 0$ results in $r \in \{1-\alpha, \beta\}$ (a) being unstable equilibria in Regions I and II, and (b) being in the basin of attraction of $r = 0$ and $r = 1$, respectively, in Regions III0, IIIe, and III1. We acknowledge that this choice is arbitrary in nature, and that defining $v(1-\alpha; \alpha)$ and $p(\beta; \beta)$ differently would slightly change the vector field of Equation (1) at the points $r \in \{1-\alpha, \beta\}$. Nevertheless, Appendix B shows that essentially the same dynamics occur when the step functions in the visibility and policing terms are relaxed to generic sigmoid-type transitions between 0 and 1. Thus, we confirm that approximating the rapid transitions in the visibility and policing terms by step functions is justified by greatly simplifying Equation (1) and its analysis, without substantially affecting its dynamics or interpretation.

4. Interpretation and Application to Arab Spring

In this section we first provide an interpretation of the classification of parameter regions (Figure 3) in terms of political regime types and their stability and potential for revolutionary events (as indicated in Figure 4). We then investigate the application of the model to the Arab Spring revolutions, discussing the Arab Spring context and events, and societal factors relevant for the Arab Spring that have been identified in the political science literature. Finally, we discuss applying the classification resulting from the model to the current political situation in some other countries with regimes that employ censorship and police repression.

4.1. Interpretation of Classification

We first provide an interpretation of the parameter regions of the model, previously summarized in Figures 3 and 4, in the context of dictatorial regimes and their stability.

States with parameters in Region II have uncountably many stable equilibria between β and $1-\alpha$, which occur because the police capacity β of the regime is too low to clear the protesters and the visibility α is too low to attract new protesters, thus preventing any one group from taking control (see Region II in Figure 4). We therefore interpret Region II to correspond to *failed states*. In the context of our model, we investigate the application of the failed state parameter region to the case of Somalia, in Section 4.3.3.

Regions III0, IIIe, and III1 differ only in the stability of the interval $(1-\alpha, \beta)$. For Region III0 the interval $(1-\alpha, \beta)$ lies in the basin of attraction of total state control $r=0$. Because of the contribution of $(1-\alpha, \beta)$ to the stability of the regime, we refer to Region III0 as a *stable police state*. Examples of states that, in the context of our model, are consistent with the parameter regime of Region III0 may include Tunisia and Egypt prior to 2010 (see Section 4.2.4) and Iran in 2009 (see Section 4.3.1). Analogous to the case of Region III0, we refer to Region III1 as an *unstable police state*, since $(1-\alpha, \beta)$ lies in the basin of attraction of the realized revolution $r=1$. Region IIIe introduces an intermediate locally asymptotically stable equilibrium $r=c^*$, which lies between the equilibria of total state control $r=0$ and of the realized revolution $r=1$, and which has the interval $(1-\alpha, \beta)$ as its basin of attraction. We refer to $r=c^*$ as the equilibrium of *civil unrest* and to Region IIIe as a *meta-stable police state*. We hypothesize that Egypt and Tunisia transitioned to Region IIIe, and possibly to Region III1, in Section 4.2.4. Section 4.3.2 provides an additional example of a country, China, that in the context of our model shows characteristics of countries that would be classified in Region IIIe.

The examples given above of countries that could potentially be classified according to the above interpretation are discussed in detail in Sections 4.2.4 and 4.3, and are summarized conceptually in Figure 5. We note that as the situation in a particular country evolves the regime may pass from one parameter region to another. We discuss how adoption of new media may affect the parameter region of a country in Section 4.2.2. Section 4.2.3 then discusses how moving from one parameter region to another affects the likelihood of a realized revolution.

4.2. Application to Arab Spring in Tunisia and Egypt

We now investigate the application of the model to the Arab Spring revolutions in Tunisia and Egypt. We first provide a timeline of the main events in the revolutions, followed by discussions on the effects of new media on the model parameters, and the effects of the model parameters on model stability. We then investigate how the model can be related to the events of the Arab Spring revolutions in Tunisia and Egypt.

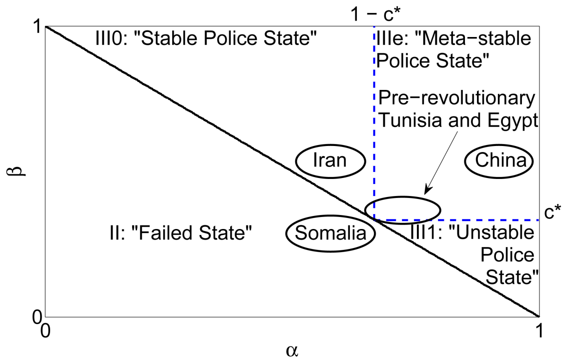

Figure 5: Division of $\alpha - \beta$ parameter space into Regions I, II, IIIe, III0, and III1, and conceptual summary of the case studies of Tunisia, Egypt, Iran, China, and Somalia.

