mini-course:

Numerical Magnetohydrodynamics with Application to Space Physics Flows

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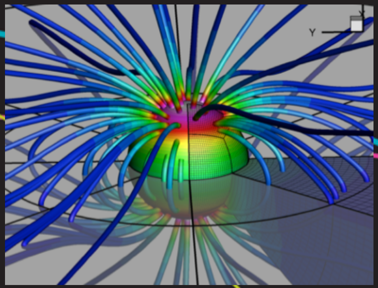

Workshop on Numerical Methods for Fluid Dynamics

Fields Institute – Carleton University, August 2013

Lecture 1: Structure of MHD as a Hyperbolic System

compressible ideal magnetohydrodynamics (MHD)

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\vec{v} \\ \frac{\rho v^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2} \\ \vec{B} \end{bmatrix} + \nabla\cdot\begin{bmatrix} \rho\vec{v} \\ \rho\vec{v}\vec{v}+\left(p+\frac{B^2}{2}\right)\vec{I}-\vec{B}\vec{B} \\ \left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+p\right)\vec{v}-(\vec{v}\times\vec{B})\times\vec{B} \\ \vec{v}\vec{B}-\vec{B}\vec{v} \end{bmatrix} = 0$$

- nonlinear system of PDEs
- describes ‘perfectly conducting fluid’ (ideal gas)
- 8 equations in 8 unknowns
- 8 unknown functions of (3D) space and time

compressible ideal MHD

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\vec{v} \\ \frac{\rho v^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2} \\ \vec{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\vec{v} \\ \rho\vec{v}\vec{v}+\left(p+\frac{B^2}{2}\right)\vec{I}-\vec{B}\vec{B} \\ \left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+p\right)\vec{v}-(\vec{v}\times\vec{B})\times\vec{B} \\ \vec{v}\vec{B}-\vec{B}\vec{v}\end{bmatrix} = 0$$

- mass density ρ
- velocity vector $\vec{v}$

(adiabatic constant γ)

- pressure p
- magnetic field vector $\vec{B}$

(➔ 8 unknown functions)

compressible ideal MHD

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\vec{v} \\ \frac{\rho v^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2} \\ \vec{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\vec{v} \\ \rho\vec{v}\vec{v}+\left(p+\frac{B^2}{2}\right)\vec{I}-\vec{B}\vec{B} \\ \left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+p\right)\vec{v}-(\vec{v}\times\vec{B})\times\vec{B} \\ \vec{v}\vec{B}-\vec{B}\vec{v}\end{bmatrix}=0$$

- nonlinear hyperbolic conservation law
- conservation of mass, momentum, total energy, magnetic field
- no dissipation (‘ideal’), perfectly conducting fluid
- for $\vec{B}=0$: compressible gas dynamics (Euler)

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compressible ideal MHD

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\vec{v} \\ \frac{\rho v^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2} \\ \vec{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\vec{v} \\ \rho\vec{v}\vec{v}+\left(p+\frac{B^2}{2}\right)\vec{I}-\vec{B}\vec{B} \\ \left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+p\right)\vec{v}-(\vec{v}\times\vec{B})\times\vec{B} \\ \vec{v}\vec{B}-\vec{B}\vec{v}\end{bmatrix}=0$$

- MHD = combination of Euler and Maxwell (current $\vec{J}=\nabla\times\vec{B}$ and $\vec{J}\times\vec{B}$ force)
- non-relativistic, low-frequency limit
- constraint $\nabla\cdot\vec{B}=0$ (initial condition, satisfied for all times)

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compressible ideal MHD – ‘physical form’

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$ (mass conservation)

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + (\nabla \times \vec{B}) \times \vec{B}$$ (Newton)

$$\frac{\partial p}{\partial t} + (\vec{v} \cdot \nabla) p + \gamma p \nabla \cdot \vec{v} = 0$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$ (Faraday)

current $\vec{J} = \nabla \times \vec{B}$ (Ampere)

entropy $s = \frac{p}{\rho^\gamma}$ $\frac{ds}{dt} = 0$

Ohm $\vec{E} = -\vec{v} \times \vec{B}$

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \vec{v} \\ \frac{\rho v^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{2} \\ \vec{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \vec{v} \\ \rho \vec{v}\vec{v} + \left(p + \frac{B^2}{2}\right) \vec{I} - \vec{B}\vec{B} \\ \left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1} + p\right) \vec{v} - (\vec{v} \times \vec{B}) \times \vec{B} \\ \vec{v}\vec{B} - \vec{B}\vec{v} \end{bmatrix} = 0$$

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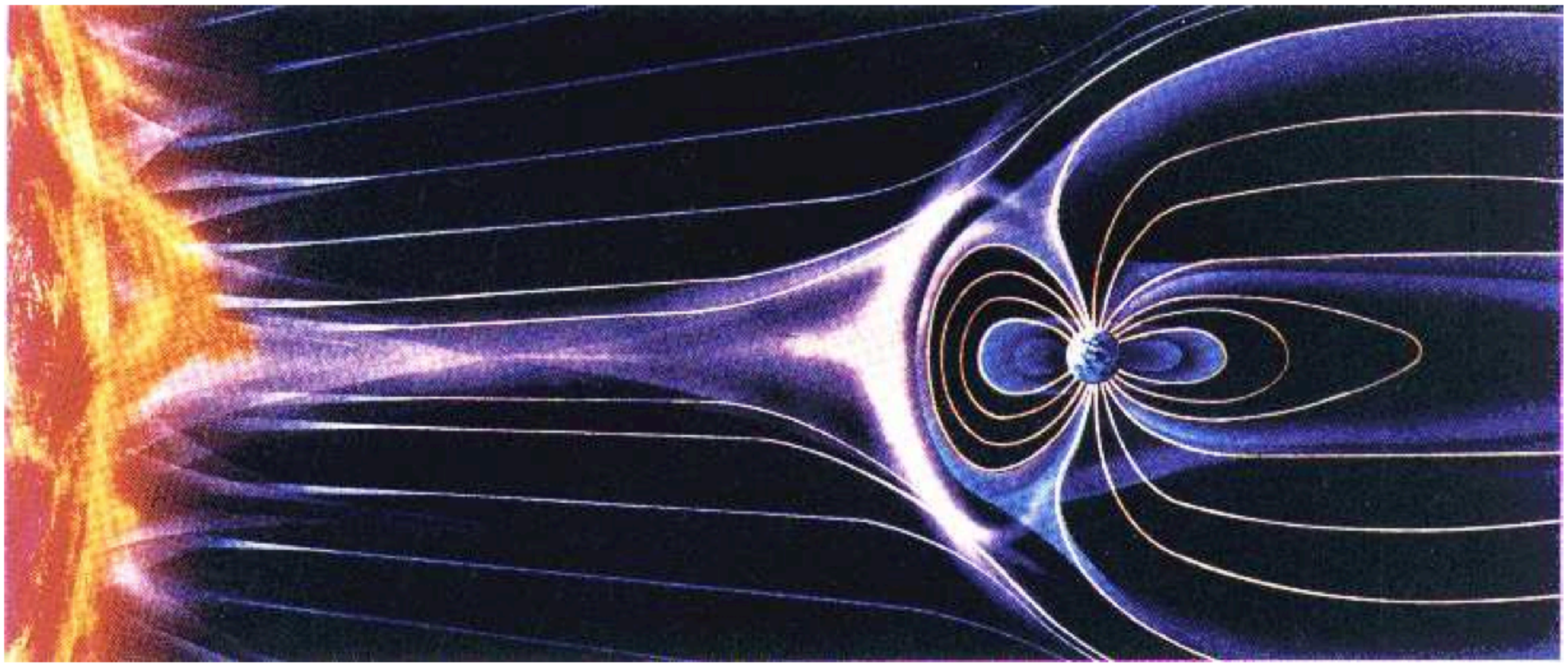
ideal MHD

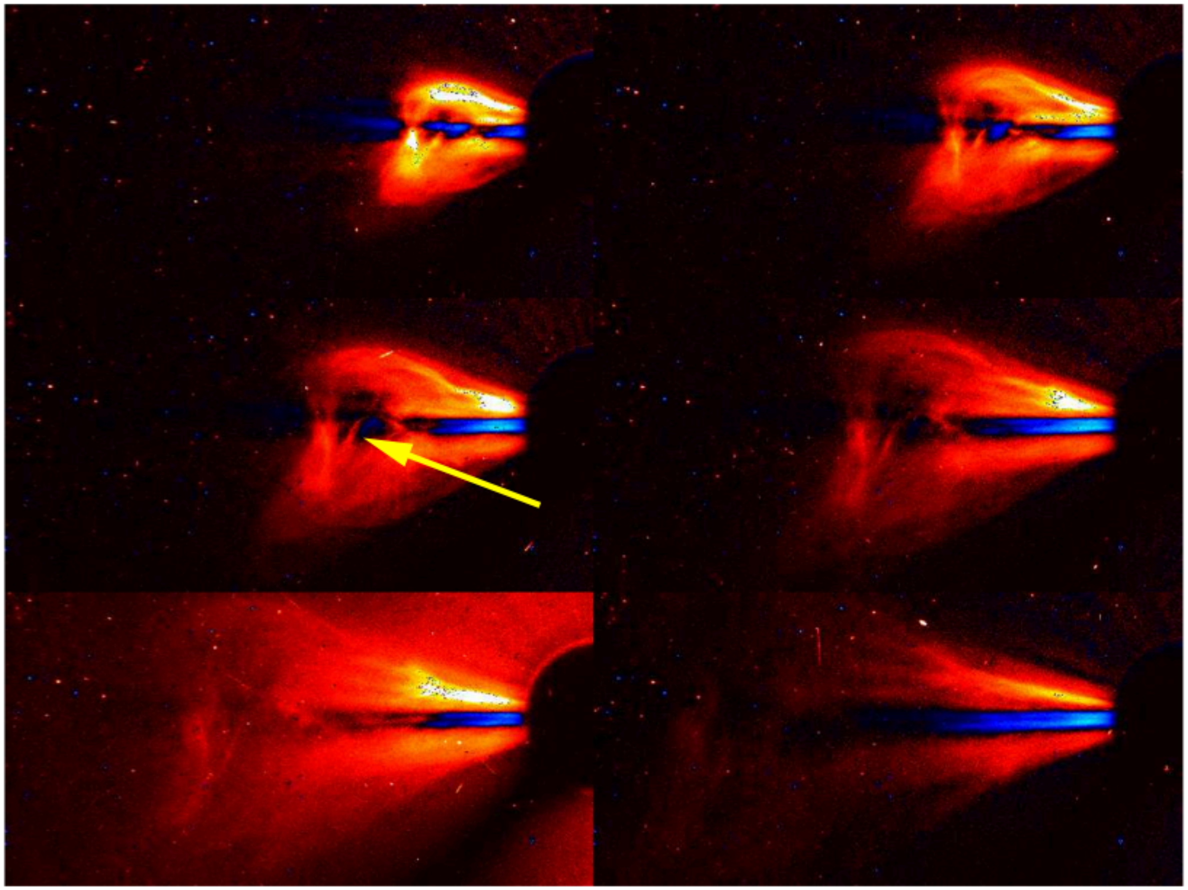
- ‘perfectly conducting fluid’ models space and fusion plasmas (ionized gas) in certain parameter regimes

