Adaptive Algebraic Multigrid for Canonical Tensor Decomposition

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uwaterloo.ca

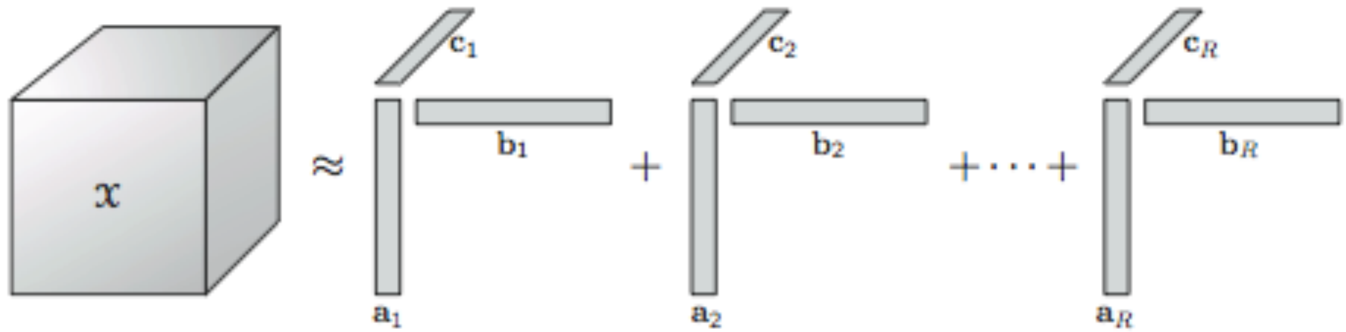
Hans De Sterck and Killian Miller

Department of Applied Mathematics
University of Waterloo, Canada

Weizmann Workshop 2013 on Multilevel Computational Methods and Optimization

(1) canonical tensor decomposition

- tensor = element of tensor product of real vector spaces (*N*-dimensional array)
- *N*=3:



(from "Tensor Decompositions and Applications", Kolda and Bader, SIAM Rev., 2009 [1])

- canonical decomposition: decompose tensor in sum of *R* rank-one terms (approximately)

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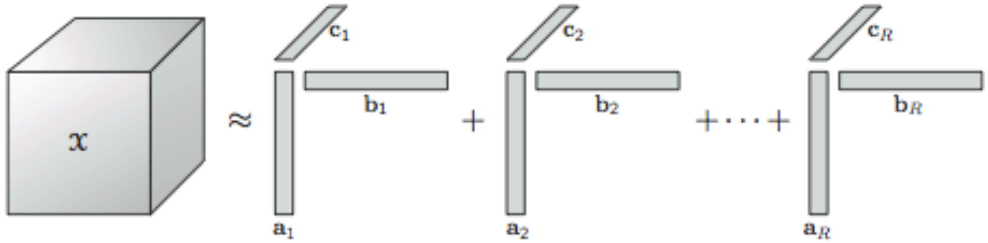
canonical tensor decomposition



OPTIMIZATION PROBLEM

given tensor $\mathcal{T} \in \mathbb{R}^{I_1 \times \ldots \times I_N}$, find rank-$R$ canonical tensor $\mathcal{A}_R \in \mathbb{R}^{I_1 \times \ldots \times I_N}$ that minimizes

$$f(\mathcal{A}_R) = \frac{1}{2} \|\mathcal{T} - \mathcal{A}_R\|_F^2.$$

FIRST-ORDER OPTIMALITY EQUATIONS

$$\nabla f(\mathcal{A}_R) = \mathbf{g}(\mathcal{A}_R) = 0.$$

(problem is non-convex, multiple (local) minima, solution may not exist (ill-posed), ... ; but smooth, and we assume there is a local minimum)

(de Silva and Lim, SIMAX, 2009)

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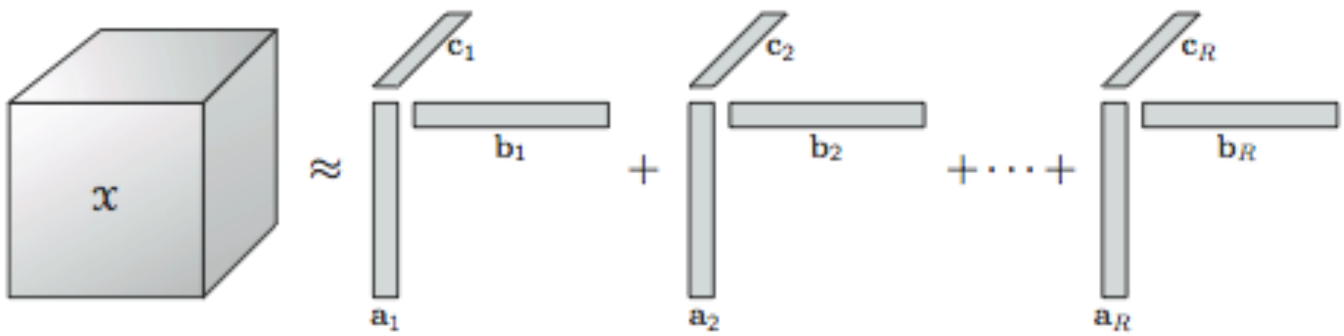
link with singular value decomposition

- SVD of $A \in \mathbb{R}^{m \times n}$ $\qquad m \geq n$

$$A = U\,\Sigma\,V^t = \sigma_1\,u_1\,v_1^T + \ldots + \sigma_n\,u_n\,v_n^T$$

- canonical decomposition of tensor



a difference with the SVD

truncated SVD is best rank-*R* approximation:

$$A = \sigma_1 \, u_1 \, v_1^T + \ldots + \sigma_R \, u_R \, v_R^T + \sigma_{R+1} \, u_{R+1} \, v_{R+1}^T + \ldots + \sigma_n \, u_n \, v_n^T$$

$$\underset{B \text{ with rank } \leq R}{\arg\min} \; \|A - B\|_F = \sigma_1 \, u_1 \, v_1^T + \ldots + \sigma_R \, u_R \, v_R^T$$

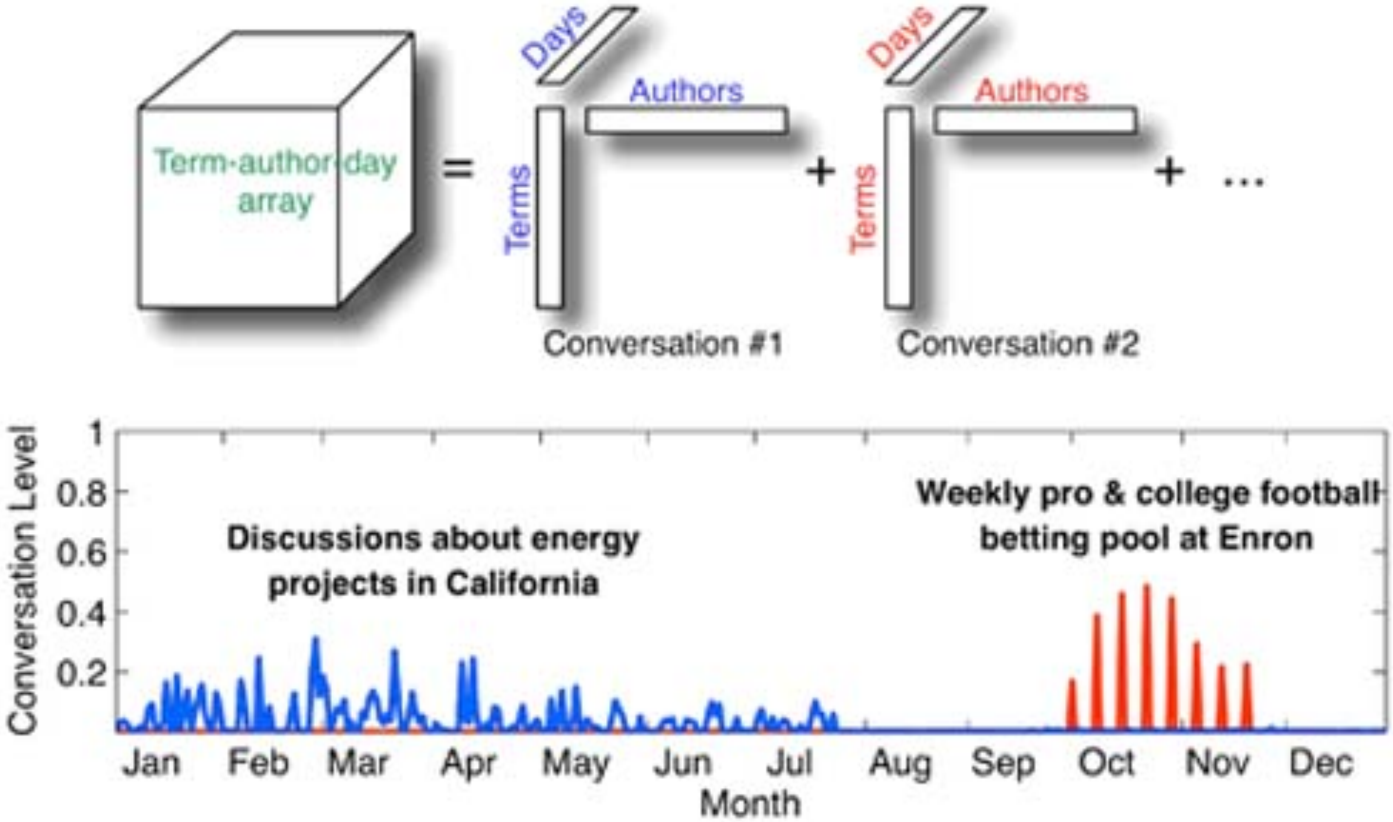
BUT best rank-*R* tensor cannot be obtained by truncation: different optimization problems for different *R*!