4.2.1. Timeline of Arab Spring in Tunisia and Egypt

To aid in the interpretation of our model in the cases of the Tunisian and Egyptian Arab Spring we provide the following rough timeline of major events [24, 25, 21, 4, 7, 9, 26].

- Dec. 17, 2010: Mohamed Bouazizi self-immolates in the Tunisian city of Sidi Bouzid.
- Dec. 18, 2010: Protests erupt in Sidi Bouzid. Protesters begin recording and uploading videos of the protests and police response to the Internet.
- Dec. 27 - 28, 2010: Protests break out in the Tunisian capital, Tunis. Tunisian President Ben Ali denounces protests in a televised address.
- Jan. 5, 2011: Mohamed Bouazizi dies from burn injuries.
- Jan. 14 - 15, 2011: Ben Ali resigns and flees to Saudi Arabia. An interim government is formed.
- Jan. 25, 2011: The "Day of Protest" in Tahrir Square, Cairo, is the first major Arab Spring protest in Egypt.
- Jan. 26, 2011: Egyptian police clear Tahrir Square.
- Jan. 28, 2011: Protesters occupy Tahrir Square, Egyptian President Mubarak addresses the nation, and major Internet disruptions begin.

- Feb. 1, 2011: US President Obama withdraws support for the Mubarak regime, the army refuses to act against protesters, and major Internet disruptions end.
- Feb. 2, 2011: State vandals and thugs attack protesters in Tahrir Square. Army officers intervene on behalf of protesters.
- Feb. 11, 2011: President Mubarak resigns.

4.2.2. Effects of New Media on Protesters' Enthusiasm c_1 and Visibility α

As outlined in Section 1, our development of model (1) was motivated by our interest in providing a basic dynamic model for Arab Spring revolutions and studying the effect of new media on censorship and the stability of dictatorial regimes as in the Arab Spring. We propose that the Internet, social media, satellite TV, and cell phone communication technologies may have empowered protesters by enhancing their

(1) capacity for organization and coordination [7, 9, 27],
(2) ability to assess the current public support for the revolution [6, 7, 8, 9, 12], and
(3) awareness of the nature and severity of government repression [7, 28].

(1) The decision of whether or not to protest is a coordination problem [18], the realization of which led activists to use the Internet to coordinate protests in Tunisia and Egypt [7, 8, 9, 12]. For example, technologies such as SMS and Twitter messaging were used between co-revolutionaries to communicate which streets were the most/least obstructed by security forces [7]. This enhanced the speed with which revolutionaries mobilized, and in the context of our model this corresponds to a increase in protesters' enthusiasm c_1. Social media and the Internet also contributed to the relatively leaderless way in which the Arab Spring revolutions developed. Compared to revolutions with a more hierarchical leadership structure, a leaderless revolution is difficult if not impossible to disrupt by targeting only a handful of individuals [7, 27]. This increased resilience also corresponds to an increase in protesters' enthusiasm c_1 in our model.

(2) Dictatorial regimes attempt to control protests through censorship by lowering visibility α in order to ensure that protests remain virtually invisible to the general population. The Internet, social media, satellite TV, and cell phones all work towards increasing visibility α by disrupting the regime's monopoly on the distribution of information. For example, in Tunisia the Internet and social media created a virtual space where Tunisians could express their true opinions with minimal censorial oversight or fear of reprisal [7, 8, 9]. In the microscopic network model of Appendix A, an increase in the number of links (average degree in the graph) along with a decrease in the threshold for action result in a shift of the sharp increase in participation toward smaller fractions of the population (see Figure A.7, bottom panel). Cell phones and social media sites vastly increased the speed with which information travelled, allowing Tunisians - and the entire world - to follow the revolution with unprecedented detail and speed [5, 7, 12]. Satellite TV further enhanced visibility α by corroborating and then rebroadcasting stories relating to the size of the revolution and the regime's brutal response [6].

(3) Awareness of the brutality and severity of the government's reaction may increase both protesters' enthusiams c_1 and visiblity α. An increase in protesters' enthusiasm c_1 may be a direct result of increasing resentment of the regime. In contrast, the increase of visibility α is likely to be indirect. Specifically, otherwise apolitical individuals may be induced to join the revolution [7, 28], presumably by lowering their personal thresholds for participation.