Earth’s bow shock



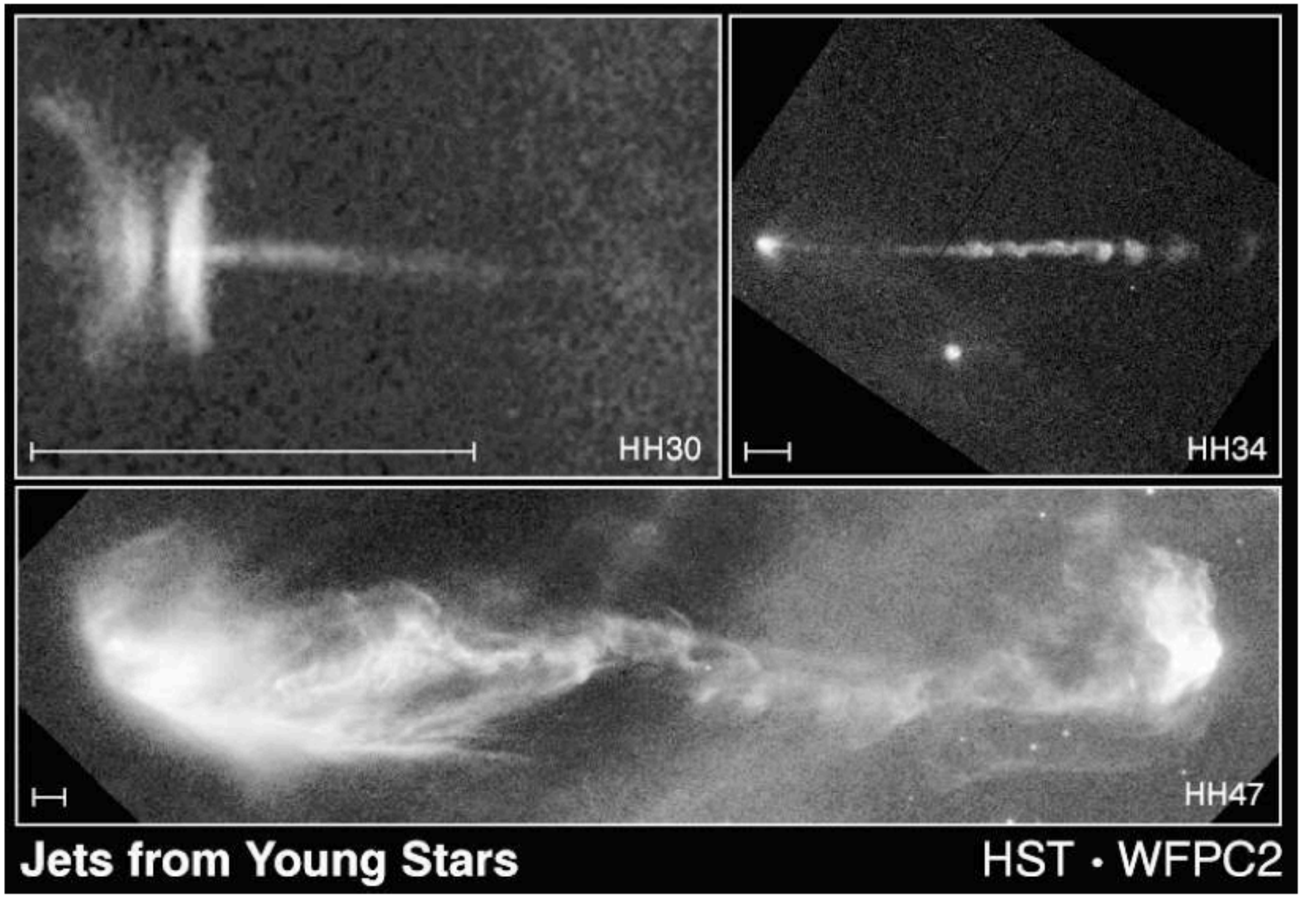
ideal MHD

Solar Coronal Mass Ejections



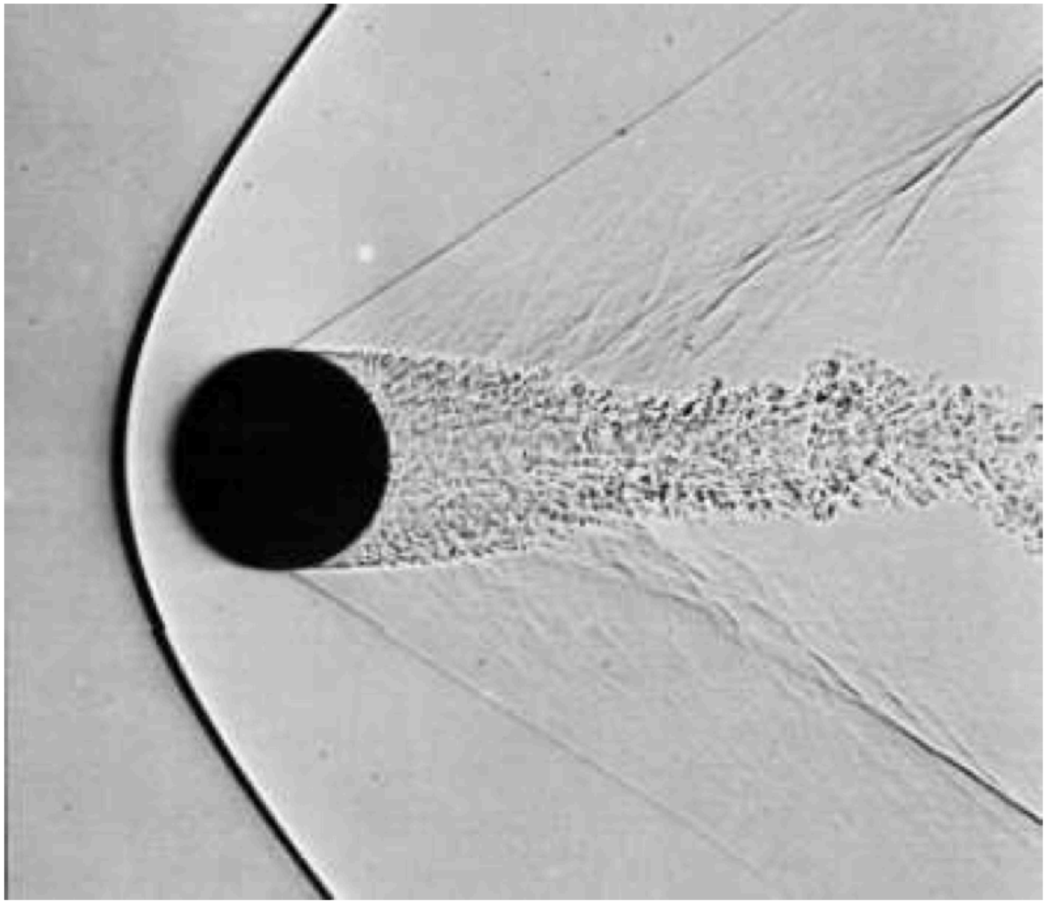
ideal MHD

Astrophysical jets



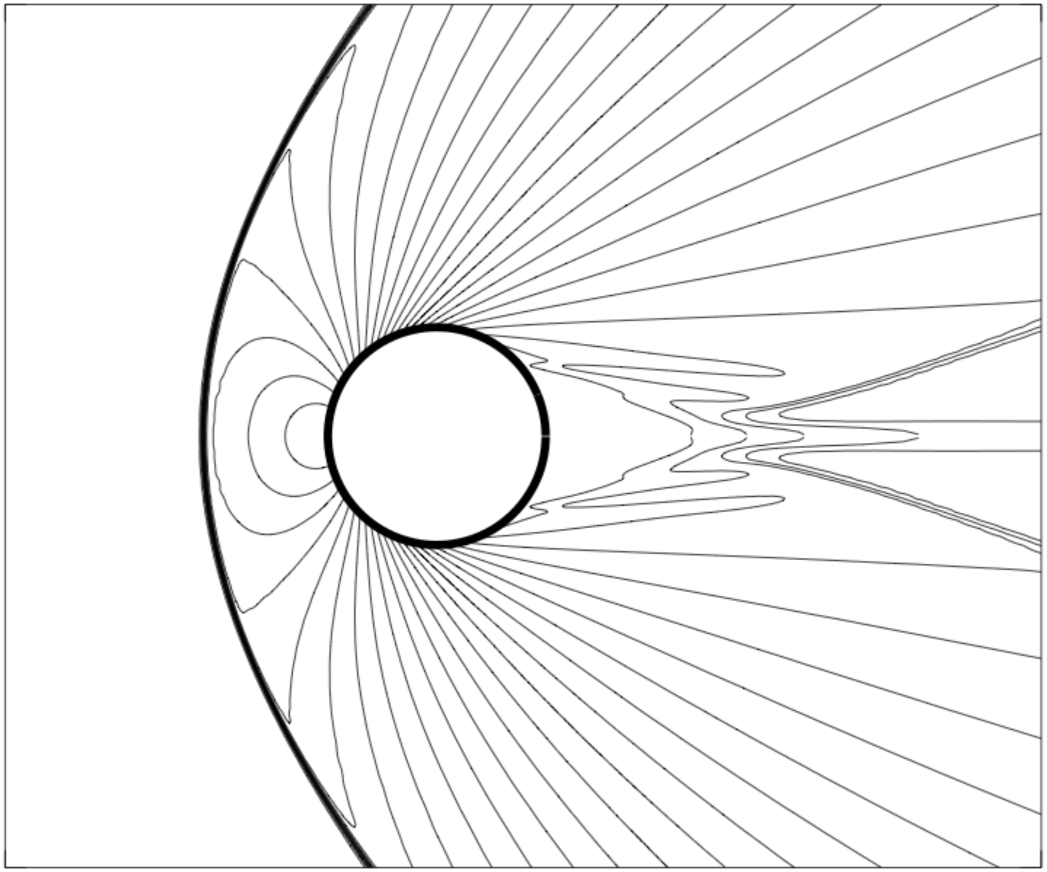
ideal MHD

Supersonic airflow over sphere



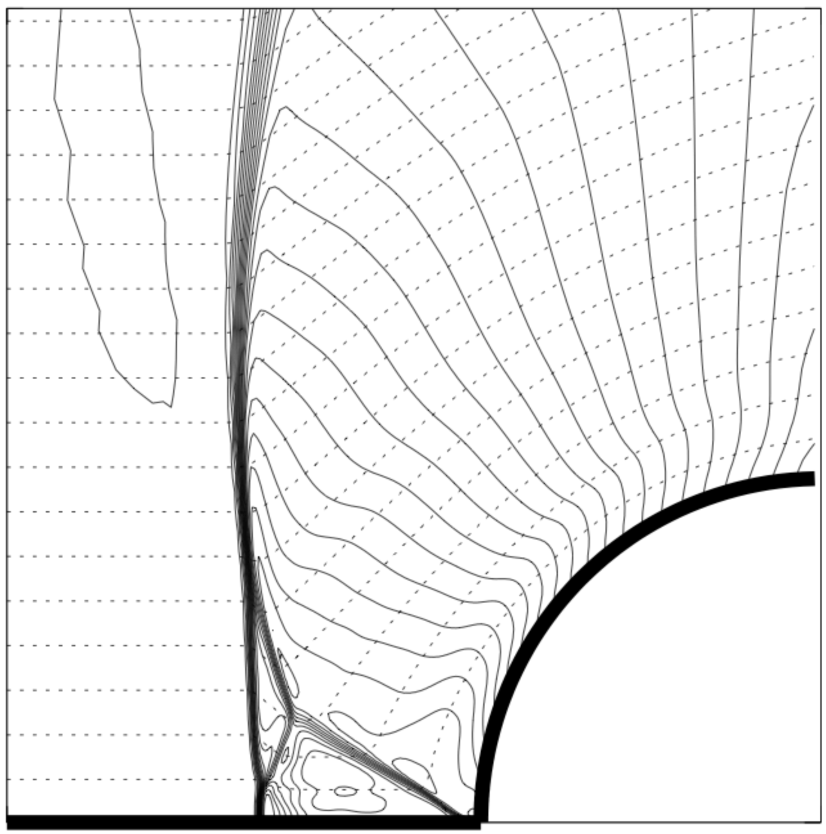
ideal MHD

Numerical simulation of bow shock (gasdynamic)

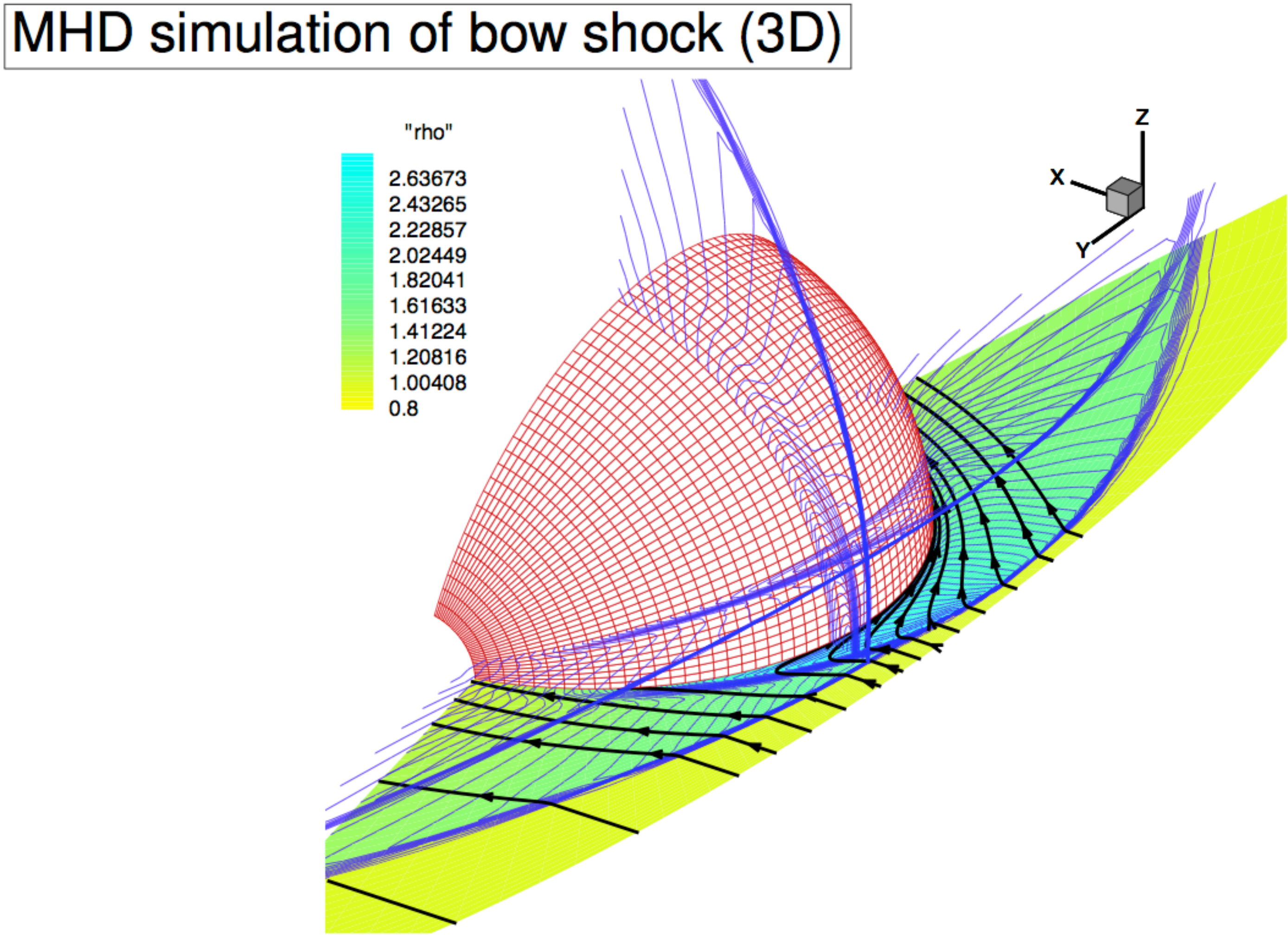


ideal MHD

MHD simulation of bow shock (2D)



ideal MHD


MHD simulation of bow shock (3D)
"rho"
2.63673
2.43265
2.22857
2.02449
1.82041
1.61633
1.41224
1.20816
1.00408
0.8
Z
X
Y

this mini-course

"Numerical Magnetohydrodynamics with Application to Space Physics Flows"

- lecture 1: Structure of MHD as a Hyperbolic System
  (conservation, waves, shocks; differences with Euler)
- lecture 2: Finite Volume Methods for MHD
  (FV methods, divergence constraint, high-order methods, adaptive cubed-sphere grids)
- lecture 3: Numerical Methods for Transonic Solutions
  (transitions from supersonic to subsonic flow (e.g., solar wind), critical points, dynamical systems methods)

some references

- hyperbolic conservation laws (and finite volume methods):
  - Leveque, Numerical methods for conservation laws, Birkhauser, 1992 (no MHD)
  - Leveque, Finite Volume Methods for Hyperbolic Problems, 2002 (Cambridge Texts in Applied Mathematics)
  - Courant and Hilbert, Methods of mathematical physics, vol. 2, Interscience, 1962
  - Courant and Friedrichs, Supersonic flow and shock waves, Interscience, 1948
  - Whitham, Linear and nonlinear waves, Wiley-Interscience, 1974

some references

- MHD:
  - Landau and Lifshitz, Electrodynamics of Continuous Media, Pergamon, 1984
  - Jeffrey and Taniuti, Nonlinear wave propagation, Academic, Press, 1964
  - Anderson, Magnetohydrodynamic shock waves, MIT Press, 1963
  - Goedbloed and Poedts, Principles of Magnetohydrodynamics, Cambridge University Press, 2004
  - Goedbloed, Keppens and Poedts, Advanced Magnetohydrodynamics, Cambridge University Press, 2010
  - (also De Sterck, PhD thesis, 1999)

lecture 1: Structure of MHD as a Hyperbolic System

1.1 hyperbolic conservation laws

1.2 MHD waves

1.3 MHD shocks

1.1 hyperbolic conservation laws

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\vec{v} \\ \frac{\rho v^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2} \\ \vec{B} \end{bmatrix} + \nabla\cdot\begin{bmatrix} \rho\vec{v} \\ \rho\vec{v}\vec{v}+\left(p+\frac{B^2}{2}\right)\vec{I}-\vec{B}\vec{B} \\ \left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+p\right)\vec{v}-(\vec{v}\times\vec{B})\times\vec{B} \\ \vec{v}\vec{B}-\vec{B}\vec{v} \end{bmatrix} = 0$$