given tensor $\mathcal{T} \in \mathbb{R}^{I_1 \times \ldots \times I_N}$, find rank-$R$ canonical tensor $\mathcal{A}_R \in \mathbb{R}^{I_1 \times \ldots \times I_N}$ that minimizes

$$f(\mathcal{A}_R) = \frac{1}{2} \|\mathcal{T} - \mathcal{A}_R\|_F^2.$$

tensor approximation applications

(1) “Discussion Tracking in Enron Email Using PARAFAC” by Bader, Berry and Browne (2008) (sparse, nonnegative)



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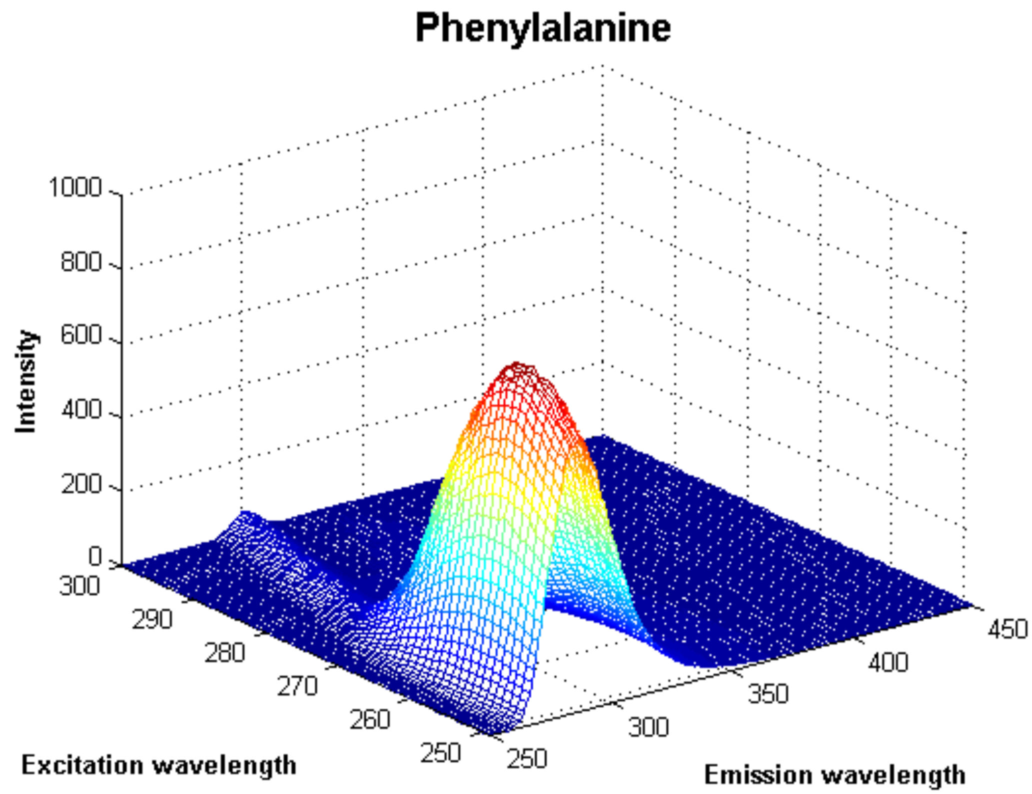
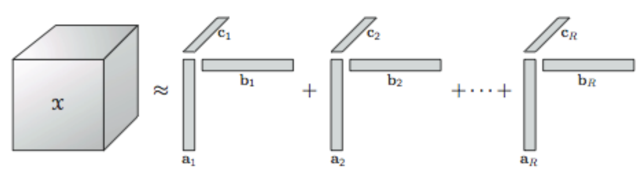
tensor approximation applications

(2) chemometrics: analyze spectrofluorometer data (dense) (Bro et al., http://www.models.life.ku.dk/nwaydata1)

- 5 x 201 x 61 tensor: 5 samples (with different mixtures of three amino acids), 61 excitation wavelengths, 201 emission wavelengths
- goal: recover emission spectra of the three amino acids (to determine what was in each sample, and in which concentration)





(from [1])

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tensor approximation applications

(2) chemometrics: analyze spectrofluorometer data (dense) (Bro et al.,

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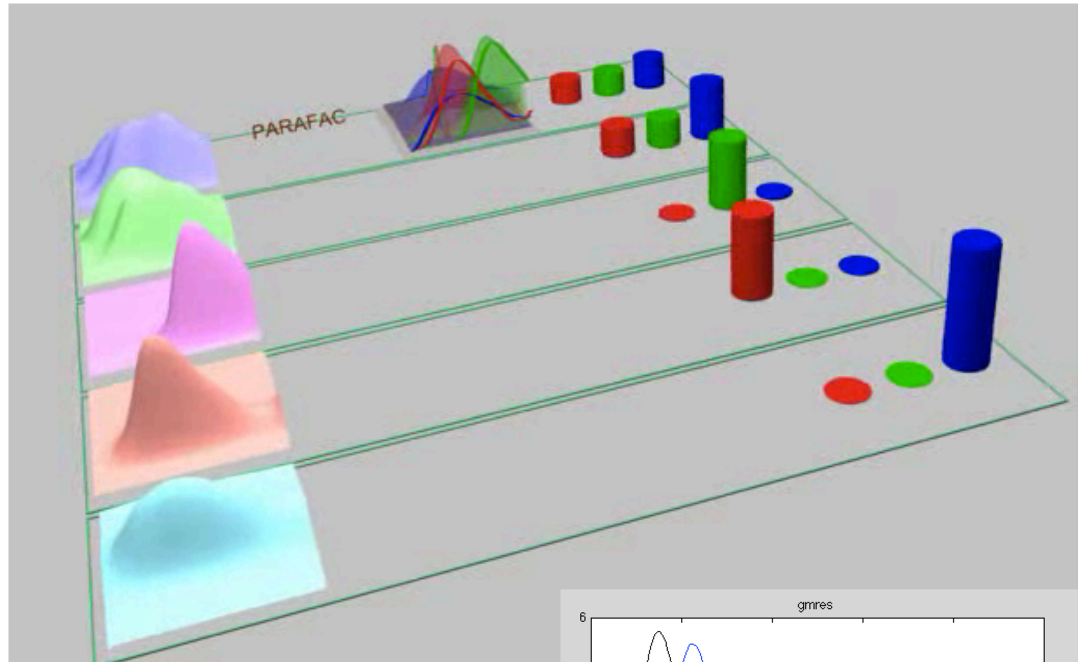
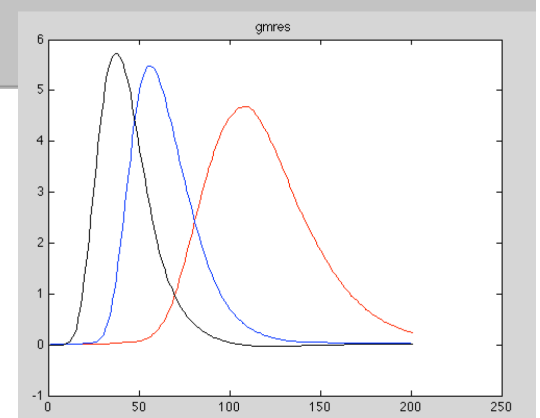
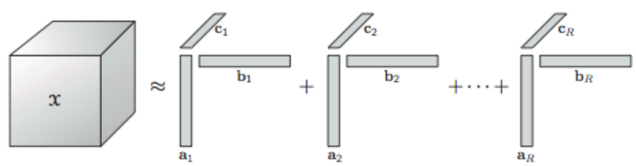




- 5 x 201 x 61 tensor: 5 samples (with different mixtures of three amino acids), 61 excitation wavelengths, 201 emission wavelengths
- goal: recover emission spectra of the three amino acids (to determine what was in each sample, and in which concentration)
- also: psychometrics, ..