4.2.3. Sensitivity of Model (1) *Stability to Protesters' Enthusiasm c_1 and Visibility α*

The Internet and social media had at best modest penetratration in countries of the Arab Spring prior to the self-immolation of Mohamed Bouazizi. Indeed, approximately 25% of Tunisians and 10% of Egyptians had used the Internet at least once prior to the Arab Spring [4]. Given this fact, it is reasonable to ask how a modest level of adoption of new technologies might have a significant impact on the outcome of a revolution. Recall that in Section 2.2 we argued that keeping our model simple requires introducing the concept of external shocks. An analysis of how changes in parameters may affect the outcome of a revolution must take into account these shocks.

A small increase in visibility α or protesters' enthusiasm c_1 can move parameters from the stable (Region III0) to the meta-stable (Region IIIe) police state (see Figure 5). For parameters in Region

III0 a shock $\Delta r > \beta$ is required to pass from total state control $r = 0$ to the basin of attraction of the realized revolution $r = 1$ (see Figure 6). In contrast, for parameters in Region IIIe passing from total state control $r = 0$ to the basin of attraction of the realized revolution $r = 1$ can result from two smaller shocks $\Delta r_1 > 1 - \alpha$ and $\Delta r_2 > \beta - c^*$, where $\Delta r_1 + \Delta r_2$ may be significantly smaller than β (see Figure 6 bottom line). Note that for parameters in Region IIIe the lower bounds for sufficiently strong Δr_1 and Δr_2 decrease with increasing visibility α and protesters' enthusiasm c_1, respectively. If shocks occur distributed according to some probability distribution, then it is reasonable to assume that shocks of sufficient magnitude to mobilize large fractions of the population lie in the tail of this distribution. For many reasonable probability distributions, halving the size of shock necessary to trigger a revolution more (and potentially much more) than doubles the likelihood of a revolution occurring in any given amount of time. Thus, small increases in visibility α or protesters' enthusiasm c_1 resulting from modest Internet penetration or social media usage can have a large impact on the expected amount of time one has to wait until a revolution is triggered.

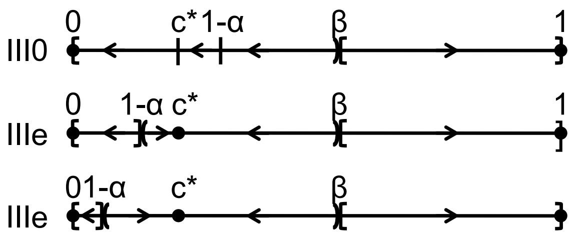

Figure 6: The effect of increasing α. Closed circles represent locally asymptotically stable equilibria. Left (right) arrows indicate regions where $\dot{r} < 0$ ($\dot{r} > 0$). Square ("[" and "]") or curved ("(" and ")") brackets indicate boundaries of basins of attraction that contain or do not contain their boundary points, respectively. Short vertical lines indicate special values of r that are not equilibria or boundaries of basins of attraction.

The sensitivity of the final outcome of a revolution to the protesters' enthusiasm c_1 and visibility α also provides us with a potential explanation for why the revolutions of the Arab Spring, and indeed revolutions in general, come as such a surprise to so many regimes and political observers [6, 4, 7, 18]. Increasing either protesters' enthusiasm c_1 or visiblity α eventually decreases the size of the basin of attraction of total state control $r = 0$, and does so in a discontinuous fashion. (See, e.g., Figure 6: when α increases such that, at first, c^* is just below $1 - \alpha$, and then becomes slightly larger than $1 - \alpha$, the basin of attraction of $r = 0$ changes from $[0, \beta)$ to $[0, 1 - \alpha)$ in a discontinuous fashion.) This undermines the regime by decreasing the size of shock necessary to trigger a revolution. However, since (a) $r = 0$ always remains locally asymptotically stable, (b) large shocks are exceedingly rare, and (c) determining the precise extent to which a regime controls the society through media control and policing (corresponding to determining accurate values for the parameters in a model like ours) is very difficult [23], the exact size of shock necessary to trigger a revolution is impossible to determine until such a shock occurs. It follows that for someone observing a regime before and after the adoption of social media there would be few, if any, outward signs of instability: the regime appears stable until it isn't.

4.2.4. Dynamics of Arab Spring Revolutions in Tunisia and Egypt

Guided by the discussion in Sections 4.2.2 and 4.2.3, it is interesting to consider hypotheses for the Tunisian and Egyptian Arab Spring in the context of our model. It is a reasonable hypothesis that Tunisia and Egypt were, in the context of our model, in parameter Region III0 (stable police state) for a long time in the years before 2010 (Ben Ali was in power in Tunisia for 24 years (1987-2011) and Mubarak was in power in Egypt for 30 years (1981-2011)). However, the realized revolutions of 2010-2011 appear to indicate that Tunisia and Egypt had evolved to significantly less stable regimes (Region IIIe or III1) by 2010, see Figure 5. Once in Region IIIe or III0, we observe that there are many potential candidates for shocks, including but not limited to those listed in Section 4.2.1, that may, within the context of our model, have moved the regime to a state of civil unrest or realized revolution.