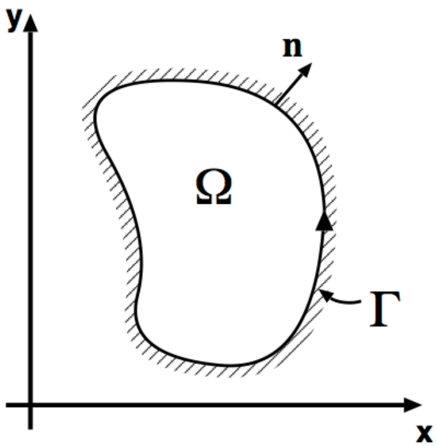
- conservation law: $$\frac{\partial U}{\partial t}+\nabla\cdot\vec{F}(U)=0$$

(nonlinear flux function $\vec{F}(U)$)

integral form of conservation law

- conservation law: $$\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0$$

- integral form:



$$\frac{\partial \overline{U}}{\partial t} + \oint_{\partial\Omega} \vec{F}(U) \cdot \vec{n} \, dA = 0$$

with $$\overline{U} = \int_{\Omega} U \, dV$$

quasi-linear form in 1D

$$\boxed{\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0}$$

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

$$\Downarrow$$

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial U}\frac{\partial U}{\partial x} = 0$$

$$\Downarrow \frac{\partial F(U)}{\partial U} = \mathsf{A}(U)$$

$$\boxed{\frac{\partial U}{\partial t} + \mathsf{A}(U) \cdot \frac{\partial U}{\partial x} = 0 \quad (1)}$$

with $U(x,t) = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in R^n$

and $F(U) = \begin{bmatrix} f_1(U) \\ \vdots \\ f_n(U) \end{bmatrix}$

hyperbolic conservation law

$$\boxed{\frac{\partial U}{\partial t} + \mathsf{A}(U) \cdot \frac{\partial U}{\partial x} = 0 \quad (1)}$$

(1) is a hyperbolic system of equations

$$\Updownarrow$$

$\mathsf{A}(U)$ has n real eigenvalues and a complete set of eigenvectors $\forall U$

$$\Updownarrow$$

the system has n real characteristic curves

$$\Rightarrow \mathsf{A}(U) = \mathsf{R}(U) \cdot \Lambda(U) \cdot \mathsf{L}(U)$$

wave speeds – characteristic variables

$$\frac{\partial U}{\partial t} + \mathsf{R}(U) \cdot \Lambda(U) \cdot \mathsf{L}(U) \cdot \frac{\partial U}{\partial x} = 0$$

$$\frac{\mathsf{L}(U) \cdot \partial U}{\partial t} + \Lambda(U) \cdot \frac{\mathsf{L}(U) \cdot \partial U}{\partial x} = 0$$

define (formally) $\boxed{\partial W = \mathsf{L}(U) \cdot \partial U : \quad \text{characteristic variables } \partial W}$

$$\boxed{\frac{\partial W}{\partial t} + \Lambda(U) \cdot \frac{\partial W}{\partial x} = 0}$$

n wave modes with wave speeds λ_i

1.2 MHD waves

- start with the Euler equations (1D, 2 velocity components)

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0}$$

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho\, e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p \\ \rho v_x v_y \\ (\rho\, e + p) v_x \end{bmatrix}$$

$$\rho\, e = \frac{p}{(\gamma - 1)} + \frac{1}{2}\rho v^2$$

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conservative and primitive variables

- vector of conservative variables:

$$U = \begin{bmatrix} \rho \\ m_x \\ m_y \\ E \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho\, e = \frac{p}{(\gamma-1)} + \frac{1}{2}\rho v^2 \end{bmatrix} \text{ in } \frac{\partial U}{\partial t} + \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial x} = 0$$

- vector of primitive variables: $V = \begin{bmatrix} \rho \\ v_x \\ v_y \\ p \end{bmatrix}$

- transformation: $\partial U = \dfrac{\partial U}{\partial V} \cdot \partial V$

conservative and primitive variables

$$\frac{\partial U}{\partial V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_x & \rho & 0 & 0 \\ v_y & 0 & 1 & 0 \\ v^2/2 & \rho\, v_x & \rho\, v_y & 1/(\gamma - 1) \end{bmatrix}$$

$$\frac{\partial V}{\partial U} = \frac{\partial U}{\partial V}^{-1}$$

$$\Rightarrow \frac{\partial U}{\partial V} \cdot \frac{\partial V}{\partial t} + \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial V} \cdot \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial U} \cdot \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial V} \cdot \frac{\partial V}{\partial x} = 0$$

define $\mathsf{A}(V)_V = \frac{\partial V}{\partial U} \cdot \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial V}$

with property $\lambda(\mathsf{A}(V)_V) = \lambda(\mathsf{A}(U)_U)$ (similarity transformation)

$$\Rightarrow \frac{\partial V}{\partial t} + \mathsf{A}(V)_V \cdot \frac{\partial V}{\partial x} = 0$$

Euler waves

- $\frac{\partial V}{\partial t} + \mathsf{A}(V)_V \cdot \frac{\partial V}{\partial x} = 0$

  with $V = \begin{bmatrix} \rho \\ v_x \\ v_y \\ p \end{bmatrix}$ and $\mathsf{A}_V = \begin{bmatrix} v_x & \rho & 0 & 0 \\ 0 & v_x & 0 & 1/\rho \\ 0 & 0 & v_x & 0 \\ 0 & c^2\rho & 0 & v_x \end{bmatrix}$

  $c = \sqrt{\frac{\gamma p}{\rho}}$

  $\Rightarrow$

  $\lambda_1 = v_x$: entropy wave

  $\lambda_2 = v_x$: shear wave

  $\lambda_3 = v_x + c$: sound wave, right traveling

  $\lambda_4 = v_x - c$: sound wave, left traveling

Euler waves

$$\Rightarrow \mathsf{R} = \left[\begin{array}{c|c|c|c} 1 & 0 & \rho & \rho \\ 0 & 0 & c & -c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho\, c^2 & \rho\, c^2 \end{array}\right]$$

$$\Rightarrow \mathsf{L} = \left[\begin{array}{cccc} 1 & 0 & 0 & -1/c^2 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1/(2c) & 0 & 1/(2\rho c^2) \\ \hline 0 & -1/(2c) & 0 & 1/(2\rho c^2) \end{array}\right]$$

$\Rightarrow$ hyperbolic system

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wave properties

$\lambda_1 = v_x$: entropy wave

$\lambda_2 = v_x$: shear wave

$\lambda_3 = v_x + c$: sound wave, right traveling

$\lambda_4 = v_x - c$: sound wave, left traveling

- entropy wave and shear wave are linearly degenerate waves:

$$\nabla \lambda_k(U) \cdot R_k(U) = 0 \qquad \forall U$$

  - do not steepen nor rarify; arbitrary profile exactly preserved

- sound waves are genuinely nonlinear waves:

$$\nabla \lambda_k(U) \cdot R_k(U) \neq 0 \qquad \forall U$$

  - steepen into shocks; rarefactions

Euler Riemann problem

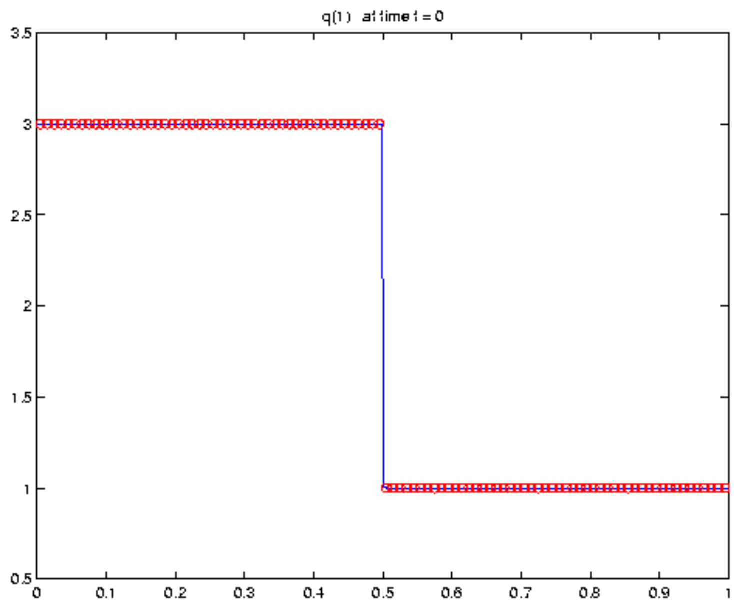
- Sod shock tube problem

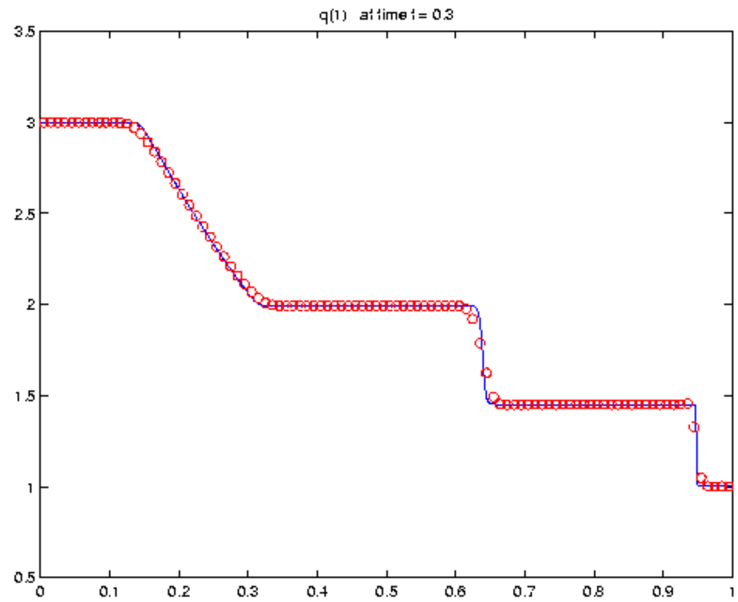
$\lambda_1 = v_x$: entropy wave

$\lambda_2 = v_x$: shear wave

$\lambda_3 = v_x + c$: sound wave, right traveling

$\lambda_4 = v_x - c$: sound wave, left traveling





(clawpack)

MHD waves

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ \rho e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho\, v_x \\ \rho\, v_x^2 + p + B^2/2 - B_x^2 \\ \rho\, v_x v_y - B_x B_y \\ \rho\, v_x v_z - B_x B_z \\ 0 \\ B_y v_x - B_x v_y \\ B_z v_x - B_x v_z \\ (\rho\, e + p + B^2/2) v_x - B_x\, (\vec{v} \cdot \vec{B}) \end{bmatrix}$$

$$\rho\, e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2 + \frac{1}{2} B^2$$

conservative variables

- vector of conservative variables:

$$U = \begin{bmatrix} \rho \\ m_x \\ m_y \\ m_z \\ B_x \\ B_y \\ B_z \\ E \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ \rho\, e = \frac{p}{(\gamma-1)} + \frac{1}{2}\rho v^2 + \frac{1}{2}B^2 \end{bmatrix} \text{ in } \frac{\partial U}{\partial t} + \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial x} = 0$$

primitive variables

- vector of primitive variables: $V = \begin{bmatrix} \rho \\ v_x \\ v_y \\ v_z \\ B_x \\ B_y \\ B_z \\ p \end{bmatrix}$
- transformation: $\partial U = \frac{\partial U}{\partial V} \cdot \partial V$

$$\Rightarrow \frac{\partial V}{\partial t} + \mathsf{A}(V)_V \cdot \frac{\partial V}{\partial x} = 0$$

hyperbolic system

- Hyperbolic system:

with $A_V =$
$$\begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & 0 & B_y/\rho & B_z/\rho & 1/\rho \\ 0 & 0 & v_x & 0 & 0 & -B_x/\rho & 0 & 0 \\ 0 & 0 & 0 & v_x & 0 & 0 & -B_x/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_y & -B_x & 0 & 0 & v_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & 0 & v_x & 0 \\ 0 & c^2\rho & 0 & 0 & 0 & 0 & 0 & v_x \end{bmatrix} \quad \left(c = \sqrt{\frac{\gamma p}{\rho}}\right)$$

$\lambda_1 = v_x + c_{fx}$: fast wave, right

$\lambda_2 = v_x - c_{fx}$: fast wave, left

$\lambda_3 = v_x + c_{Ax}$: Alfvén wave, right

$\lambda_4 = v_x - c_{Ax}$: Alfvén wave, left

$\lambda_5 = v_x + c_{sx}$: slow wave, right

$\lambda_6 = v_x - c_{sx}$: slow wave, left

$\lambda_7 = v_x$: entropy wave

$\lambda_8 = 0$: not Galilean invariant!!