(from [1])

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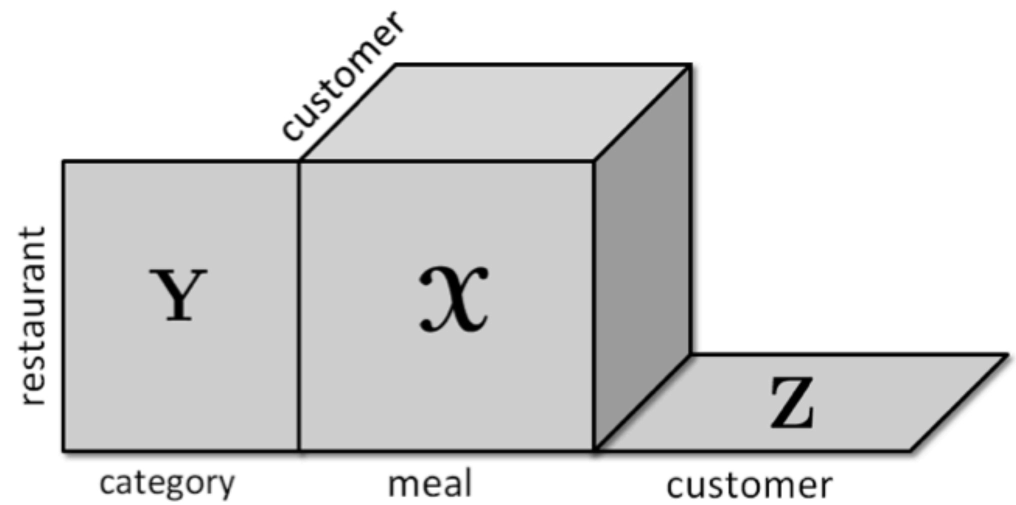
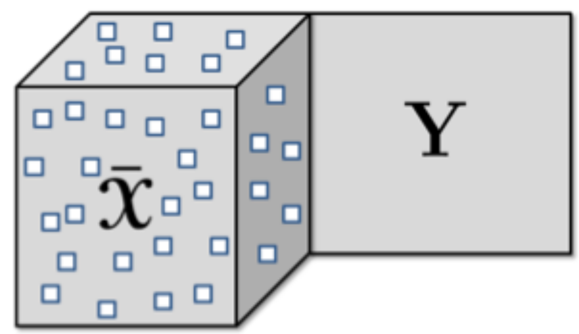
tensor approximation applications

(3) “All-at-once Optimization for Coupled Matrix and Tensor Factorizations” by Acar, Kolda and Dunlavy (2011)





$$\left\| \mathcal{W} * \left(\mathcal{X} - [\![\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}]\!] \right) \right\|^2 + \frac{1}{2} \left\| \mathbf{Y} - \mathbf{A}^{(n)} \mathbf{V}^{\mathsf{T}} \right\|^2$$

$$f(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{V}) = \| \mathcal{X} - [\![\mathbf{A}, \mathbf{B}, \mathbf{C}]\!] \|^2 + \left\| \mathbf{Y} - \mathbf{A}\mathbf{V}^{\mathsf{T}} \right\|^2$$

(2) ‘workhorse’ algorithm: alternating least squares (ALS)

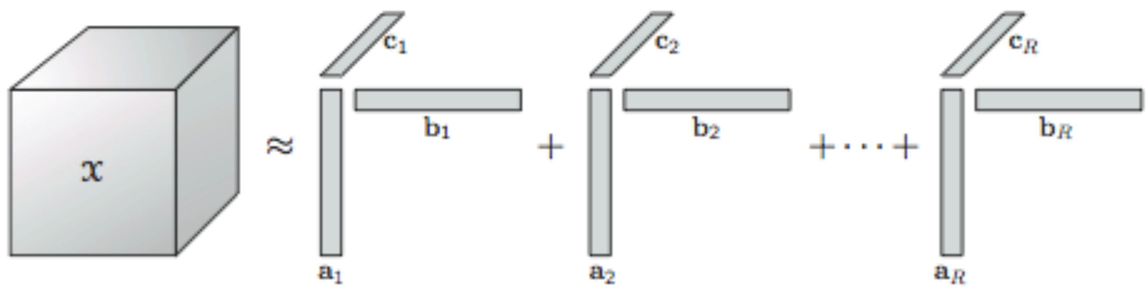
$$f(\mathcal{A}_R) = \frac{1}{2} \left\| \mathcal{T} - \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ a_r^{(3)} \right\|_F^2$$

(1) freeze all $a_r^{(2)}$, $a_r^{(3)}$, compute optimal $a_r^{(1)}$ via a least-squares solution (linear, overdetermined)

(2) freeze $a_r^{(1)}$, $a_r^{(3)}$, compute $a_r^{(2)}$

(3) freeze $a_r^{(1)}$, $a_r^{(2)}$, compute $a_r^{(3)}$

- repeat



(from [1])

alternating least squares (ALS)

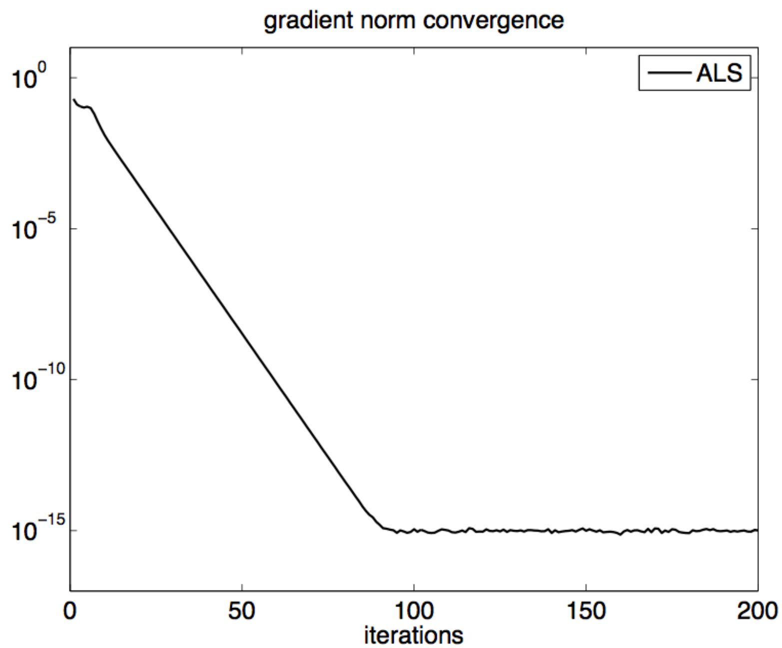
$$f(\mathcal{A}_R) = \frac{1}{2} \left\| \mathcal{T} - \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ a_r^{(3)} \right\|_F^2$$

- “simple iterative optimization method”
- ALS is block nonlinear Gauss-Seidel
- ALS is monotone
- ALS is sometimes fast, but can also be extremely slow (depending on problem and initial condition)

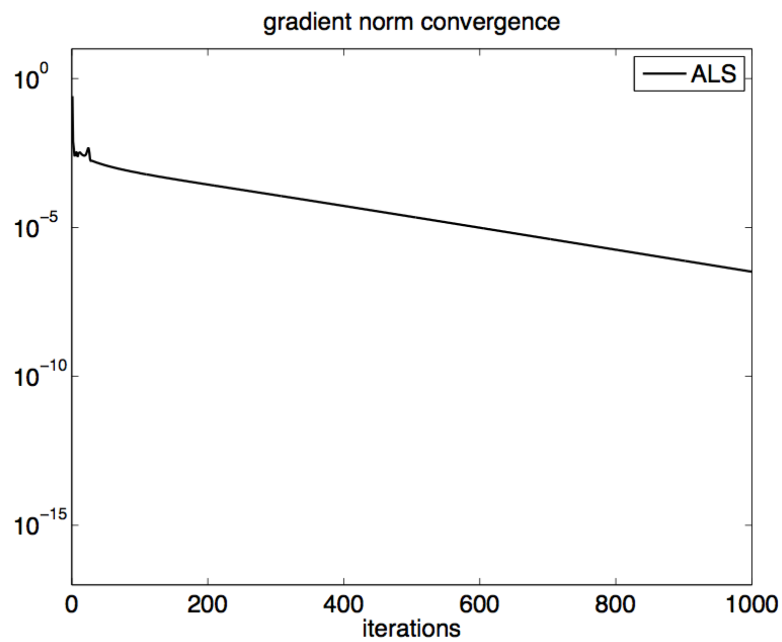
alternating least squares (ALS)

$$f(\mathcal{A}_R) = \frac{1}{2}\left\| \mathcal{T} - \sum_{r=1}^{R} a_r^{(1)} \circ a_r^{(2)} \circ a_r^{(3)} \right\|_F^2$$

fast case



slow case



(we used Matlab with Tensor Toolbox (Bader and Kolda) and Poblano Toolbox (Dunlavy et al.) for all computations)

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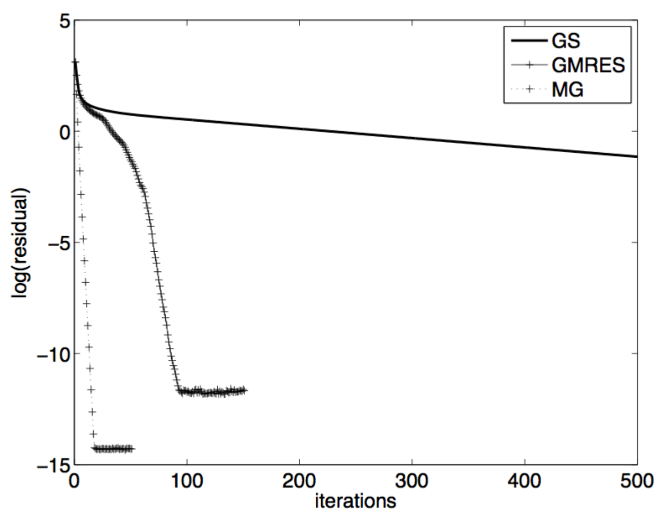
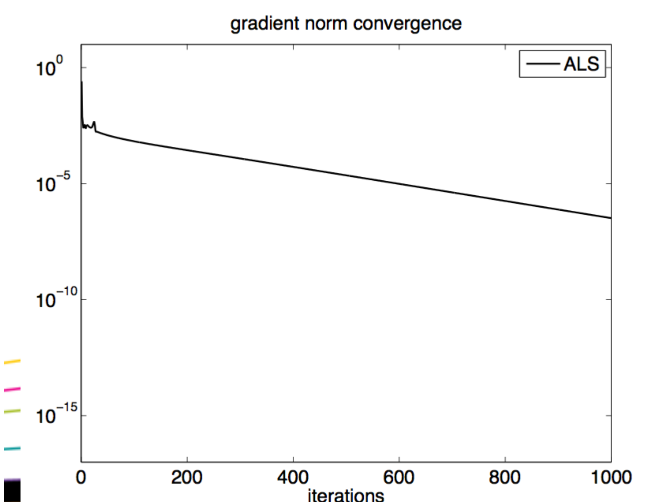
convergence acceleration for ALS (nonlinear optimization)