While it is clear that, in the context of our Arab Spring revolution model, reliable time series measurements of quantities that would correspond to the fraction of protesters in the revolution are not available, and while it is thus not realistic to consider fitting the model and its parameters to observed time series, it is interesting to note that there are recent efforts that attempt to gather quantitative data that can be used for social science research using, for example, Blogs and online social media platforms like Twitter. See, e.g., [4, 5] for the case of the Arab Spring revolutions. Unfortunately, with the currently available data this type of comparison can only be done at a superficial level, but it is an intriguing prospect that this kind of approach may offer new ways to quantitatively test models in social science in the future when more and higher-quality quantitative data of this type may become available.

4.3. Application to non-Arab Spring Countries

Although our model was developed with the specific goal of modelling revolutions in dictatorial regimes that employ censorship and police repression initially applied to the Arab Spring, it is interesting to consider its application to various situations in other countries. In this section we consider the cases of the failed 2009 "Green Revolution" in Iran, and of present-day China and Somalia.

4.3.1. 2009 Iranian "Green Revolution"

The protests following Iran's 2009 election, dubbed the "Green Revolution", were ultimately put down by the regime despite widespread use of social media technology. The large amount of resources that were available to the Iranian regime and the relative novelty of applying social media for use in protest [10, 29] may be consistent with a regime with parameters, in the context of our model, in Region III0 (stable police state). This, in turn, would be consistent with the failure of the Green Revolution.

4.3.2. China

While the current regime in China differs from the pre-revolutionary regimes in Tunisia and Egypt in many aspects, it is interesting to consider how our model may apply to China in terms of the influence of state control on the media and the Internet, and police control of dissident opinion. The number of "mass group incidents" reported annually in China has been rising consistently for at least two decades [30]. Constant low levels of protest may correspond to the civil unrest equilibrium ($r = c^*$) in the meta-stable police state (Region IIIe). In our model rising levels of protest would correspond to rising c^*. We note that, in the context of our model, a continued rise of c^* through increased Internet and social media exposure may eventually result in increasing the chance of a realized revolution, as argued in Section 4.2.3.

4.3.3. Somalia

The failed state region (Region II) features low visibility α (weak media) and low police capacity β (weak government), which prevents individuals from joining any popular movements and prevents the government from reigning in existing movements, respectively. This results mathematically in an uncountable number of equilibria contained in $(\beta, 1-\alpha)$. This appears consistent with the slow and erratic rise and fall of local militia and the succession of weak central governments seen in Somalia from 1991 [31]. Our model predicts that a successful national state could arise from either (a) improving police capacity of the transitional government (increasing β), or (b) increasing social cohesion and the capacity of the media in Somalia (increasing α). Interestingly, Somalia has developed a sophisticated and affordable telecommunications sector [32], which may mean that an increased visibility α is not unrealistic.

5. Conclusion

We have introduced a simple compartmental model for the dynamics of a revolution in dictatorial regimes that employ censorship and police repression. The model features visibility and policing terms that describe rapid transitions between 0 and 1 as a function of the size of the revolution, for which we have provided conceptual and network-based mathematical justifications. The dynamical behaviour of the model was classified, leading to a division in parameter space that is interpreted naturally in terms of stability of the regime (stable police state, meta-stable police state, unstable police state, and failed state). We show in Appendix B that these dynamical properties of the model are generic for a broad class of visibility and police capacity functions that feature rapid transitions between 0 and 1. We investigated how the model can be applied to the Arab Spring revolutions in Tunisia and Egypt, taking into account the influence of the Internet and new media on the visibility of the revolution and the ensuing reduced effectivity of censorship. Within the model this leads to significant, discontinuous changes in regime stability, which greatly increases the probability of realized revolutions. These properties of the model inform possible answers to questions on causes and timing of the Arab Spring revolutions, and the role of the Internet and new media. The broader relevance of the model classification was also investigated by applying it to the current political situations in Iran, China and Somalia. We note here that our model is general enough to potentially capture a wide range of social change phenomena in democratic regimes as well. Consider, for example, social norms such as the recognition of gay marriage [33] or the practice of cremation versus burial [34]. Both of these issues have recently gained substantial support over a relatively short period, despite significant resistance. We emphasize, however, that in these cases the opposition to these issues comes not from the policing capacity of a government but from elements of society that are reluctant and/or resistant to change, and may fiercely oppose the change until the case is deemed lost.