MHD wave speeds

with

$$c_{fx}^2 = \frac{1}{2}\left(\frac{\gamma p + B^2}{\rho} + \sqrt{\left(\frac{\gamma p + B^2}{\rho}\right)^2 - 4\frac{\gamma p B_x^2}{\rho^2}}\right)$$

$$c_{Ax}^2 = \frac{B_x^2}{\rho}$$

$$c_{sx}^2 = \frac{1}{2}\left(\frac{\gamma p + B^2}{\rho} - \sqrt{\left(\frac{\gamma p + B^2}{\rho}\right)^2 - 4\frac{\gamma p B_x^2}{\rho^2}}\right)$$

wave speeds anisotropic!! (depending on angle between propagation direction x and local magnetic field $\vec{B}$)

MHD waves versus Euler waves

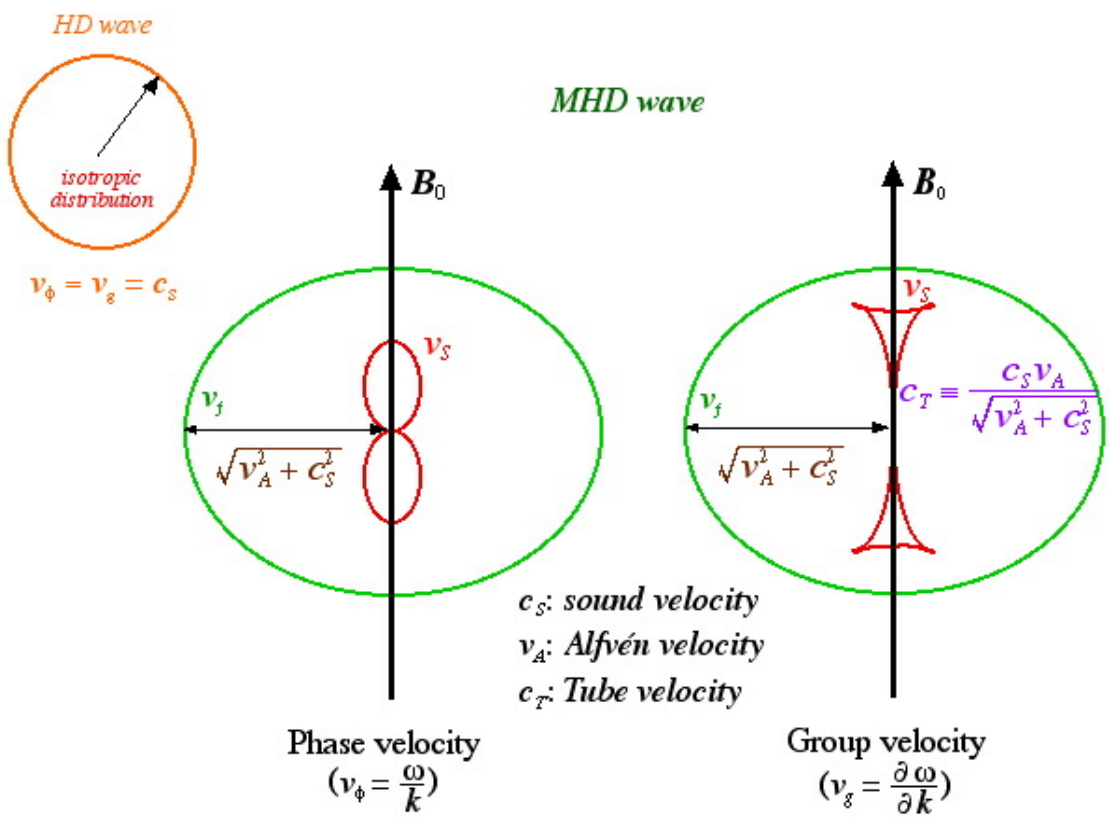
- Euler: isotropic

  phase speed =

  group speed =

  sound speed $c = \sqrt{\frac{\gamma p}{\rho}}$

- MHD: anisotropic



(solar.physics.montana.edu)

MHD wave properties

$\lambda_1 = v_x + c_{fx}$: fast wave, right — nonlinear

$\lambda_2 = v_x - c_{fx}$: fast wave, left — nonlinear

$\lambda_3 = v_x + c_{Ax}$: Alfvén wave, right — linearly degenerate

$\lambda_4 = v_x - c_{Ax}$: Alfvén wave, left — linearly degenerate

$\lambda_5 = v_x + c_{sx}$: slow wave, right — nonlinear

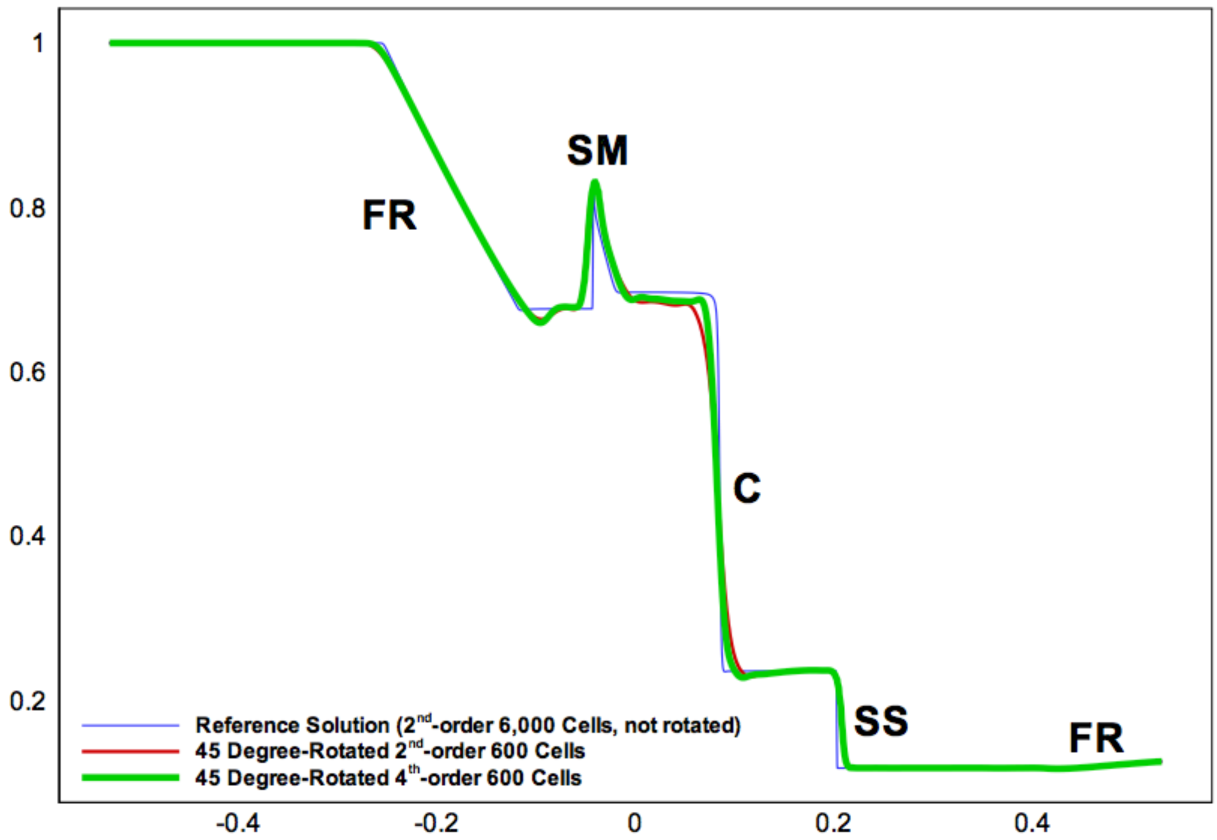
$\lambda_6 = v_x - c_{sx}$: slow wave, left — nonlinear

$\lambda_7 = v_x$: entropy wave — linearly degenerate

$\lambda_8 = 0$: not Galilean invariant!!

MHD Riemann problem

- Brio-Wu shock tube problem



$\lambda_1 = v_x + c_{fx}$: fast wave, right

$\lambda_2 = v_x - c_{fx}$: fast wave, left

$\lambda_3 = v_x + c_{Ax}$: Alfvén wave, right

$\lambda_4 = v_x - c_{Ax}$: Alfvén wave, left

$\lambda_5 = v_x + c_{sx}$: slow wave, right

$\lambda_6 = v_x - c_{sx}$: slow wave, left

$\lambda_7 = v_x$: entropy wave

$\lambda_8 = 0$: not Galilean invariant!!

problem: too many waves! (SM = shock and rarefaction?)

wave types

- linearly degenerate: $\boxed{\nabla\lambda_k(U)\cdot R_k(U)=0 \qquad \forall U}$
  - Euler: entropy (contact)
  - MHD: entropy, Alfven
- genuinely nonlinear: $\boxed{\nabla\lambda_k(U)\cdot R_k(U)\neq 0 \qquad \forall U}$
  - Euler: sound waves
- nonconvex nonlinear: $\boxed{\nabla\lambda_k(U)\cdot R_k(U)=0 \text{ for some } U}$
  - MHD: fast and slow waves
    - ➔ 'compound shocks' occur: shock with attached rarefaction

nonconvex nonlinear waves

- nonconvex nonlinear waves:

$\nabla \lambda_k(U) \cdot R_k(U) = 0$ for some U

scalar ($k = 1$): $\nabla \lambda(U) = \frac{\partial \lambda}{\partial u} = \frac{\partial^2 f(u)}{\partial u} = f''(u)$, $R_k = 1$

nonconvex $\Leftrightarrow$ $f''(u) = 0$ for some u

- consider scalar conservation law:

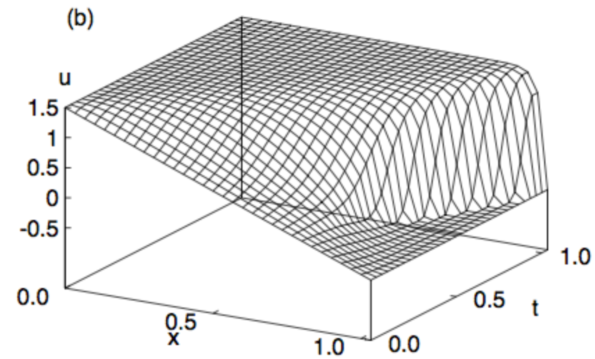
$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0 \qquad \text{with} \qquad f'(u) = \frac{\partial f(u)}{\partial u}$$

scalar conservation laws - characteristics

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

$$\frac{\partial u}{\partial t} + f'(u)\,\frac{\partial u}{\partial x} = 0 \qquad \text{with} \qquad f'(u) = \frac{\partial f(u)}{\partial u}$$

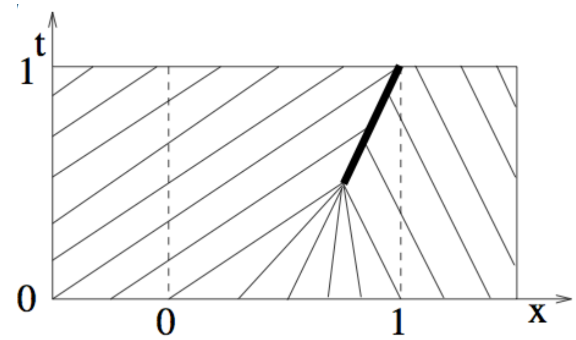


- consider curve $x(t)$ in xt-plane, how does u vary along curve?

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = \frac{\partial u(x(t),t)}{\partial t} + \frac{\partial u(x(t),t)}{\partial x}\,\frac{\partial x(t)}{\partial t}$$

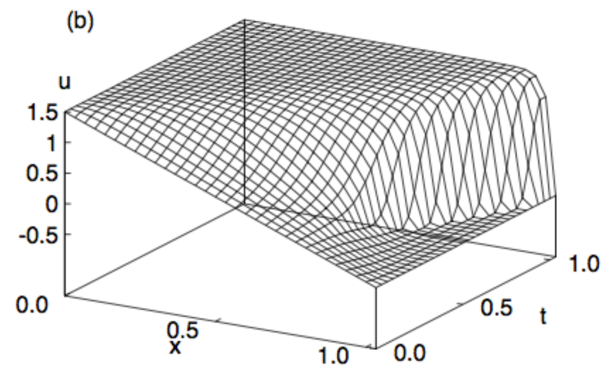
$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \equiv 0 \qquad \text{if} \qquad \frac{\partial x(t)}{\partial t} = f'(u)$$

characteristic curve: $\boxed{x(t) : \frac{\partial x(t)}{\partial t} = f'(u)}$



scalar conservation laws - characteristics

characteristic curve: $x(t) : \dfrac{\partial x(t)}{\partial t} = f'(u)$

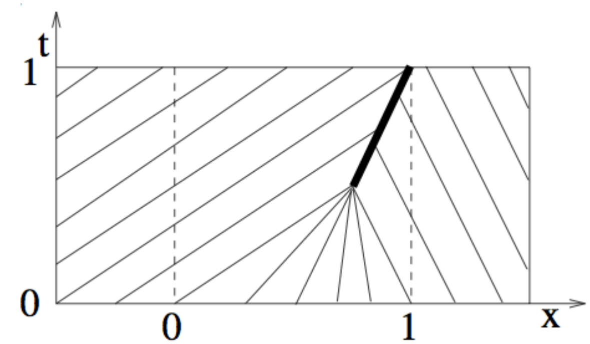


- example: (inviscid) Burgers equation

$$f(u) = \frac{u^2}{2}$$

$$f'(u) = u$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0$$



entropy-satisfying shock:
characteristics converge into shock

Rankine-Hugoniot relation: $s = \dfrac{\Delta f}{\Delta u} = \dfrac{f(u_r) - f(u_l)}{u_r - u_l}$

Burgers Riemann problem

- Riemann problem = how does initial discontinuity between two constant states u_l and u_r evolve in time?