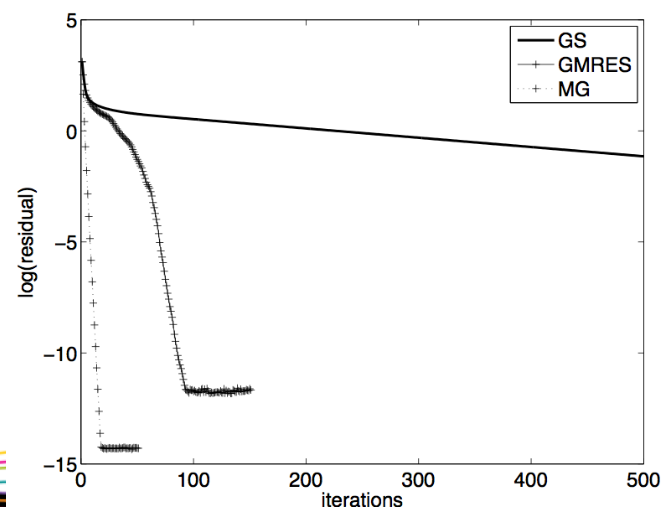
convergence acceleration for linear systems	convergence acceleration for nonlinear optimization		
P-CG	NCG, P?		
P-GMRES	?		
MG	? (MG/OPT)		

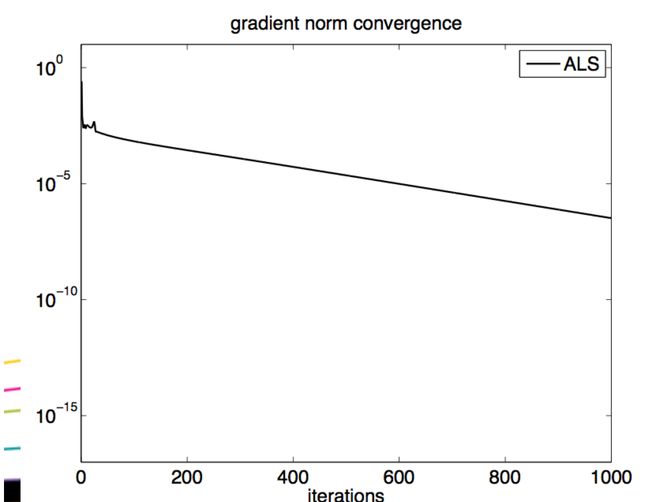




convergence acceleration for ALS (nonlinear optimization)

convergence acceleration for linear systems	convergence acceleration for nonlinear optimization	convergence acceleration for nonlinear systems	
P-CG	NCG, P?	NCG, P?	
P-GMRES	?	P-NGMRES (Washio and Oosterlee, Anderson)	
MG	? (MG/OPT)	FAS	





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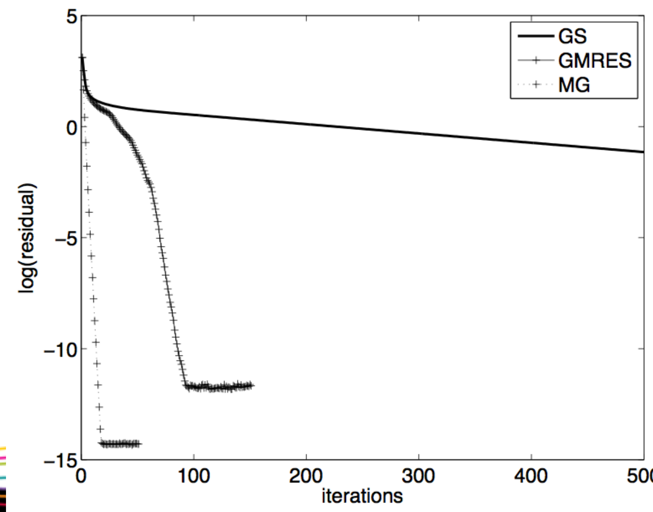
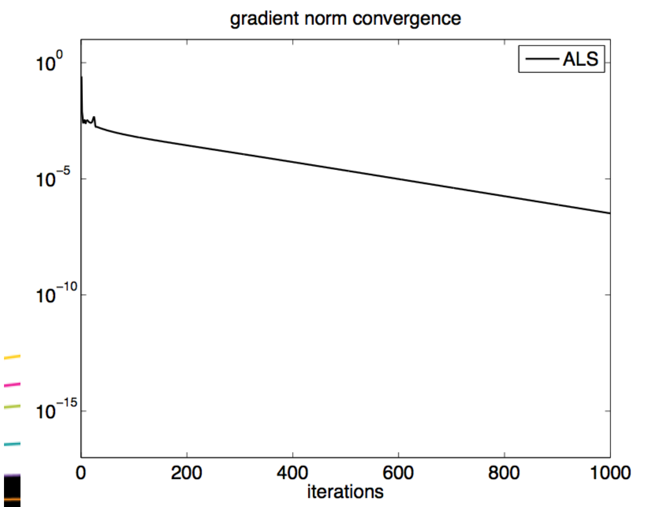
convergence acceleration for ALS (nonlinear optimization)

convergence acceleration for linear systems	convergence acceleration for nonlinear optimization	convergence acceleration for nonlinear systems	convergence acceleration for nonlinear optimization
P-CG	NCG, P?	NCG, P?	P-NCG
P-GMRES	?	P-NGMRES (Washio and Oosterlee, Anderson)	P-NGMRES for optimization
MG	?	FAS	adaptive AMG-FAS for optimization





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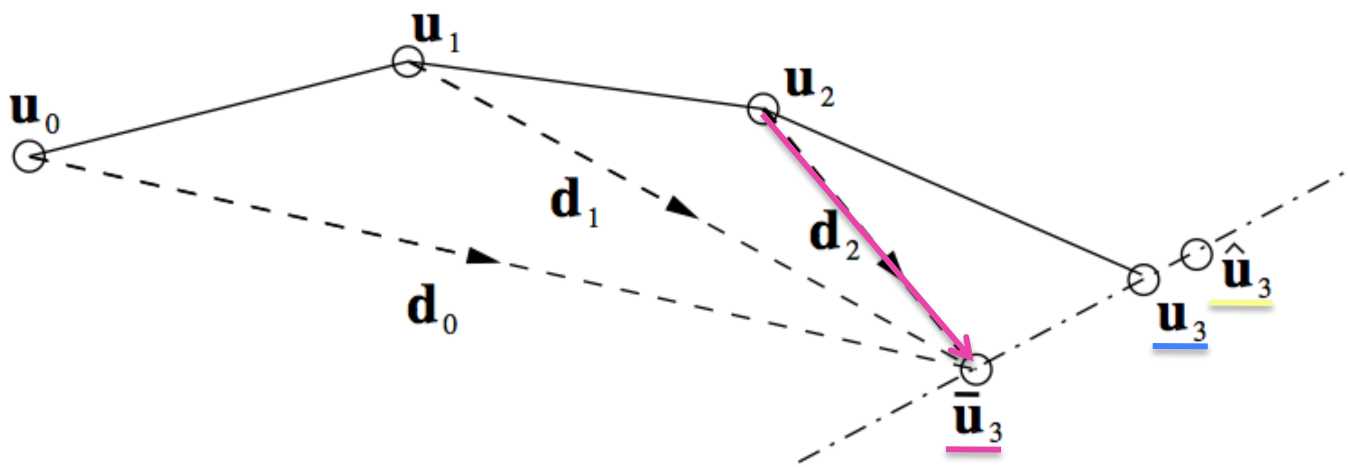
(3) nonlinear GMRES optimization method (N-GMRES)



Algorithm 1: N-GMRES optimization algorithm (window size w)

Input: w initial iterates $\mathbf{u}_0, \ldots, \mathbf{u}_{w-1}$.