Simple models like ours have the advantage of relying on just a few basic assumptions about individual and communal behaviour. More elaborate models can easily be imagined, but adding detail to a model comes at the expense of its tractability. Indeed, the very simplicity of our model is what admits a complete and rigorous mathematical classification of its dynamical behaviour, as well as an interpretation that offers interesting hypotheses about how Arab Spring-type revolutions may unfold. Of course, simple models like ours are also subject to many limitations. For example, our model is not capable of describing the Arab Spring revolution in Lybia and Syria, or the counter-revolution in present-day Egypt, because they do not correspond to our basic assumption that the population is uniform in its dislike for the current regime. In particular, tribalism in Lybia [35], religion in Syria [36], and religion, secular democracy, and the vested interests of the military in present-day Egypt [37] divide the population into factions that cannot easily be accounted for in a one-dimensional mathematical model. Moreover, events in these countries are further complicated by significant external interference and interventions. These complications, among others, must be addressed by more sophisticated models and constitute a significant avenue of future research.

6. Acknowledgements

We would like to acknowledge Thomas Homer-Dixon for interesting discussions on conflict modeling and the Arab Spring revolutions.

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Appendix A. Network-Based Justification of Rapid Transition in Visibility Term $v(r;\alpha)$

Section 2.2 introduced the decision of an individual to participate in a revolution as a collective action problem. If individuals protest individually then the state is capable of severe retaliation, however, if individuals protest in sufficient numbers then the state loses its ability to punish. It is a reasonable assumption, therefore, that each individual i in a network has a threshold $\theta_i \in (0,1)$ which determines his or her participation. In particular, an individual is *considering joining the revolution* only if the fraction of the individual's neighbors participating in the revolution is at least θ_i [18]. We use this basic framework to justify the choice of $v(r;\alpha)$ as a rapid transition between 0 and 1 (modelled by a step function in our basic model) by considering the following derivation.

Let individuals in a population be represented by nodes V in a network $G = G(V,E)$, where edges E linking individuals represent social interactions. We define the average degree and threshold of individuals V to be ϕ and θ, respectively. Suppose that in the time interval $(t, t+\Delta t)$ an individual $v \in V$ who is considering joining the revolution joins with probability $c_1 \Delta t$. Equivalently, taking $\Delta t \to 0$, a node that is considering joining the revolution will join at the first arrival time of a Poisson process with rate c_1.

In order to estimate the fraction of the average individual's neighbours that are participating in the revolution, without having to specify additional information about the structure of the underlying network, it is necessary to make a simplifying assumption. Specifically, following [38, 39, 40] we assume that the states of an individual's neighbors are uncorrelated. It now follows that the probability that an average individual considers joining the revolution is

$$\begin{aligned}&\sum_{k=\lceil\theta\phi\rceil}^{\lfloor\phi\rfloor} \binom{\lfloor\phi\rfloor}{k} r^k (1-r)^{\lfloor\phi\rfloor-k}\\ &= \sum_{k=0}^{\lfloor\phi\rfloor-\lceil\theta\phi\rceil} \binom{\lfloor\phi\rfloor}{k} (1-r)^k r^{\lfloor\phi\rfloor-k}\\ &= \mathrm{BinCDF}(\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r),\end{aligned}$$

where $\mathrm{BinCDF}(x;n,p)$ is the cumulative distribution function for the binomial distribution with n trials and probability of success p evaluated at x. An approximation for the number of nodes considering joining the revolution is thus

$$(1-r)\ \mathrm{BinCDF}(\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r).$$

It follows that

$$\Delta r \approx c_1\ \Delta t\ (1-r)\ \mathrm{BinCDF}(\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r).$$

Dividing by Δt and taking the limit $\Delta t \to 0$ yields

$$\frac{dr}{dt} \approx c_1\ (1-r)\ \mathrm{BinCDF}(\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r),$$

where $\mathrm{BinCDF}(\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r)$ corresponds to the visibility term $v(r;\alpha)$ of model (1).

Individuals only consider joining the revolution if it has already grown to a certain extent, but an individual's ability to determine the extent of the revolution is constrained by the number of neighbors the individual has. Specifically, the more neighbors one has the more certain one can be about the true extent of the revolution. We emphasize that determining the true extent of the revolution before deciding to act is of critical importance because if one overestimates the support for a revolution then one risks acting unilaterally. We would, therefore, expect the average threshold θ and average degree ϕ to be negatively correlated (increased ϕ corresponds to decreased θ).