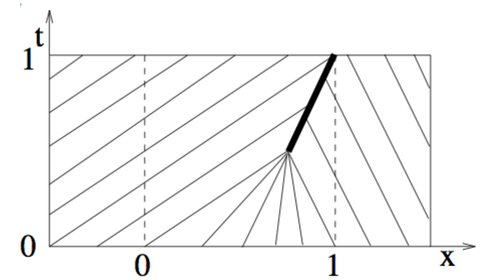
$\Rightarrow$ Burgers:

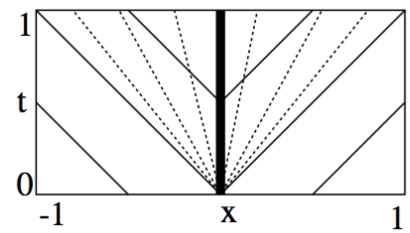
- $f'(u_l) > f'(u_r)$, characteristics converge

$\Rightarrow$ shock with shock speed s from RH relation

- $f'(u_l) < f'(u_r)$, characteristics diverge

$\Rightarrow$ (continuous) rarefaction wave





- for Burgers: every Riemann problem results in either a shock or a rarefaction

  - this is true for all convex flux functions $f(u)$

  - $f(u)$ is convex $\Leftrightarrow f'(u)$ is monotone $\Leftrightarrow f''(u)$ does not change sign

  - $f(u)$ is convex $\Rightarrow f'(u_l) > s > f'(u_r)$ when characteristics converge

  reason: $s = \dfrac{f(u_r) - f(u_l)}{u_r - u_l} = f'(c)$ with c between u_l and u_r due to mean value theorem, and $f'(u)$ is monotone

  - Burgers is convex ($f(u) = u^2/2$)

- if $f(u)$ is non-convex: Riemann problems may have more complicated solutions than just a shock or a rarefaction

non-convex case

- example: $f(u) = \frac{u^3}{3}$, $f'(u) = 3\,u^2$

$\Rightarrow$ non-convex

Riemann problem: $u_l = 1$ and $u_r = -0.75$

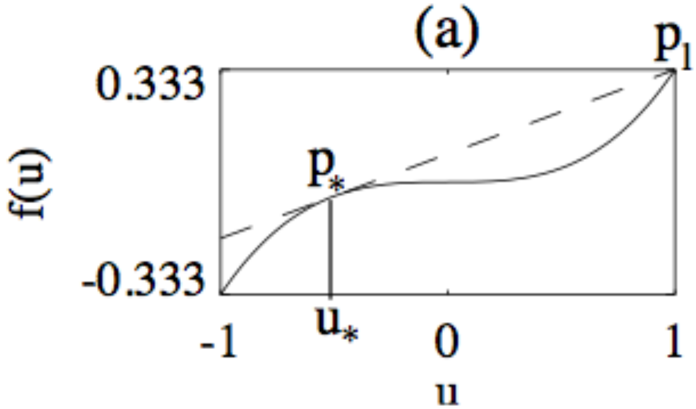
RH: $f'(u_l) > f'(u_r) > s = 37/168$

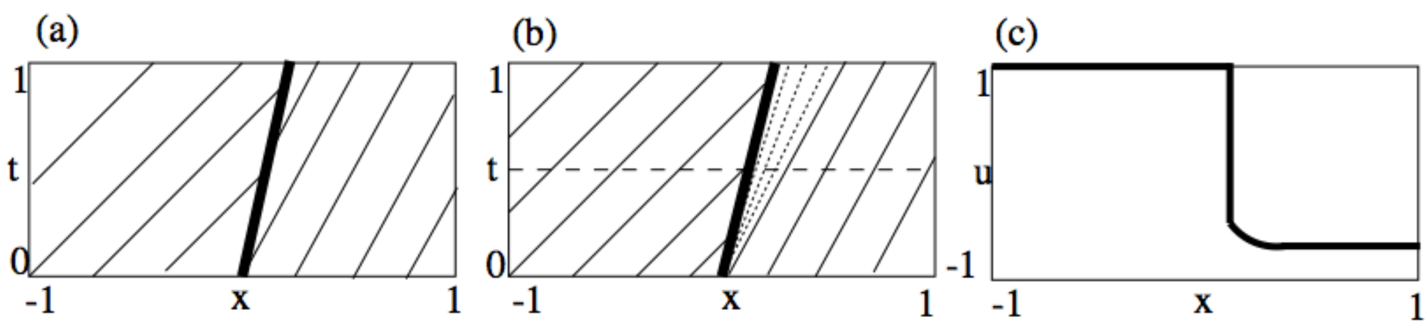
right characteristics diverge from shock : unstable

solution: rarefaction starting from right state until right characteristic becomes parallel to shock (with speed $\neq s$!)

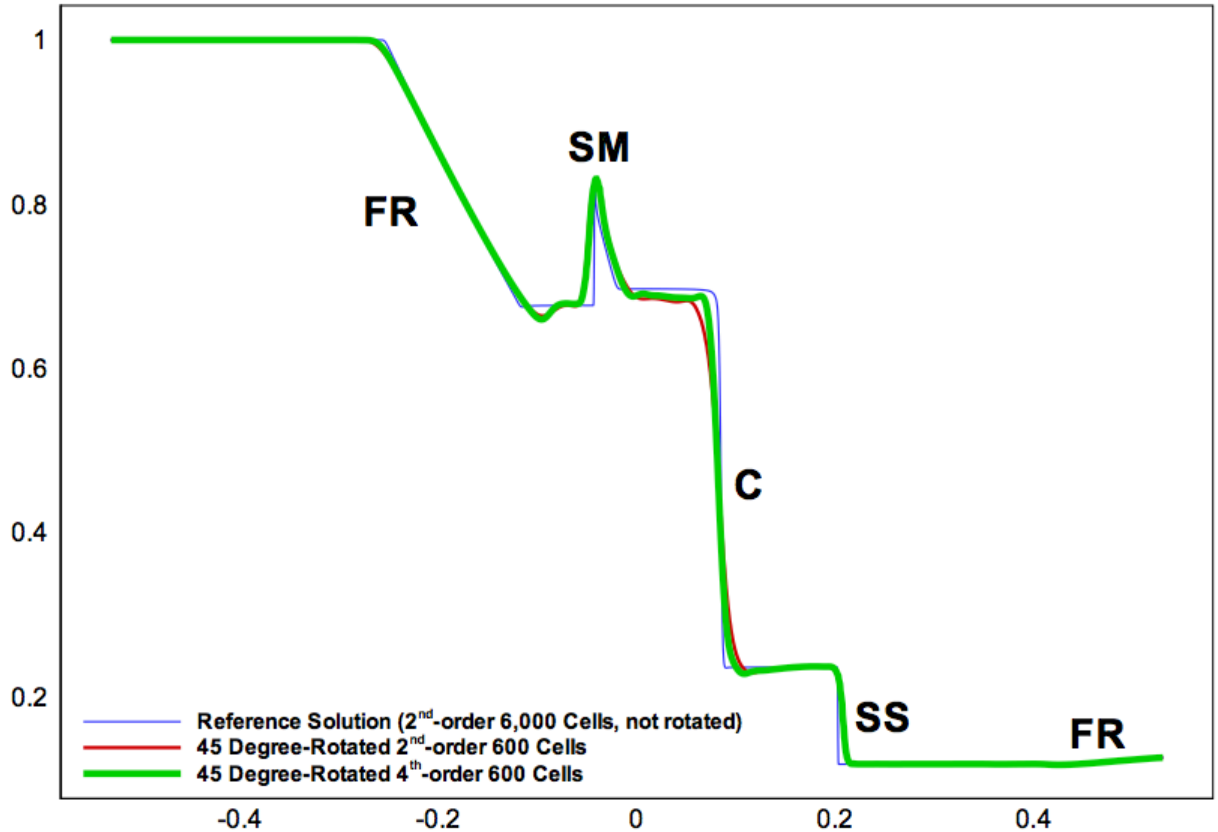
condition: $f'(u^*) = s^* = \frac{f(u^*) - f(u_l)}{u^* - u_l}$

tangent hull construction





MHD compound shocks



$\lambda_1 = v_x + c_{fx}$: fast wave, right

$\lambda_2 = v_x - c_{fx}$: fast wave, left

$\lambda_3 = v_x + c_{Ax}$: Alfvén wave, right

$\lambda_4 = v_x - c_{Ax}$: Alfvén wave, left

$\lambda_5 = v_x + c_{sx}$: slow wave, right

$\lambda_6 = v_x - c_{sx}$: slow wave, left

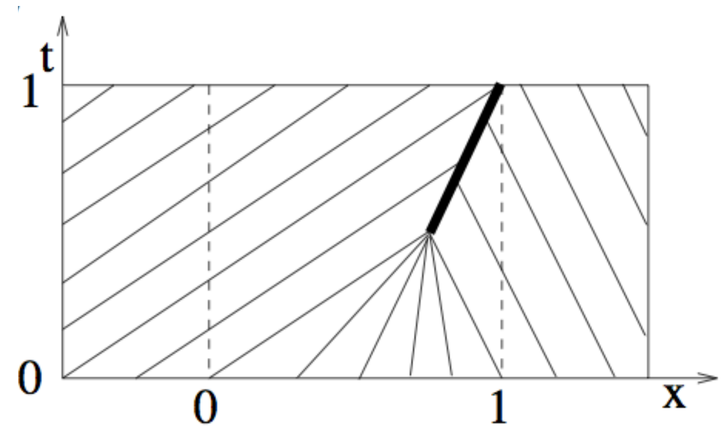
$\lambda_7 = v_x$: entropy wave

$\lambda_8 = 0$: not Galilean invariant!!

1.3 MHD shocks

first consider scalar case:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \qquad u(x,t)$$



$$\int\int \left(\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x}\right) dx\, dt = 0$$

$$\Downarrow$$

$$\int_{t_0}^{t_1} \frac{\partial \overline{u}(t)}{\partial t} dt + \int_{x_0}^{x_1} \frac{\partial \overline{\overline{f}}(x)}{\partial x} dx = 0$$

$$\Downarrow$$

$$\overline{u}(t_1) - \overline{u}(t_0) + \overline{\overline{f}}(x_1) - \overline{\overline{f}}(x_0) = 0$$

with $\overline{u}(t) = \int_{x_0}^{x_1} u(x,t)dx$

$$\overline{\overline{f}}(x) = \int_{t_0}^{t_1} f(u(x,t))dt$$

shocks for scalar conservation laws

$$\int\int\left(\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}\right)dx\,dt=0$$

$$\Downarrow$$

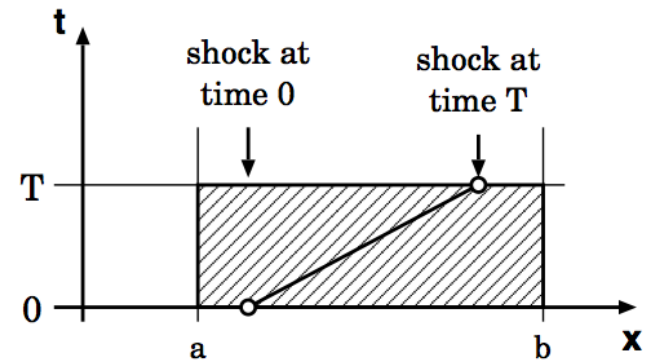
$$\int_{t_0}^{t_1}\frac{\partial\overline{u}(t)}{\partial t}dt+\int_{x_0}^{x_1}\frac{\partial\overline{\overline{f}}(x)}{\partial x}dx=0$$

$$\Downarrow$$

$$\boxed{\overline{u}(t_1)-\overline{u}(t_0)+\overline{\overline{f}}(x_1)-\overline{\overline{f}}(x_0)=0}$$

with $\overline{u}(t)=\int_{x_0}^{x_1}u(x,t)dx$

$\overline{\overline{f}}(x)=\int_{t_0}^{t_1}f(u(x,t))dt$

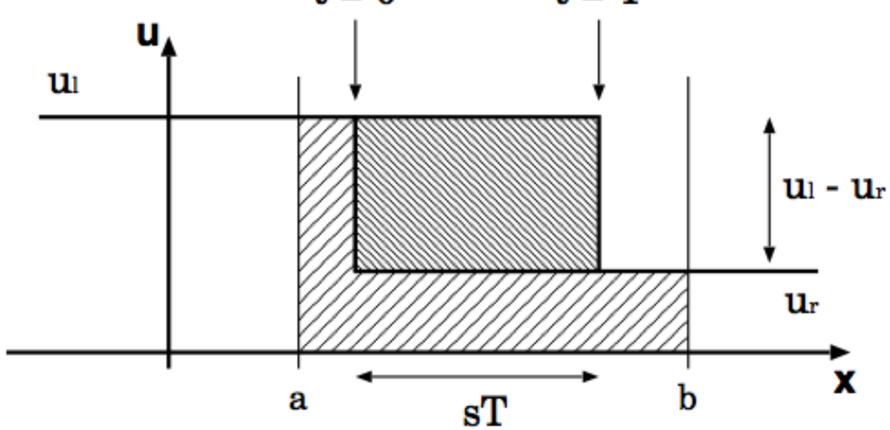


$$\Downarrow$$

$$-s\,\Delta t\,(u_r-u_l)+\Delta t\,(f(u_r)-f(u_l))$$

$$-s\,\Delta u+\Delta f=0$$

$$\boxed{s=\frac{\Delta f}{\Delta u}=\frac{f(u_r)-f(u_l)}{u_r-u_l}}$$



shocks for hyperbolic systems

Rankine-Hugoniot relation:

$$\boxed{s(U_r - U_l) = F(U_r) - F(U_l)}$$

- nonlinear
- small-amplitude shocks: linearize

$s(U_r - U_l) = \mathsf{A}(U_l)\,(U_r - U_l)$

given U_l, there are n small-amplitude shocks, with speeds λ_i, and $U_r - U_l = R_i$

- in general: n shock curves in phase plane (see Leveque)

MHD shocks

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ \rho\, e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho\, v_x \\ \rho\, v_x^2 + p + B^2/2 - B_x^2 \\ \rho\, v_x v_y - B_x B_y \\ \rho\, v_x v_z - B_x B_z \\ 0 \\ B_y v_x - B_x v_y \\ B_z v_x - B_x v_z \\ (\rho\, e + p + B^2/2) v_x - B_x\, (\vec{v} \cdot \vec{B}) \end{bmatrix}$$

MHD RH relations

$$s(U_r - U_l) = F(U_r) - F(U_l)$$

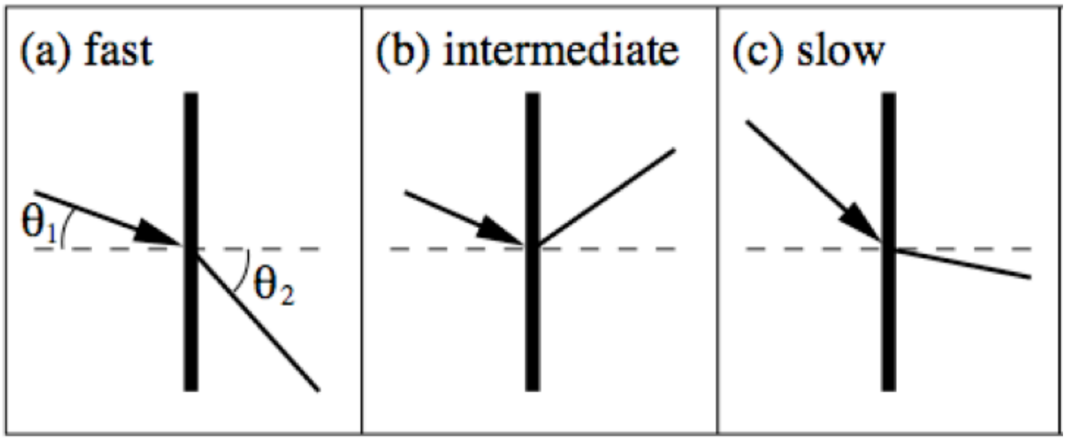
- given U_l, find curves $U(\xi)$ in phase space of points that can be connected to U_l by a k-discontinuity with speed $s(\xi)$

$$s(\xi)\,(U(\xi) - U_l) = F(U(\xi)) - F(U_l)$$

- Riemann problem: given U_1, U_2, find intermediate states U^{*i} such that U_1 and U_2 can be connected by shocks and rarefactions
- given U_l, s: find U_r, and how many U_r exist for a given s and U_l?