$i = w - 1$

repeat

STEP I: *(generate preliminary iterate by one-step update process $M(.)$)* (ALS)

$\bar{\mathbf{u}}_{i+1} = M(\mathbf{u}_i)$

STEP II: *(generate accelerated iterate by nonlinear GMRES step)*

$\hat{\mathbf{u}}_{i+1} = \text{gmres}(\mathbf{u}_{i-w+1}, \ldots, \mathbf{u}_i; \bar{\mathbf{u}}_{i+1})$

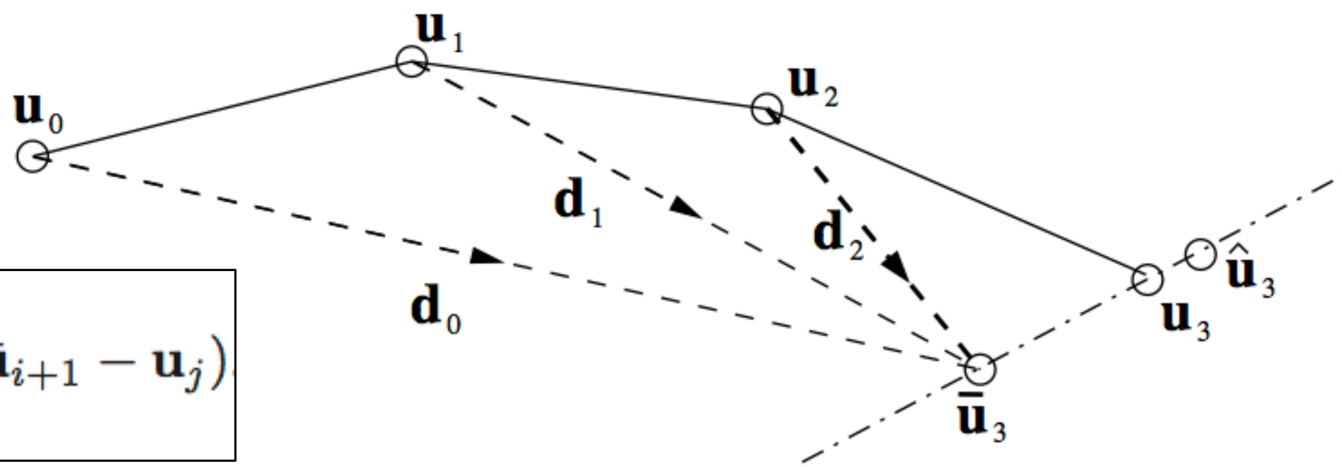
STEP III: *(generate new iterate by line search process)* (Moré-Thuente line search, satisfies Wolfe conditions)

$\mathbf{u}_{i+1} = \text{linesearch}(\bar{\mathbf{u}}_{i+1} + \beta(\hat{\mathbf{u}}_{i+1} - \bar{\mathbf{u}}_{i+1}))$

$i = i + 1$

until *convergence criterion satisfied*

step II: N-GMRES acceleration: $\nabla f(\mathcal{A}_R) = \mathbf{g}(\mathcal{A}_R) = 0$



$$\boxed{\hat{\mathbf{u}}_{i+1} = \bar{\mathbf{u}}_{i+1} + \sum_{j=0}^{i} \alpha_j \left(\bar{\mathbf{u}}_{i+1} - \mathbf{u}_j\right)}$$

$$\mathbf{g}(\hat{\mathbf{u}}_{i+1}) \approx \mathbf{g}(\bar{\mathbf{u}}_{i+1}) + \sum_{j=0}^{i} \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{u}}_{i+1}} \alpha_j \left(\bar{\mathbf{u}}_{i+1} - \mathbf{u}_j\right)$$

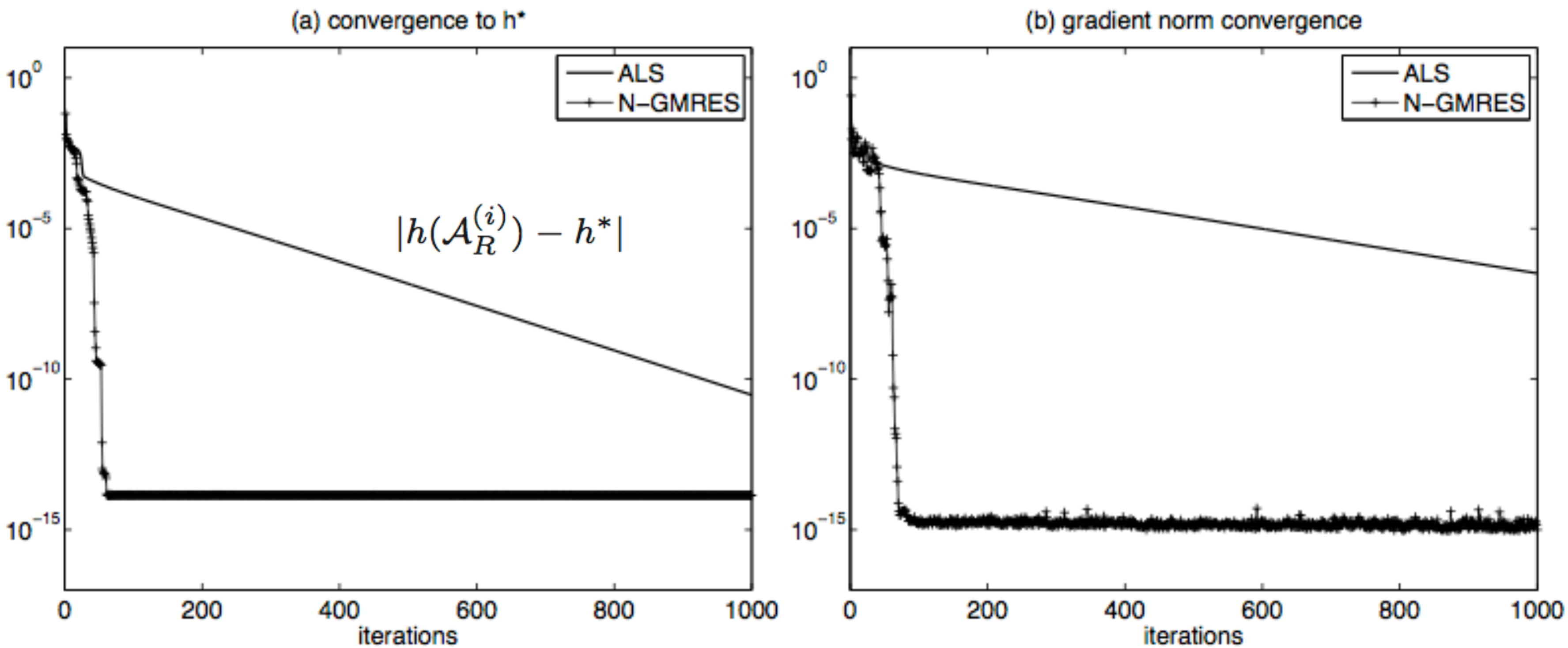
$$\approx \mathbf{g}(\bar{\mathbf{u}}_{i+1}) + \sum_{j=0}^{i} \alpha_j \left(\mathbf{g}(\bar{\mathbf{u}}_{i+1}) - \mathbf{g}(\mathbf{u}_j)\right)$$

find coefficients $(\alpha_0, \ldots, \alpha_i)$ that minimize

$$\left\|\mathbf{g}(\bar{\mathbf{u}}_{i+1}) + \sum_{j=0}^{i} \alpha_j \left(\mathbf{g}(\bar{\mathbf{u}}_{i+1}) - \mathbf{g}(\mathbf{u}_j)\right)\right\|_2.$$

numerical results for ALS-preconditioned N-GMRES applied to tensor problem

- dense test problem (from Tomasi and Bro; Acar et al.):
  - random rank-R tensor modified to obtain specific column collinearity, with added noise



$$h(\mathcal{A}_R^{(i)}) = \frac{\|\mathcal{T} - \mathcal{A}_R^{(i)}\|_F}{\|\mathcal{T}\|_F}$$

nonlinearly preconditioned N-GMRES for nonlinear optimization

- works well for canonical tensor decomposition (N-GMRES accelerates ALS)
- also promising as a general way to accelerate 'simple' optimization methods

- Hans De Sterck, *'A Nonlinear GMRES Optimization Algorithm for Canonical Tensor Decomposition'*, SIAM J. Sci. Comp., 2012
- Hans De Sterck, *'Steepest Descent Preconditioning for Nonlinear GMRES Optimization'*, Numer. Linear Algebra Appl., 2012

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convergence acceleration for ALS (nonlinear optimization)

convergence acceleration for linear systems	convergence acceleration for nonlinear optimization	convergence acceleration for nonlinear systems	convergence acceleration for nonlinear optimization
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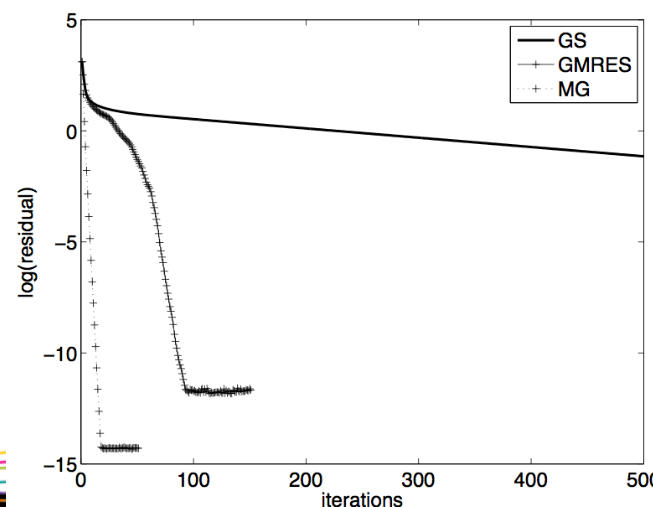