We illustrate the dependence of $\mathrm{BinCDF}(\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r)$ on the parameters θ and ϕ in Figure A.7. All panels of Figure A.7 show that $\mathrm{BinCDF}(\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r)$ is a function of r with a steep transition from 0 to 1, with the transition becoming steeper as ϕ increases, which lends justification to our choice of a rapid transition from 0 to 1 for $v(r;\alpha)$. Although we do not know the exact relationship between θ and ϕ, Figure A.7 illustrates how when θ and ϕ are negatively correlated the function $\mathrm{BinCDF}(\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r)$ can be approximated by a steep transition from 0 to 1 that has one parameter determining where the transition occurs (corresponding to parameter α in the visibility term $v(r;\alpha)$). In our basic model we model this steep transition from 0 to 1 as a step function, while more general sigmoid-type transitions are considered in Appendix B. Note that, in a macroscopic view, the average number of effective neighbors ϕ that individuals can interact with is increased significantly by new media, which corroborates our interpretation of $v(r;\alpha)$ as a *visibility term*.

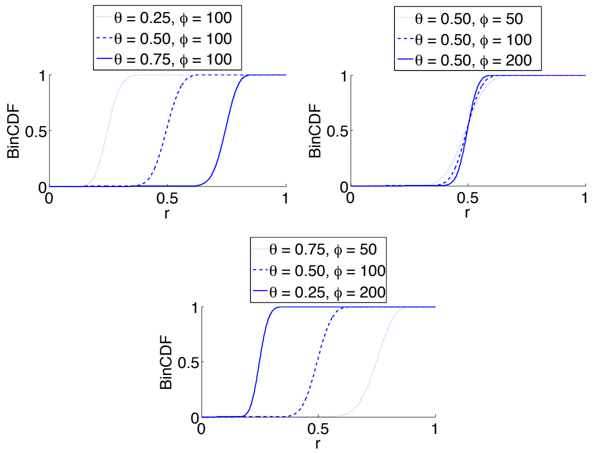

Figure A.7: Dependence of BinCDF($\lfloor\phi\rfloor - \lceil\theta\phi\rceil; \lfloor\phi\rfloor, 1-r$) on (top left) θ, (top right) ϕ, and (bottom) negatively correlated θ and ϕ.

We conclude this section with a few remarks regarding the assumption made above that the states of an individual's neighbors are uncorrelated. Given that the process we are studying spreads over social networks, which tend to have high clustering coefficients, this assumption would benefit from additional justification. The primary justification we offer for this assumption is empirical. First, we note that this assumption is applied in many models of spreading processes, for example rumour spreading processes [38] and epidemiological processes [39, 40]. These models have been shown to be good first order approximations of higher order "effective degree" or Agent Based models that do not employ assumptions about the correlations of the states of an individual's neighbors [41]. Second, we have recorded preliminary results confirming that the visibility function in a simple model that includes effects of the network (by statistically sampling real social network data sets without assuming uncorrelated neighbors) takes the form of a rapid transition of the type illustrated in Figure A.7 also for these real networks. Further work in this direction is ongoing.

Appendix B. Model Extension

Appendix B.1. Model Specification

This section relaxes the assumption that the visibility and policing terms are step functions. We show that model (1) with step functions possesses

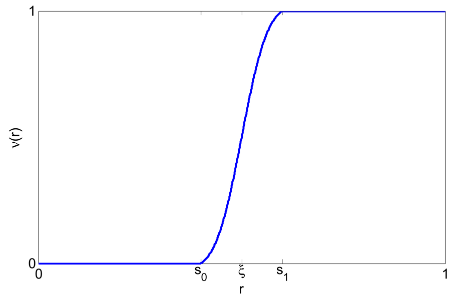

Figure B.8: Visibility function $\nu(r)$ with a fast transition from 0 to 1 that follows a sigmoidal profile.

the same dynamical behaviour as corresponding models with fast transitions from 0 to 1 of a generic sigmoidal type. Specifically, consider the equation

$$\frac{dr}{dt} = \dot{r} = \underbrace{c_1\ (1-r)\ \nu(r)}_{\gamma(r)} - \underbrace{c_2\ r\ \rho(r)}_{\delta(r)}, \tag{B.1}$$

where $\nu(r)$ and $\rho(r)$ are the visibility and policing terms, and where $\gamma(r)$ and $\delta(r)$ are the growth and decay terms, respectively. We choose $\nu(r)$ from the family of sigmoidal functions $\mathbb{S}$ comprised of (twice piecewise differentiable) functions $s : [0,1] \to [0,1]$ satisfying

(i) $\exists s_0, s_1 \in [0,1]$ such that
   - $\forall r \in (s_0, s_1) : s(r)$ is twice differentiable,
   - $\forall r \le s_0 : s(r) = 0$, and
   - $\forall r \ge s_1 : s(r) = 1$,

(ii) $\forall r \in (s_0, s_1) : s'(r) > 0$,

(iii) $\exists \xi \in (s_0, s_1)$ such that
   - $\forall r \in [s_0, \xi] : s''(r) \ge 0$ and
   - $\forall r \in [\xi, s_1] : s''(r) \le 0$,

(iv) $\forall r \in (s_0, s_1) : s''(r)s(r) \le [s'(r)]^2$, and

(v) $\lim_{r \to s_0^+} s'(r)/s(r) > (1-s_0)^{-1}$.