MHD shocks

- 3 types of MHD shocks: fast, intermediate, slow



- classification w.r.t. fast, Alfven and slow Mach numbers:

$$M_{fx} = \frac{|v_x|}{c_{fx}}, \quad M_{Ax} = \frac{|v_x|}{c_{Ax}}, \quad \text{and} \quad M_{sx} = \frac{|v_x|}{c_{sx}}$$

MHD shocks

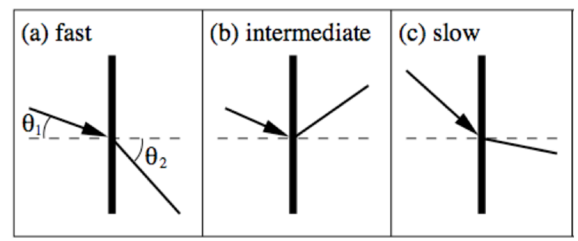
- states of type 1, 2, 3, 4 w.r.t. Mach numbers:

1	$c_{fx}^{(1)} < v_x^{(1)}$	$M_{fx}^{(1)} > 1$	$M_{Ax}^{(1)} > 1$	$M_{sx}^{(1)} > 1$
2	$c_{Ax}^{(2)} < v_x^{(2)} < c_{fx}^{(2)}$	$M_{fx}^{(2)} < 1$	$M_{Ax}^{(2)} > 1$	$M_{sx}^{(2)} > 1$
3	$c_{sx}^{(3)} < v_x^{(3)} < c_{Ax}^{(3)}$	$M_{fx}^{(3)} < 1$	$M_{Ax}^{(3)} < 1$	$M_{sx}^{(3)} > 1$
4	$v_x^{(4)} < c_{sx}^{(4)}$	$M_{fx}^{(4)} < 1$	$M_{Ax}^{(4)} < 1$	$M_{sx}^{(4)} < 1$

fast shock: 1-2

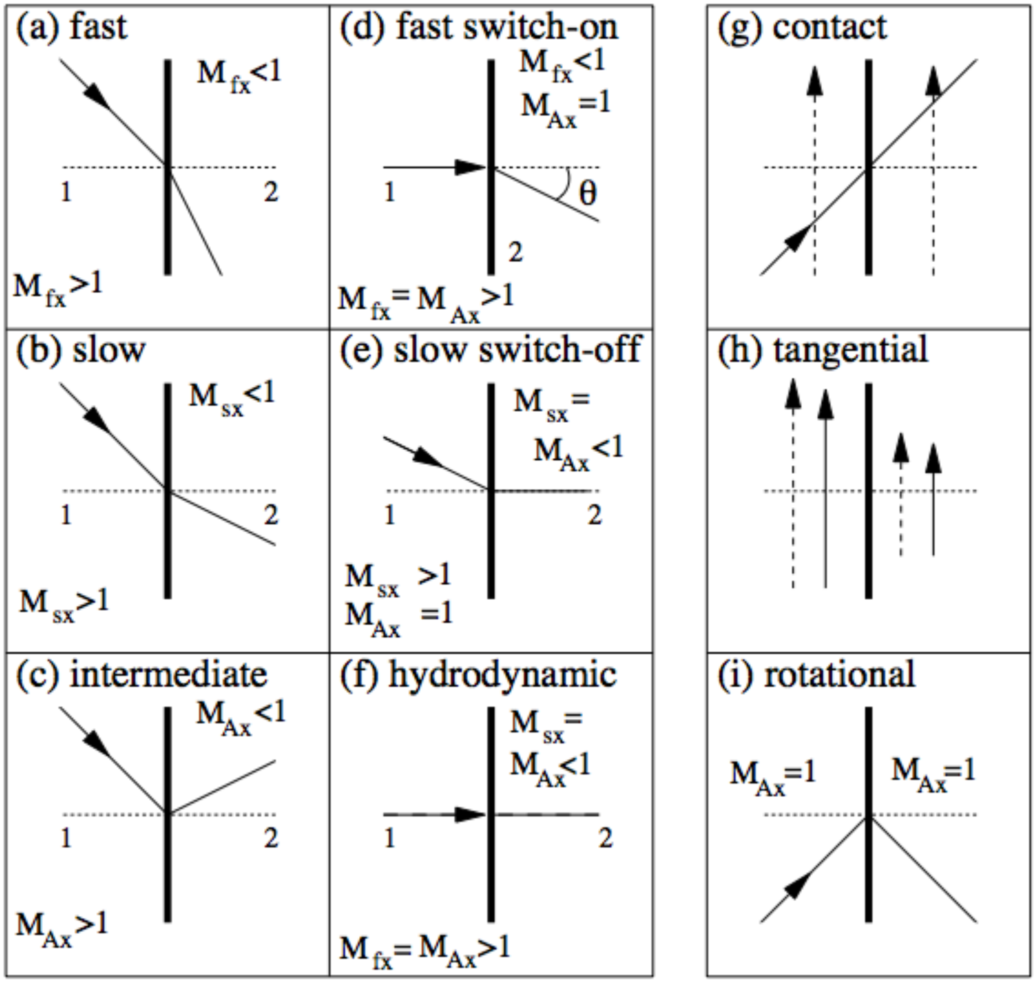
slow shock: 3-4

intermediate shock:
1-3, 1-4, 2-3, 2-4



(note: intermediate shocks have intricate stability properties; they are generically unstable for ideal MHD, but are stable for MHD with finite resistivity)

MHD discontinuities



MHD RH solutions

$$s(U_r - U_l) = F(U_r) - F(U_l)$$

given U_l, s: find U_r, and how many U_r exist for a given s and U_l?

go through succession of simplifications, using properties

- Galilean Invariance: transform to the shock frame

$$x \rightarrow x - s\,t$$

$$v \rightarrow v - s$$

$$(s \rightarrow s - s = 0)$$

$$B \rightarrow B$$

$$\rho \rightarrow \rho$$

$$p \rightarrow p$$

$$F(U) = \begin{bmatrix} \rho\, v_x \\ \rho\, v_x^2 + p + B^2/2 - B_x^2 \\ \rho\, v_x v_y - B_x B_y \\ \rho\, v_x v_z - B_x B_z \\ 0 \\ B_y v_x - B_x v_y \\ B_z v_x - B_x v_z \\ (\rho\, e + p + B^2/2) v_x - B_x\,(\vec{v} \cdot \vec{B}) \end{bmatrix}$$

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RH in the shock frame

$$\Rightarrow F(U_r) - F(U_l) = 0$$

$$[\rho\, v_x] = 0$$

$$[\rho\, v_x^2 + p + B^2/2 - B_x^2] = 0$$

$$[\rho\, v_x v_y - B_x B_y] = 0$$

$$[\rho\, v_x v_z - B_x B_z] = 0$$

$$[B_y v_x - B_x v_y] = 0$$

$$[B_z v_x - B_x v_z] = 0$$

$$[(\rho\, e + p + B^2/2) v_x - B_x\, (\vec{v} \cdot \vec{B})] = 0$$

also: $[B_x] = 0$ (follows from $\nabla \cdot \vec{B} = 0$)

given U_l, 8 equations to obtain 8 U_r state variables

but: nonlinear: 1, 2, ... , many solutions!

RH in the shock frame

- B_x constant, $m = \rho v_x$ constant, $\tau = 1/\rho$ new variables

  $\Rightarrow$ six remaining variables: $\tau, \vec{v}_t = (v_y, v_z), \vec{B}_t = (B_y, B_z), p$

$$m^2[\tau] + [p] + [\frac{\vec{B}_t^2}{2}] = 0$$

$$m[\vec{v}_t] - B_x[\vec{B}_t] = 0$$

$$m[\vec{B}_t\tau] - B_x[\vec{v}_t] = 0$$

$$m[\epsilon(p,\rho) + \frac{1}{2}m^2\tau^2 + \frac{1}{2}\vec{v}_t^2 + p\tau + \vec{B}_t^2\tau] - B_x[\vec{v}_t \cdot \vec{B}_t] = 0$$

classification of MHD discontinuities

A: $v_x = 0$: no mass flow

A1: $v_x = 0$ and $B_x \neq 0$: contact discontinuity

$\Rightarrow$ corresponds to LD entropy wave

$\Rightarrow$ $[\rho]$ arbitrary, all other variables constant

A2: $v_x = 0$ and $B_x = 0$: tangential discontinuity

$\Rightarrow$ $[\rho]$, $[\vec{v}_t]$, $[\vec{B}_t]$ arbitrary, $[p_{tot}]$ constant ($p_{tot} = p + B^2/2$)

B: $v_x \neq 0$: mass flow

$$m[\vec{v}_t] - B_x[\vec{B}_t] = 0$$

$$m[\vec{B}_t \tau] - B_x[\vec{v}_t] = 0$$

$$\Rightarrow [\vec{B}_t] \times [\vec{B}_t \tau] = 0$$

$$(\tau_r - \tau_l)(\vec{B}_{tr} \times \vec{B}_{tl}) = 0$$

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classification of MHD discontinuities

B1: $v_x \neq 0$ and $\tau_r = \tau_l$: rotational discontinuity

$\Rightarrow$ corresponds to LD Alfven wave

$$[\rho] = 0, [v_x] = 0, [p] = 0, [B^2] = 0, [v^2] = 0$$

$$v_x = \pm \frac{B_x}{\sqrt{\rho}}$$

B2: $v_x \neq 0$ and $\tau_r \neq \tau_l$: SHOCKS

$\Rightarrow (\vec{B}_{tr} \times \vec{B}_{tl}) = 0$: coplanarity theorem

choose coordinate system $B_{zl} = B_{zr} \equiv 0$

one more Galilean transformation along the shock front makes $v_{zl} = 0$, and then also $v_{zr} = 0$

MHD shocks

$\Rightarrow$ four remaining equations:

$$m^2[\tau] + [p] + [\frac{B_y^2}{2}] = 0$$

$$m[v_y] - B_x[B_y] = 0$$

$$m[B_y\tau] - B_x[v_y] = 0$$

$$m[\epsilon(p,\rho) + \frac{1}{2}m^2\tau^2 + \frac{1}{2}v_y^2 + p\tau + B_y^2\tau] - B_x[v_yB_y] = 0$$

one more Galilean transformation along the shock front makes $E_{zl} = 0$, and then also $E_{zr} = 0$ ($E_z = (\vec{v} \times \vec{B})_z = v_yB_x - v_xB_y$)

use this to eliminate v_y, three remaining equations

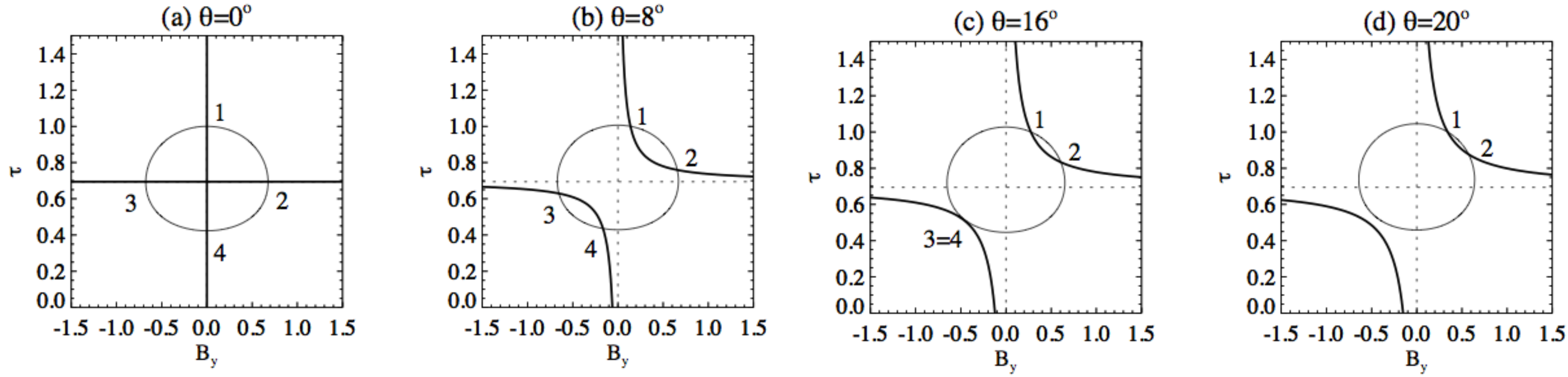
assume ideal gas, eliminate p

$\Rightarrow$ two remaining equations in τ and B_y

solve this graphically

‘magnetically dominated’ regime

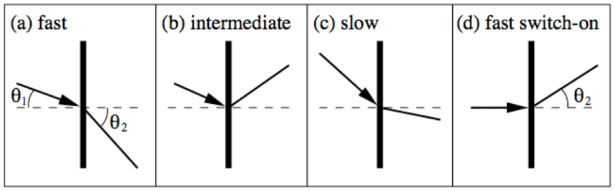
(intermediate shocks can exist)



Solutions of the RH relations for parameter values $M_A = 1.2$ and $\beta = 0.4$ for state 1 with the field aligned to the flow. These are magnetically dominated parameters for which switch-on shocks occur. θ is the angle between the fields and the shock normal. (a) For $\theta = 0^\circ$, switch-on shocks, switch-off shocks and hydrodynamic shocks arise. (b–d) For $\theta < 16^\circ$, intermediate shocks occur, but they cease to exist for larger θ.

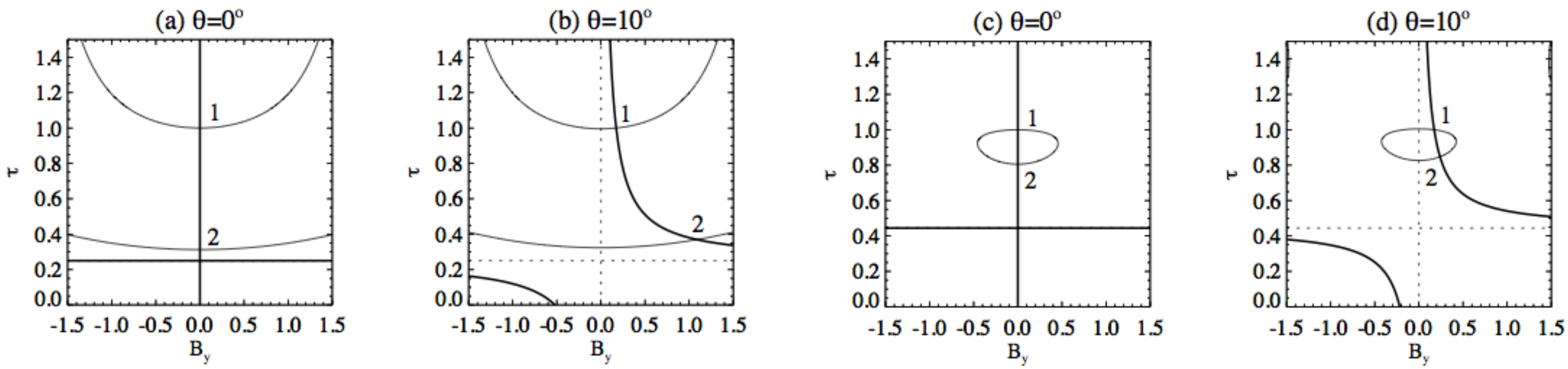
where $\beta = \dfrac{p}{B^2/2}$

(gas pressure / magnetic pressure)

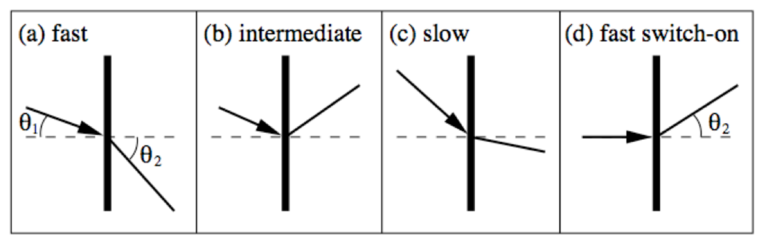


‘pressure-dominated’ regime

(intermediate shocks don’t exist)



Solutions of the RH relations for pressure-dominated parameter values for state 1. Switch-on shocks do not occur. Only 1–2 fast shocks exist. (a–b) $M_A = 2$ and $\beta = 0.4$. (c–d) $M_A = 1.5$ and $\beta = 2$.