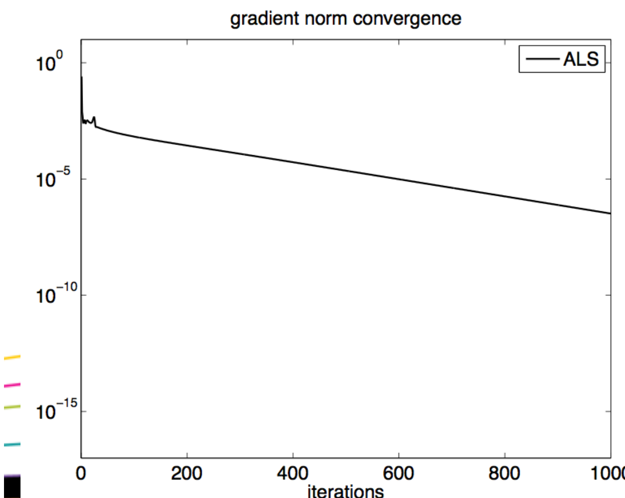

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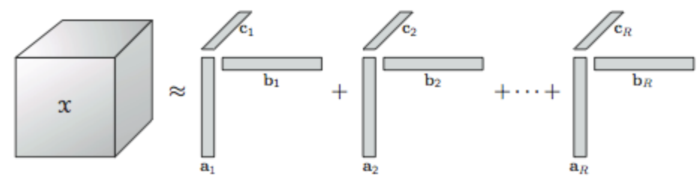
(4) adaptive AMG-FAS for accelerating ALS

- with Killian Miller
- builds on: adaptive AMG for SVD (Hans De Sterck, 'A Self-learning Algebraic Multigrid Method for Extremal Singular Triplets and Eigenpairs', SIAM J. Sci. Comput. 34, 2012)

$$A = U\,\Sigma\,V^t = \sigma_1\,u_1\,v_1^T + \ldots + \sigma_n\,u_n\,v_n^T$$



- SVD case: separate coarsening and interpolation for *u* and *v* variables

$$A \in \mathbb{R}^{m\times n} \qquad \begin{aligned} A\,v &= \sigma\,u, & u &= P\,u_c, & P^t\,A\,Q\,v_c &= \sigma\,P^t\,B\,P\,u_c,\\ A^t\,u &= \sigma\,v. & v &= Q\,v_c, & Q^t\,A^t\,P\,u_c &= \sigma\,Q^t\,C\,Q\,v_c, \end{aligned}$$

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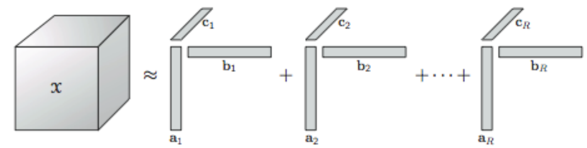
adaptive AMG-FAS for accelerating ALS

$$A = U\,\Sigma\,V^t = \sigma_1\,u_1\,v_1^T + \ldots + \sigma_n\,u_n\,v_n^T$$



- SVD case: separate coarsening and interpolation for *u* and *v* variables

$$A \in \mathbb{R}^{m\times n} \qquad \begin{aligned} A\,v &= \sigma\,u, \\ A^t\,u &= \sigma\,v. \end{aligned} \qquad \begin{aligned} u &= P\,u_c, \\ v &= Q\,v_c, \end{aligned} \qquad \begin{aligned} P^t\,A\,Q\,v_c &= \sigma\,P^t\,B\,P\,u_c, \\ Q^t\,A^t\,P\,u_c &= \sigma\,Q^t\,C\,Q\,v_c, \end{aligned}$$

(note: collective interpolation of *u* singular vectors by *P*, ...)

- we don't know the nature of the slow-to-converge components in advance: determine *P*, *Q*, *R*, ... adaptively using Bootstrap AMG in a 'multiplicative' setup phase (Brandt/Brannick/Kahl/Livshits and Kushnir/Galun/Brandt)

- use FAS in an 'additive' solve phase (Brandt/McCormick/Ruge and Borzi/Borzi)

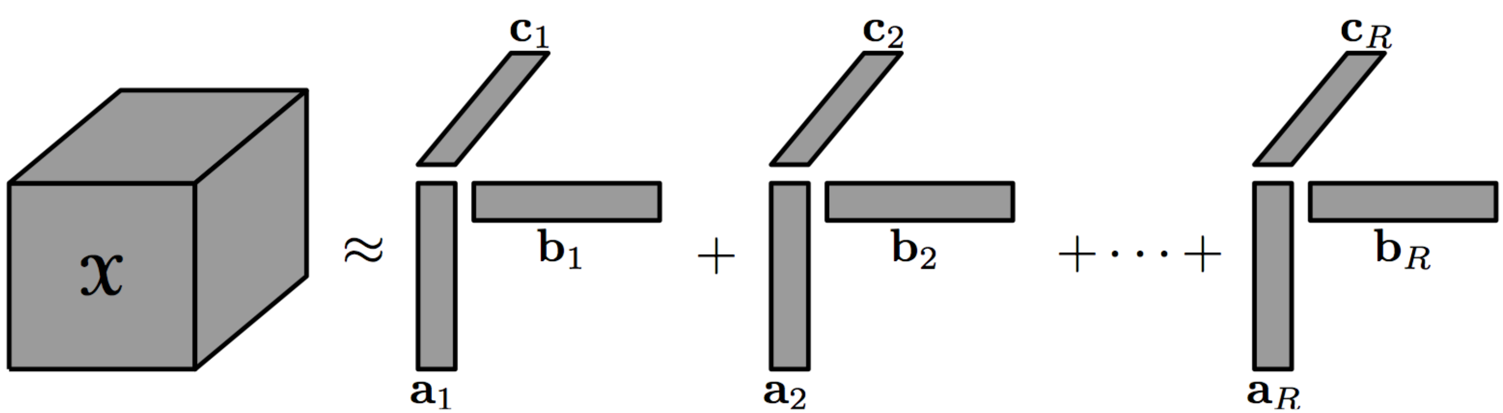
canonical tensor decomposition: notation

- Decompose a tensor as a sum of R rank-one terms
- For a third-order tensor:

$$\mathcal{X} \approx \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r = [\![\mathbf{A}, \mathbf{B}, \mathbf{C}]\!]$$

Factor matrices:

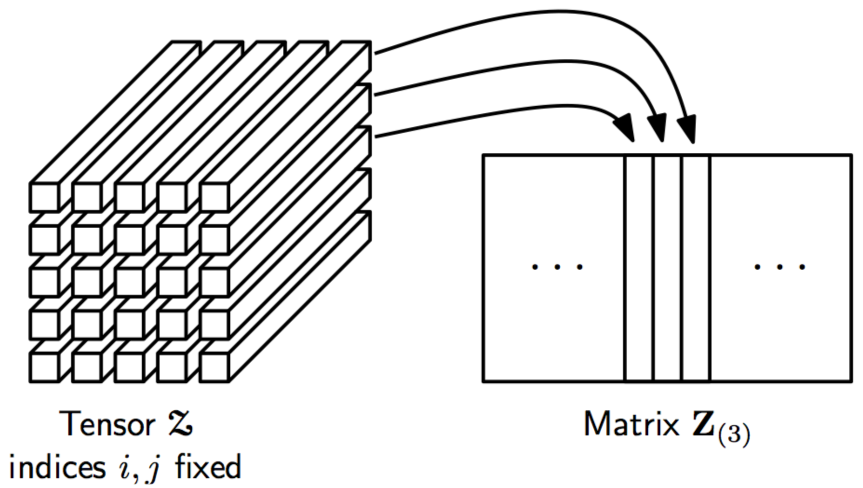
$$\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_R], \quad \mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_R], \quad \mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R]$$



some tensor machinery...

Mode-n Matricization: $\mathcal{Z} \to \mathbf{Z}_{(n)}$

- Transforms a tensor into a matrix by reordering tensor elements



Khatri-Rao product

- For $\mathbf{A} \in \mathbb{R}^{I \times J}$ and $\mathbf{B} \in \mathbb{R}^{K \times J}$

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{a}_1 \otimes \mathbf{b}_1 \quad \mathbf{a}_2 \otimes \mathbf{b}_2 \quad \cdots \quad \mathbf{a}_J \otimes \mathbf{b}_J] \in \mathbb{R}^{IK \times J}$$

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(4a) multiplicative phase: coarse equations

$$\text{minimize} \quad f(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}) := \frac{1}{2}\left\| \mathfrak{Z} - [\![\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(N)}]\!] \right\|^2$$

- Fine-level equations are the gradient equations of f:

$$\mathbf{Z}_{(n)}\left(\mathbf{A}^{(N)} \odot \cdots \odot \mathbf{A}^{(n+1)} \odot \mathbf{A}^{(n-1)} \odot \cdots \odot \mathbf{A}^{(1)}\right) = \mathbf{A}^{(n)}\mathbf{\Gamma}^{(n)}$$

- Assume that $\mathrm{range}(\mathbf{P}^{(n)})$ approximately contains $\mathbf{A}^{(n)}$, i.e.,

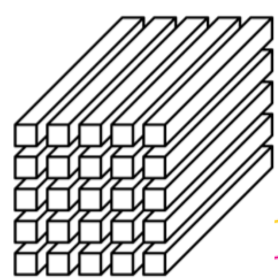
$$\mathbf{A}^{(n)} \approx \mathbf{P}^{(n)}\mathbf{A}_c^{(n)} \quad \text{for } n = 1, \ldots, N$$