Analogously, we choose $\rho(r)$ so that $\rho(1-r) \in \mathbb{S}$. The functions $s(r)$ in $\mathbb{S}$ follow a sigmoidal profile of the type illustrated in Figure B.8: in (s_0, s_1), $s(r)$ is twice continuously differentiable and strictly increasing, and $s(r)$ has at most one inflection point in (s_0, s_1), as specified by conditions (i)-(iii). Conditions (iv)-(v) are imposed to guarantee that the growth and decay terms of models (1) and (B.1) are qualitatively similar in the sense that they are all single peaked[2], see the right panels of Figures 1 and

[2] A function is single peaked if it has a unique global maximum and no other local maxima.

2, and Figure B.9. Proposition Appendix B.1 shows that the growth term $\gamma(r)$ and decay term $\delta(r)$ of model (B.1) are single-peaked if conditions (i)-(v) are satisfied. $\mathbb{S}$ describes a fairly broad class of functions $s(r)$ that transition monotonously from 0 to 1 in a way that $(1-r)s(r)$ is single-peaked. Functions in $\mathbb{S}$ may have discontinuous derivatives at s_0 and s_1. Note that condition (v) is automatically satisfied if $s(r)$ has a discontinuous first derivative at s_0, but $\mathbb{S}$ also contains functions that are smooth on the entire interval $[0,1]$. Examples of functions in $\mathbb{S}$ are scaled and translated tanh and erfc functions that are truncated at $r = s_0$ and $r = s_1$, and the linear transition from $(s_0, 0)$ to $(s_1, 1)$.

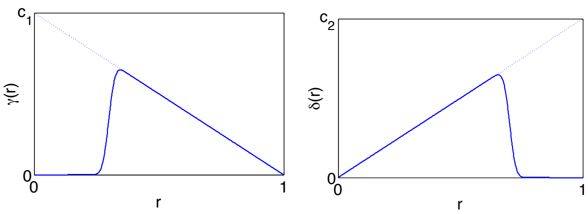

Figure B.9: (Left) Growth term $\gamma(r)$. (Right) Decay term $\delta(r)$.

Proposition Appendix B.1. *Let $s \in \mathbb{S}$. Then the function $\gamma(r) = c_1(1-r)s(r)$ is single peaked.*

Proof: Let $V = (s_0, s_1)$. Observe that $\forall r \in V$: $\gamma'(r) \geq 0$ if and only if

$$\frac{s'(r)}{s(r)} \geq \frac{1}{1-r}.$$

Now, observe that by property (iv), $\forall r \in V$:

$$\frac{d}{dr}\left[\frac{s'(r)}{s(r)}\right] = \frac{s''(r)s(r) - [s'(r)]^2}{s^2(r)} \leq 0.$$

We now have that for $r \in V$ the function $s'(r)/s(r)$ is monotonically decreasing and the function $(1-r)^{-1}$ is strictly monotonically increasing. Recall property (v)

$$\lim_{r \to s_0^+} \frac{s'(r)}{s(r)} > \left.\frac{1}{1-r}\right|_{r=s_0}.$$

If

$$\lim_{r \to s_1^-} \frac{1}{1-r} > \left.\frac{s'(r)}{s(r)}\right|_{r=s_1}$$

then there exists a unique point $r^* \in V$ where curves $s'(r)/s(r)$ and $(1-r)^{-1}$ intersect such that $\gamma'(r^*) = 0$, at which $\gamma(r)$ achieves its global maximum. Otherwise, $\forall r \in V$: $\gamma'(r) > 0$ and $\gamma(r)$ achieves its global maximum at $r = s_1$.

□

Appendix B.2. Model Analysis

Proposition Appendix B.1 allows us to sketch the growth and decay terms, see Figure B.9. The number of intersections between the growth and decay functions determines the number of equilibria in model (B.1). In what follows, we explain how the dynamics of model (B.1) with generic sigmoidal visibility and policing functions closely follows the corresponding dynamics of model (1) in Region II (an open interval of equlibria $\subset (0,1)$), Regions III0 and III1 (one equilibrium $\in (0,1)$), and Region IIIe (three equilibria $\in (0,1)$), thus establishing the equivalence in terms of dynamic behaviour of models (1) (with step functions) and (B.1) (with sigmoidal functions). We summarize the phase portraits of the different regions for model (B.1) in Figure B.10.