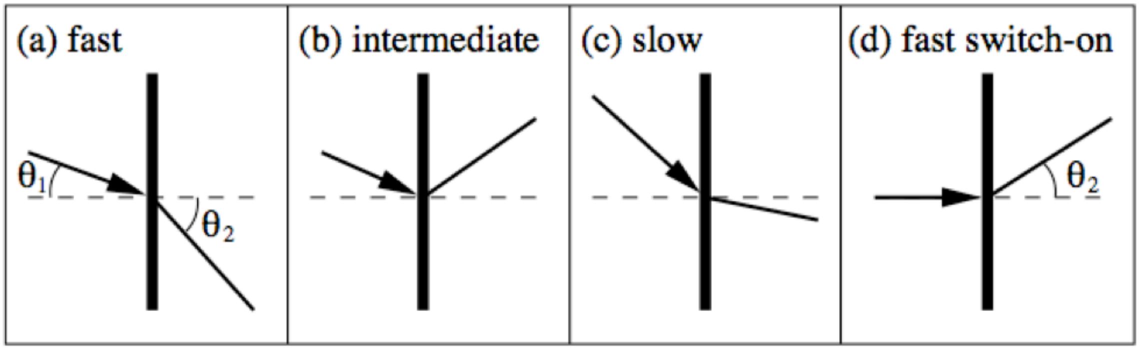


‘magnetically dominated’ regime

or: switch-on regime

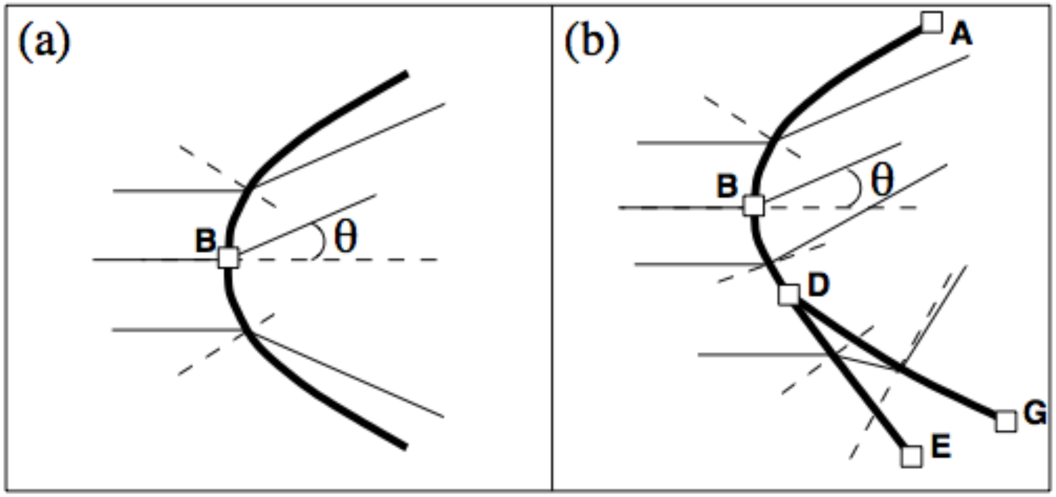
magnetic field is sufficiently strong and sufficiently close to the shock normal to make

$$\beta_1 < 2/\gamma \quad \text{and} \quad c_{A,1} < v_{x,1} < c_{A,1}\sqrt{\frac{\gamma(1-\beta_1)+1}{\gamma-1}} = v_{crit}$$

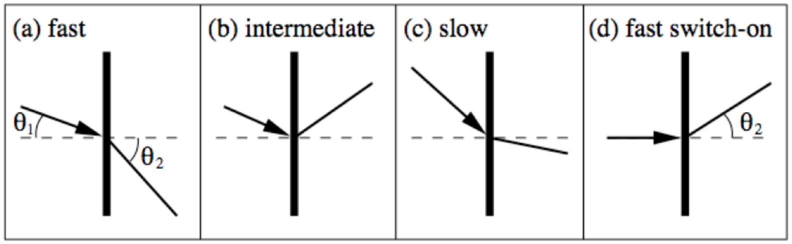


MHD bow shock topology

- magnetically dominated regime
- usual bow shock topology is ‘discontinuous’

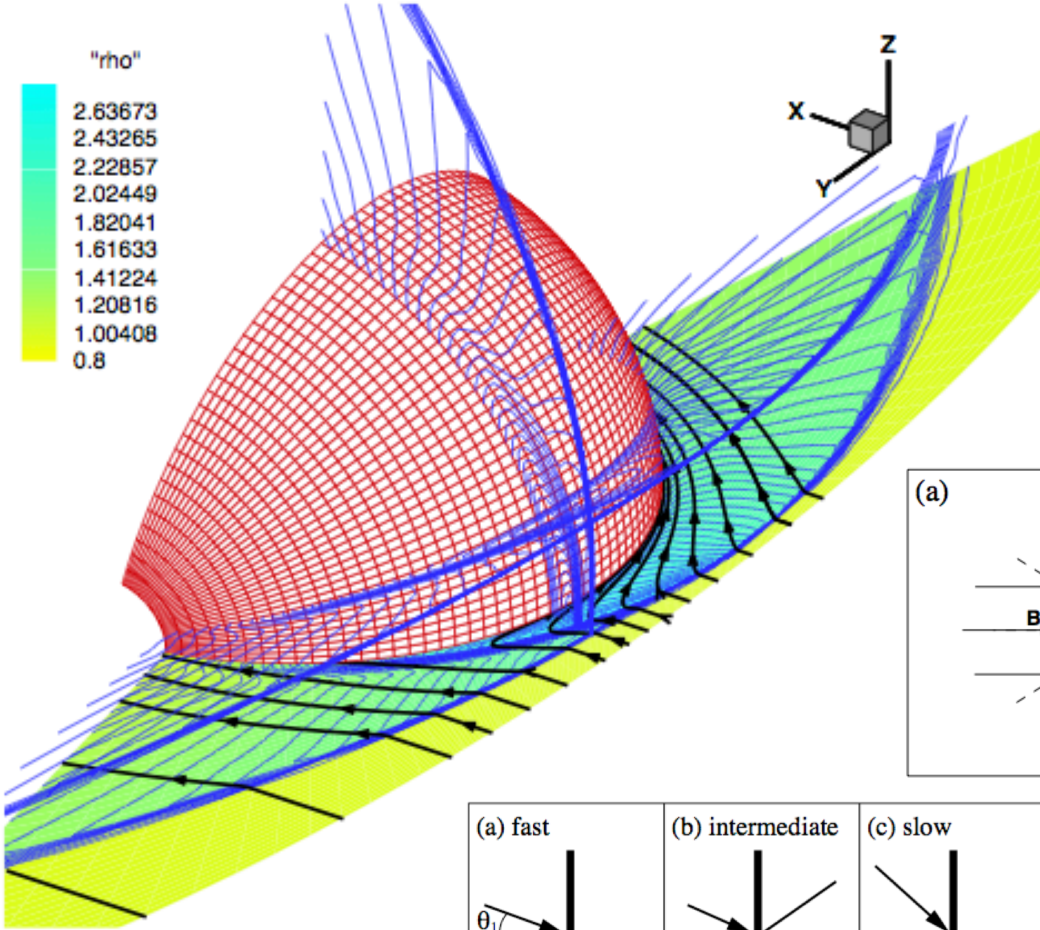


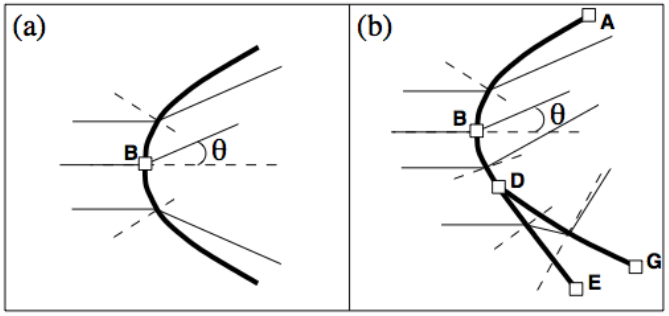
Two proposed topologies for a shock front with a switch-on shock at perpendicular point B for the case of a magnetically dominated upstream flow. Thick lines are shock fronts, thin lines are magnetic field lines, and shock normals are dashed. Point B is a perpendicular point, where the magnetic field is normal to the shock front. The shock at point B is a switch-on shock, which deflects the magnetic field with downstream angle θ. (a) The shock front cannot entirely be of the 1–2 fast type. (b) A complex shock topology is necessary to channel the flow. Shock segment AB is of 1–2 fast type, BD is 1–3 intermediate, DE is 1–2 fast, and DG is 2–4 intermediate, evolving into 2=3–4 switch-off and 3–4 slow along the front.