- Coarse-level equations:

$$\mathbf{Z}^c_{(n)}\left(\mathbf{A}_c^{(N)} \odot \cdots \odot \mathbf{A}_c^{(n+1)} \odot \mathbf{A}_c^{(n-1)} \odot \cdots \odot \mathbf{A}_c^{(1)}\right) = \mathbf{B}^{(n)}\mathbf{A}_c^{(n)}\mathbf{\Gamma}_c^{(n)}$$

$$\mathbf{B}^{(n)} = \mathbf{P}^{(n)^{\mathrm{T}}}\mathbf{P}^{(n)} \ (\mathrm{SPD})$$

$$\mathfrak{Z}^c = \mathfrak{Z} \times_1 \mathbf{P}^{(1)^{\mathrm{T}}} \times_2 \mathbf{P}^{(2)^{\mathrm{T}}} \cdots \times_N \mathbf{P}^{(N)^{\mathrm{T}}}$$



Tensor $\mathfrak{Z}$

multiplicative phase: coarse equations

- Compute Cholesky factorization: $\mathbf{B}^{(n)} = \mathbf{L}^{(n)}\mathbf{L}^{(n)\mathrm{T}}$
- Let $\hat{\mathbf{A}}_c^{(n)} = \mathbf{L}^{(n)\mathrm{T}}\mathbf{A}_c^{(n)}$ for $n = 1, \ldots, N$
- Transformed coarse-level equations:

$$\boxed{\hat{\mathbf{Z}}^c_{(n)}\left(\hat{\mathbf{A}}_c^{(N)} \odot \cdots \odot \hat{\mathbf{A}}_c^{(n+1)} \odot \hat{\mathbf{A}}_c^{(n-1)} \odot \cdots \odot \hat{\mathbf{A}}_c^{(1)}\right) = \hat{\mathbf{A}}_c^{(n)}\hat{\mathbf{\Gamma}}_c^{(n)}}$$

$$\hat{\mathfrak{Z}}^c = \mathfrak{Z} \times_1 \hat{\mathbf{P}}^{(1)\mathrm{T}} \times_2 \hat{\mathbf{P}}^{(2)\mathrm{T}} \cdots \times_N \hat{\mathbf{P}}^{(N)\mathrm{T}}$$

$$\hat{\mathbf{P}}^{(n)} = \mathbf{P}^{(n)}\mathbf{L}^{(n)-\mathrm{T}}, \quad \hat{\mathbf{R}}^{(n)} = \hat{\mathbf{P}}^{(n)\mathrm{T}}$$

- Transformed coarse-level equations are gradient equations of

$$\hat{f}_c(\hat{\mathbf{A}}_c^{(1)}, \ldots, \hat{\mathbf{A}}_c^{(N)}) := \frac{1}{2}\left\|\hat{\mathfrak{Z}}^c - [\![\hat{\mathbf{A}}_c^{(1)}, \ldots, \hat{\mathbf{A}}_c^{(N)}]\!]\right\|^2$$

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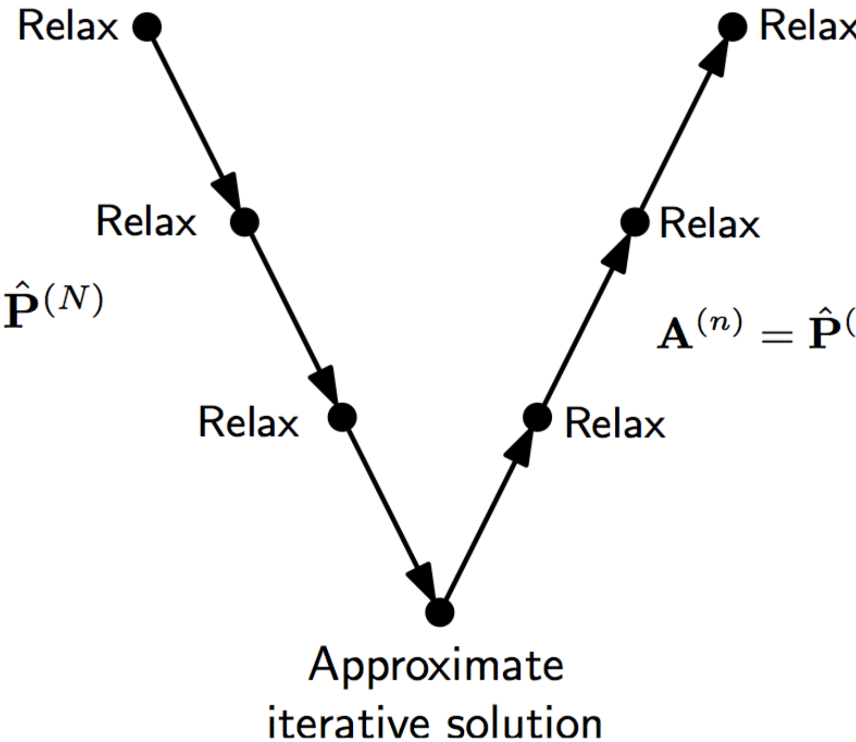
multiplicative phase: bootstrap V-cycle

- Build $\hat{\mathbf{P}}^{(1)}, \ldots, \hat{\mathbf{P}}^{(N)}$
- Build $\hat{\boldsymbol{\mathfrak{z}}}^{c}$

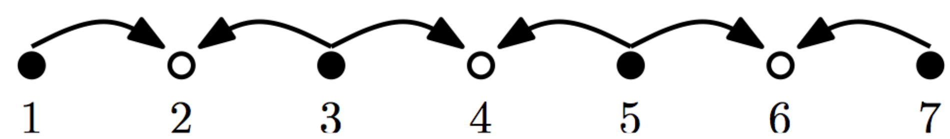


$$\mathbf{A}^{(n)} = \hat{\mathbf{P}}^{(n)} \hat{\mathbf{A}}_c^{(n)}, \; n = 1, \ldots, N$$

multiplicative phase: geometric coarsening

- Partition $\Omega_n = \{1, \dots, I_n\}$ into coarse points $\mathcal{C}_n$ and fine points $\mathcal{F}_n$
- Standard geometric coarsening:

$$\mathcal{C}_n = \{\text{odd points in } \Omega_n\}, \quad \mathcal{F}_n = \{\text{even points in } \Omega_n\}$$



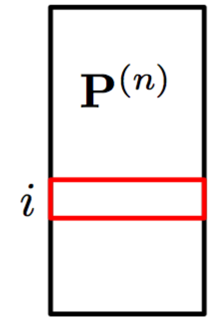
$$p_{ij}^{(n)} = \begin{cases} w_{ij}^{(n)}, & i \in \mathcal{F}_n,\ j \in \{i/2, i/2+1\} \\ 1, & i \in \mathcal{C}_n,\ j = (i+1)/2 \\ 0, & \text{otherwise,} \end{cases} \qquad \mathbf{P}^{(n)} = \begin{bmatrix} 1 & & & \\ * & * & & \\ & 1 & & \\ & * & * & \\ & & 1 & \\ & & * & * \\ & & & 1 \end{bmatrix}$$

multiplicative phase: coarse equations

- Determine $\mathbf{P}^{(n)}$ via weighted least squares fitting of the test/boot factors, *injected* to the coarse level
- Let $\mathbf{U} = [\mathbf{A}_{t,1}^{(n)}, \ldots, \mathbf{A}_{t,n_t}^{(n)} \mid \mathbf{A}_b^{(n)}]$
- For each point $i \in \mathcal{F}_n$

$$\mu_k u_{ik} = \sum_{j \in \mathcal{C}_n^i} \mu_k w_{ij}^{(n)} (\mathbf{u}_{k,c})_j$$

for $k = 1, \ldots, n_f = R(n_t + 1)$.



(one equation per test or boot factor)

- Choose n_t so system is overdetermined
- Weights μ_k proportional to reciprocal of gradient norm

(4b) additive phase: FAS

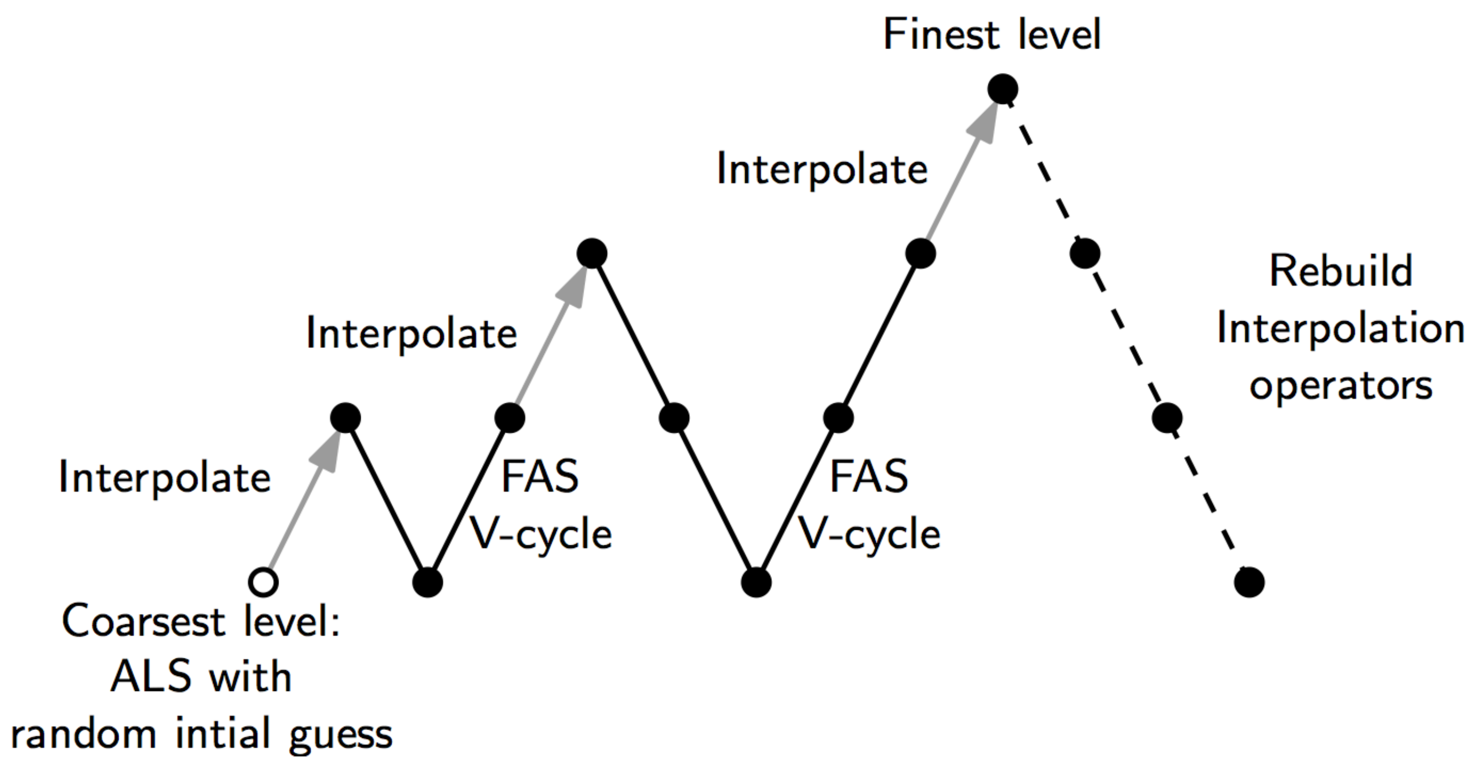
- we apply the FAS to the set of nonlinear equations