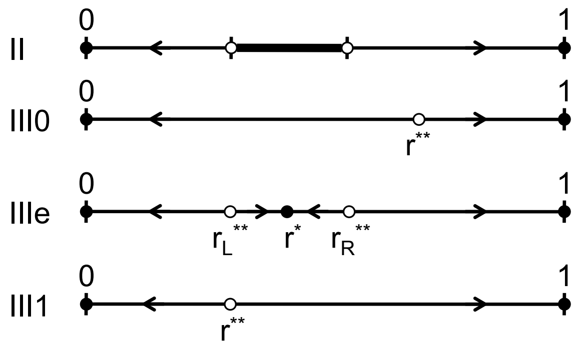

Figure B.10: Equilibria, stability and basins of attraction on the r-axis ($r \in [0,1]$) for model (B.1). Closed (open) circles represent locally asymptotically stable (unstable) equilibria. Left (right) arrows indicate regions where $\dot{r} < 0$ ($\dot{r} > 0$).

Appendix B.2.1. Region II

Suppose that the support of $\gamma(r)$ and $\delta(r)$ have no points in common. In this situation we have (a) $r = 0$ and $r = 1$ are locally asymptotically stable equilibria, (b) there exists an open interval $I \subset (0,1)$ of locally stable equilibria, and (c) the infimum and supremum of I are unstable equilibria. This situation is analogous to Region II of model (1).

Appendix B.2.2. Regions III0 and III1

Consider the situations depicted in Figure B.11. In both panels, $r = 0$ and $r = 1$ are locally asymptotically stable equilibria whose basins of attraction are separated by an unstable equilibrium point $r = r^{**}$. The left panel of Figure B.11 depicts the situation where $\max_r \gamma(r) < \max_r \delta(r)$. Because the decay term is dominant, we consider this situation to be analogous to Region III0 of model (1). Similarly, we consider the right panel of Figure B.11 to depict a situation analogous to Region III1 of model (1).

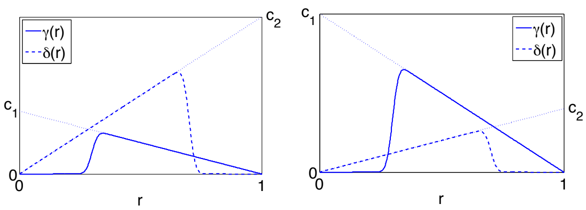

Figure B.11: $\gamma(r)$ and $\delta(r)$ with one interior intersection. (Left) Region III0. (Right) Region III1.

Appendix B.2.3. Region IIIe

Consider the situation depicted in Figure B.12 in which there are three locally asymptotically stable equilibria $r = 0$, $r = r^*$ and $r = 1$ $(0 < r^* < 1)$ whose basins of attraction are separated by two unstable equilibria $r = r_L^{**}$ and $r = r_R^{**}$ $(r_L^{**} < r_R^{**})$. This situation is analogous to Region IIIe of model (1).

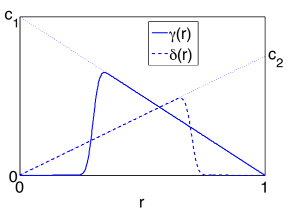

Figure B.12: $\gamma(r)$ and $\delta(r)$ with three interior intersections. Region IIIe.

Note that, corresponding to Region I of model (1), there are also special limiting cases of parameter choices for (B.1) in which, for example, some of the equilibria in Figure B.10 may coincide. We choose to neglect these cases in our analysis due to the unlikelihood of them being manifested (generic small perturbations of $\gamma(r)$, $\delta(r)$, c_1, or c_2 would take us away from such a special case).

Appendix B.3. Discussion

Figures 4 and B.10 show that models (1) and (B.1) are dynamically equivalent in the sense that they have equivalent phase portraits. This means that the simplifying assumption of model (1) of representing the steep increase in the visibility and policing terms by a step function (which is desirable because it gives a model with fewer parameters) is not limiting in the sense that it has dynamical behaviour that is equivalent to a more complicated model in which the steep increases are modelled by generic sigmoidal functions. This justifies the choice of step functions in model (1), since it leads to the simplest model that captures the relevant dynamics. In addition, it indicates that the dynamics we identified for model (1) will also occur in more complicated models for the dynamics of revolutions of type (B.1) that feature visibility and policing terms that change rapidly between 0 and 1 in a sigmoidal fashion.