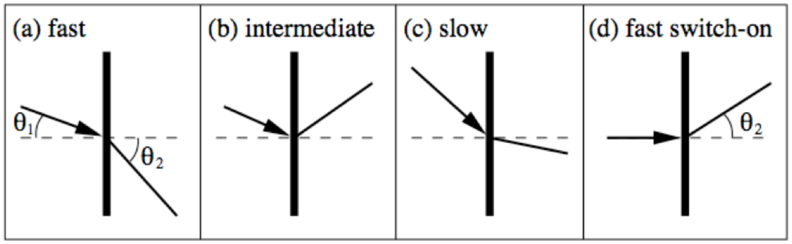


MHD bow shock simulation

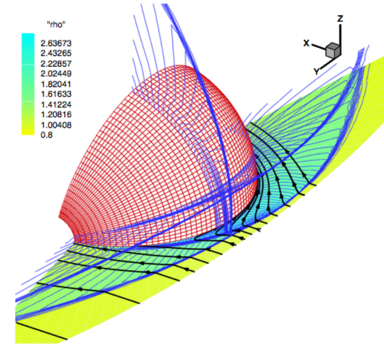
magnetically dominated regime

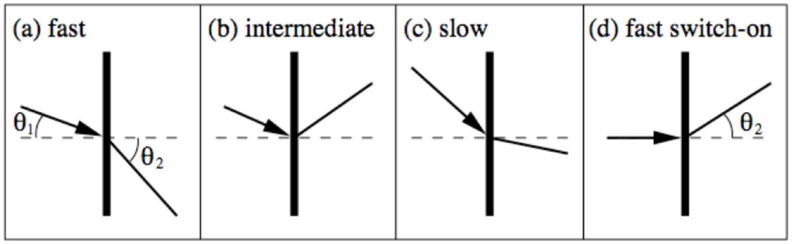






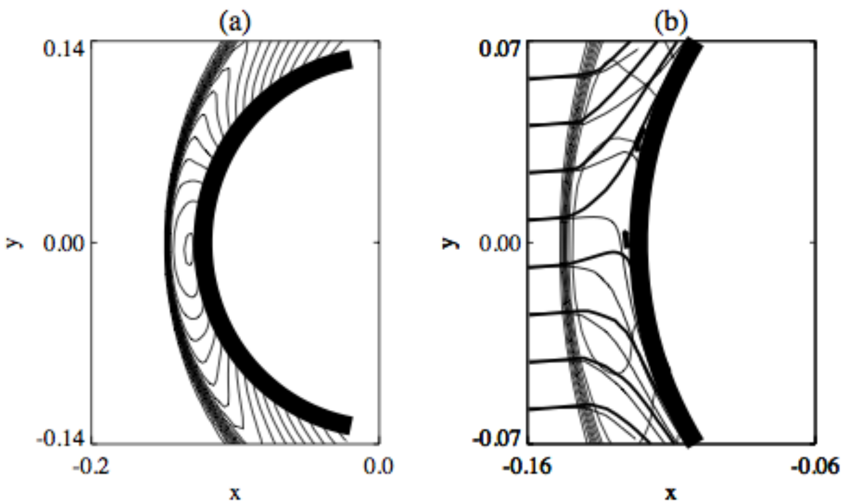
MHD bow shocks

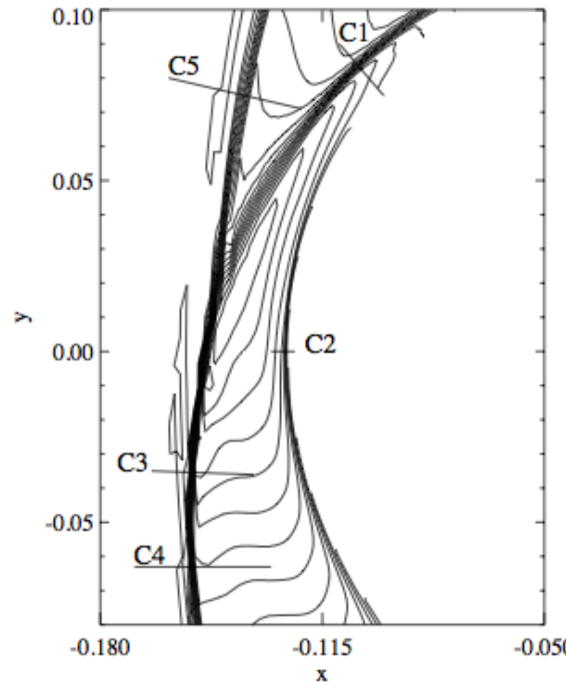


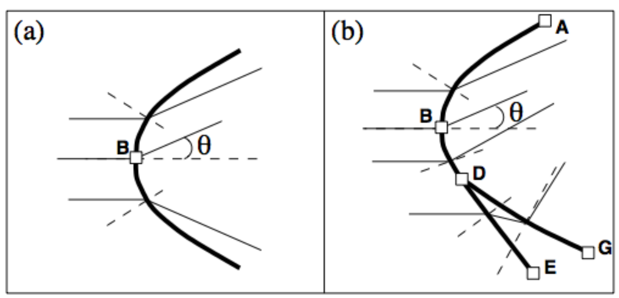


pressure-dominated

magnetically dominated







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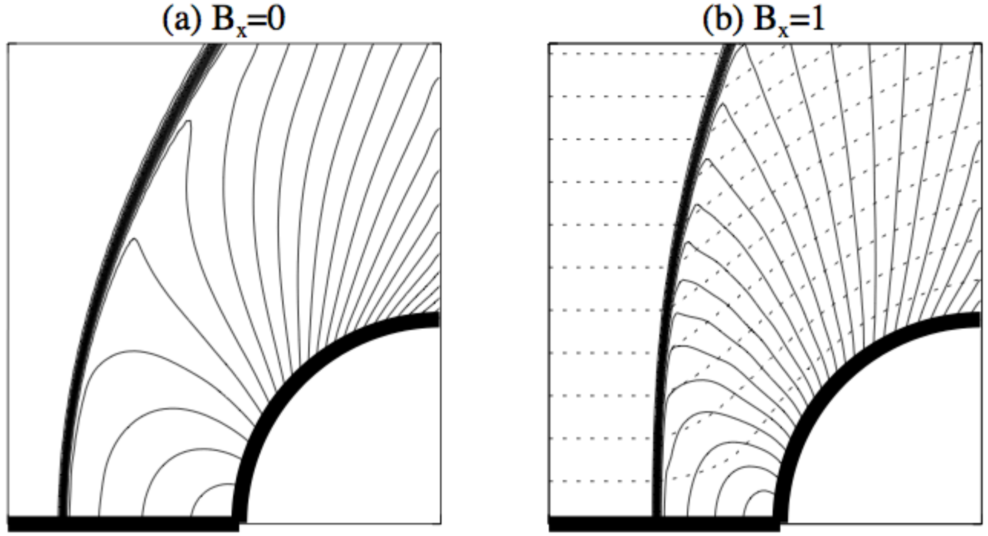
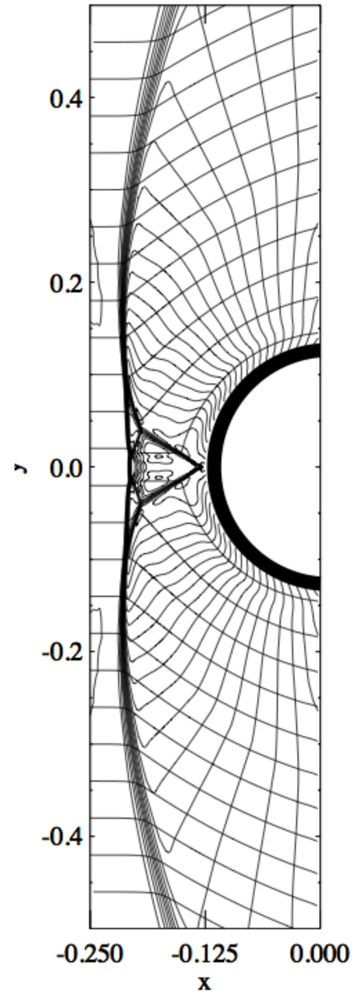
symmetric case

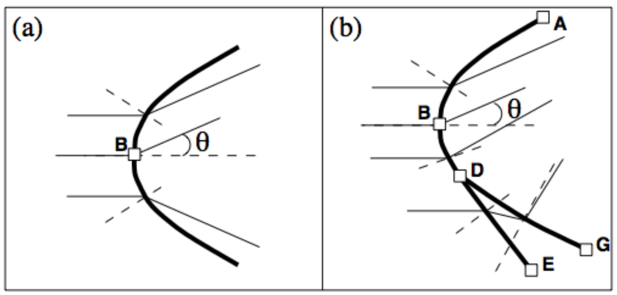
‘field-aligned’ flow over a cylinder (2D)

pressure-dominated



magnetically dominated





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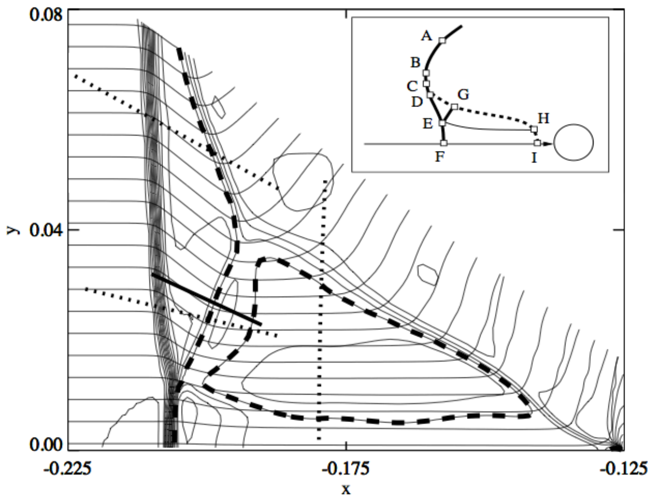
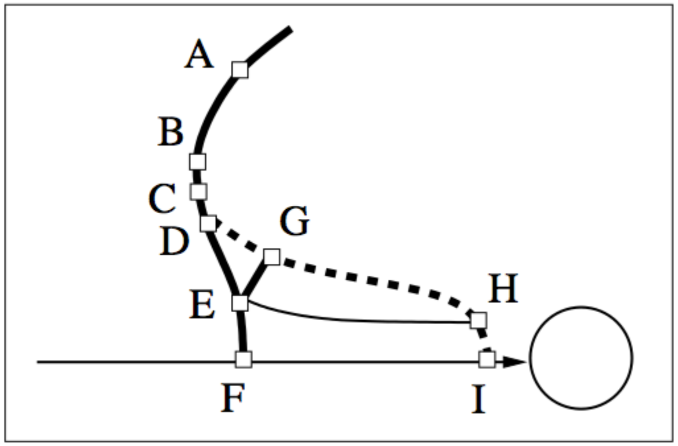
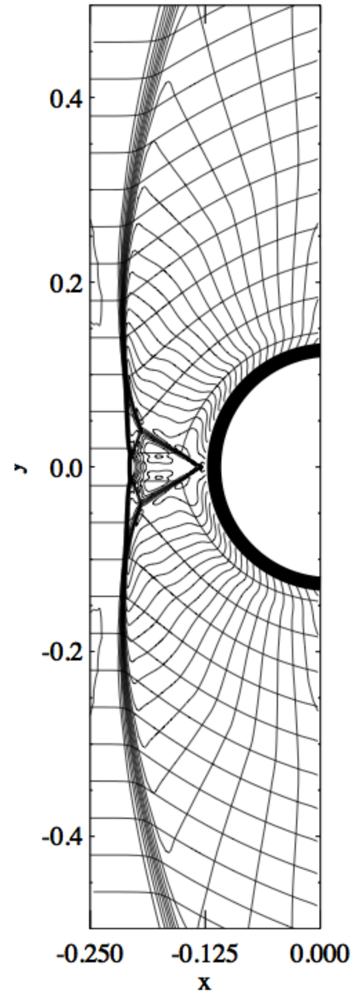
symmetric case (magnetically dominated)

‘field-aligned’ flow over a cylinder (2D)







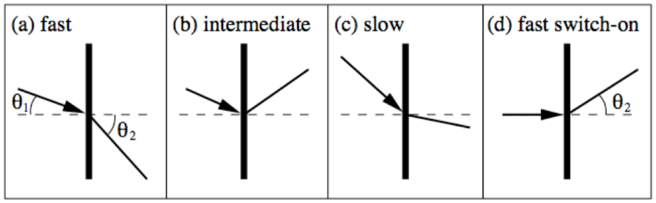


Figure 6.7: *Sketch of the complex multiple-front bow shock topology. In the leading front, shock AB is a 1—2 fast shock, BCD is 1—3 intermediate, DE is 1—2 fast, and EF is 1—4 hydrodynamic. The second front DGHI is 2—4 intermediate (2=3—4 slow switch-off or 3–4 slow at places). EG is a 1=2—3=4 intermediate shock which is sonic both upstream and downstream. EH is a tangential discontinuity, and tangential discontinuities also stretch out from points D, G, and H, along the streamlines towards infinity. At perpendicular point B the shock is a 1–2=3 switch-on shock, at points D and E the shock is of 1–3=4 intermediate type, and at perpendicular point F the shock is 1–4 hydrodynamic. The topology near the perpendicular point B is the topology of Fig. 6.4b.*

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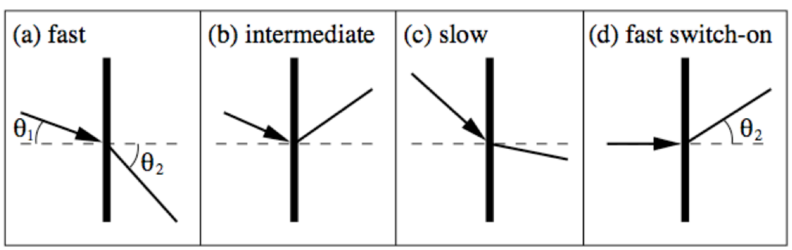
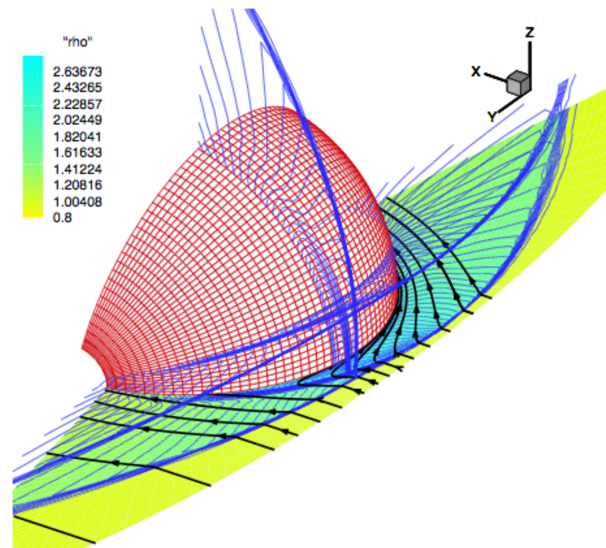
conclusions

- as a nonlinear hyperbolic system, MHD has a richer structure than Euler
  – three families of waves, anisotropic, non-strictly hyperbolic
  – many types of ‘exotic’ shock waves (switch-on, ...)
- this rich structure leads to interesting ‘predicted’ flow phenomena (consistent with theoretical expectations) even in simple flow configurations
  – interesting: not observed yet!
- numerical methods build on hyperbolic structure of MHD





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Euler equations

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho\, e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p \\ \rho v_x v_y \\ (\rho\, e + p) v_x \end{bmatrix}$$

$$\rho\, e = \rho\, \epsilon(\rho, p) + \frac{1}{2}\rho v^2 = E$$

$E = \rho\, e$: volumetric total energy (J/m^3)

$\epsilon(\rho, p)$: specific internal energy (J/kg)

perfect (ideal) gas EOS: $\epsilon(\rho, p) = \dfrac{p}{\rho\,(\gamma - 1)}$

characteristic curves and Riemann invariants

$$\frac{\partial V}{\partial t} + \mathsf{R}(V) \cdot \Lambda(V) \cdot \mathsf{L}(V) \cdot \frac{\partial V}{\partial x} = 0$$

$$\frac{\mathsf{L}(V) \cdot \partial V}{\partial t} + \Lambda(V) \cdot \frac{\mathsf{L}(V) \cdot \partial V}{\partial x} = 0$$

$\forall i$: (n characteristic fields)

- $\boxed{x_i(t) : \dfrac{\partial x_i(t)}{\partial t} = \lambda_i(V)}$: ith characteristic

- if $\exists\, w_i(V)\ :\ L_i(V) \cdot \partial V = \alpha(V)\, \partial w_i(V)$

$$\Rightarrow L_i(V) \cdot \frac{\partial V}{\partial t} + \lambda_i(V) L_i(V) \cdot \frac{\partial V}{\partial x} = 0$$

$$\alpha(V) \frac{\partial w_i(V)}{\partial t} + \lambda_i(V) \alpha(V) \frac{\partial w_i(V)}{\partial x} = 0$$

Riemann invariants

$$\Rightarrow \frac{\mathrm{d}w_i(x_i(t),t)}{\mathrm{d}t} = 0 \; : \quad w_i \text{ Riemann Invariant (RI) on characteristic } x_i$$

- $i = 1$: find $w_1(V)$ such that

$$L_1(V) \cdot \partial V = \partial\rho - \frac{1}{c^2}\partial p = \alpha(V)\,\partial w_1(V)$$

choose $w_1 = s = \dfrac{p}{\rho^\gamma}$

$$\Rightarrow \partial w_1 = -\frac{\gamma p}{\rho^{\gamma+1}}\partial\rho + \frac{\partial p}{\rho^\gamma} = \frac{-c^2}{\rho^\gamma}\left(\partial\rho - \frac{1}{c^2}\partial p\right)$$

$$\Rightarrow L_1(V) \cdot \partial V = \partial\rho - \frac{1}{c^2}\partial p = \alpha(V)\,\partial w_1(V)$$

with $\alpha(V) = \dfrac{-\rho^\gamma}{c^2}$ and $w_1 = s = \dfrac{p}{\rho^\gamma}$

$\Rightarrow s$ is a Riemann Invariant on $x_1(t)$ with $\dfrac{\partial x_1(t)}{\partial t} = v_x$

the entropy of a fluid element is conserved on its path

Riemann invariants

- $i = 2$: find $w_2(V)$ such that

  $$L_2(V) \cdot \partial V = \partial v_y = \alpha(V)\,\partial w_2(V)$$

  choose $w_2 = v_y$ and $\alpha(V) = 1$ (trivial)

$\Rightarrow v_y$ is a Riemann Invariant on $x_2(t)$ with $\dfrac{\partial x_1(t)}{\partial t} = v_x$

- $i = 3$: find $w_3(V)$ such that

  $$L_3(V) \cdot \partial V = \frac{1}{2c}\partial v_x + \frac{1}{2\rho c^2}\partial p = \alpha(V)\,\partial w_3(V)$$

  no solution, RI does not exist

- $i = 4$: no RI

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