$$\mathbf{Z}_{(n)}\left(\mathbf{A}^{(N)} \odot \cdots \odot \mathbf{A}^{(n+1)} \odot \mathbf{A}^{(n-1)} \odot \cdots \odot \mathbf{A}^{(1)}\right) = \mathbf{A}^{(n)}\mathbf{\Gamma}^{(n)}$$

  using the coarse operators and interpolation matrices computed in the setup phase

- we apply block nonlinear Gauss-Seidel as the smoother on all levels

additive phase: FMG FAS

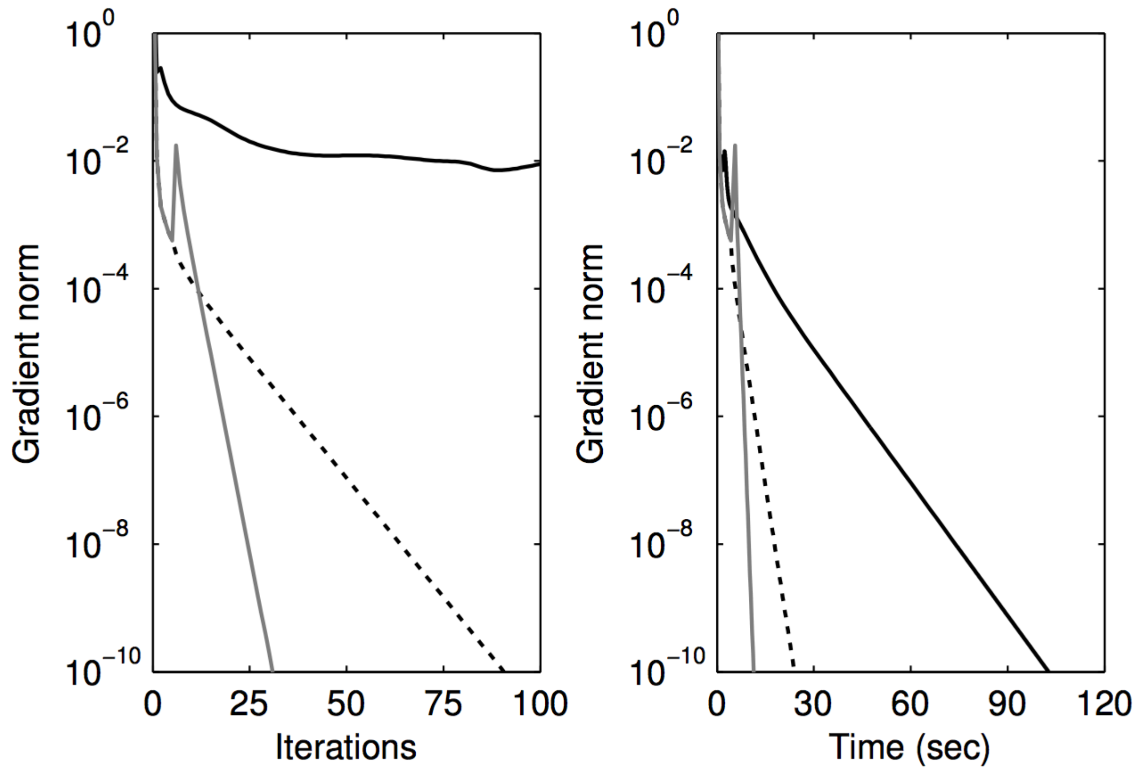


(4c) adaptive AMG numerical results

Finite Difference Laplacian Tensor

$$(N = 4,\ s = 20,\ R = 6)$$

Black: ALS, Dashed: Multilevel, Gray: Multilevel+FMG



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adaptive AMG numerical results

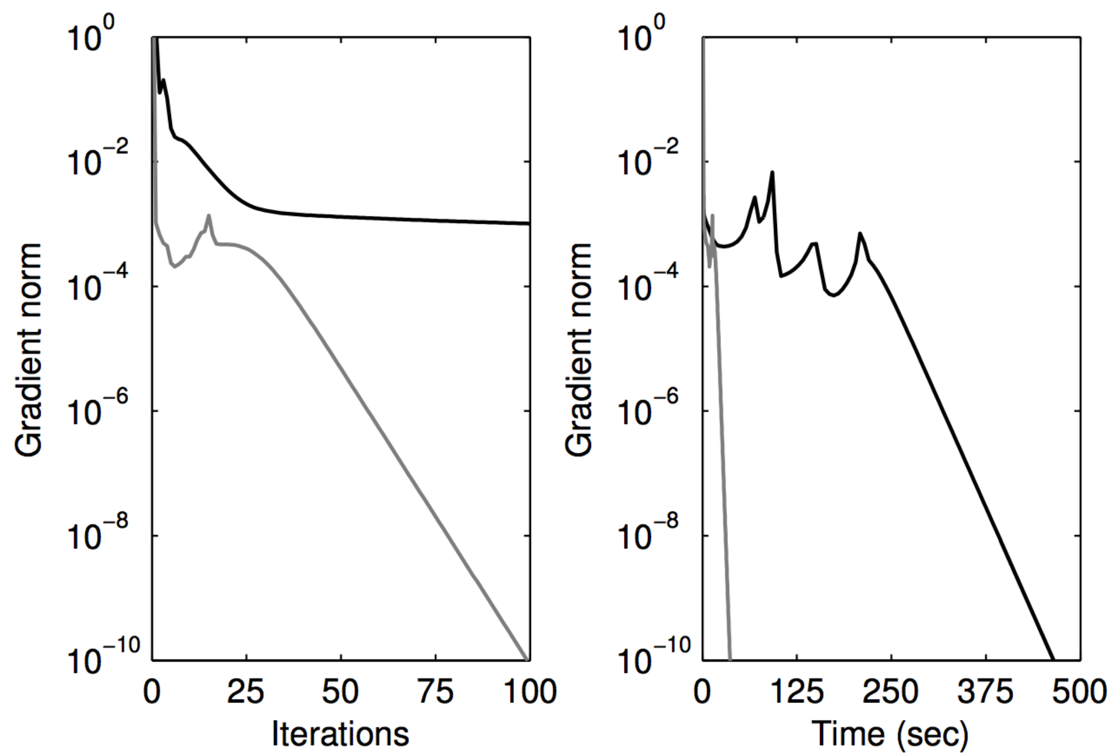
IJK Tensor

Third order tensor of size $s \times s \times s$ with elements

$$z_{ijk} = \left(i^2 + j^2 + k^2\right)^{-1/2} \quad \text{for } i, j, k = 1, \ldots, s.$$

$s = 100,\ R = 5$

Black: ALS, Gray: Multilevel+FMG



adaptive AMG numerical results

Third order tensor of size $s \times s \times s$ with elements

$$z_{ijk} = (i^2 + j^2 + k^2)^{-1/2} \quad \text{for } i, j, k = 1, \ldots, s.$$

	ALS			Multilevel + FMG			
problem parameters	it	time	ns	it	time	spd	ns
$s = 50,\ R = 2$	161	0.7	10	7	2.2	0.3	10
$s = 50,\ R = 3$	2435	11.8	10	15	2.7	4.7	10
$s = 50,\ R = 4$	4838	26.1	5	97	10.1	4.4	7
$s = 50,\ R = 5$				301	30.0		3
$s = 100,\ R = 2$	253	10.7	10	7	7.9	1.4	10
$s = 100,\ R = 3$	1695	80.2	9	9	8.1	10.1	10
$s = 100,\ R = 4$	3836	202.2	6	82	27.5	14.1	9
$s = 100,\ R = 5$	7854	455.2	2	192	62.0	9.0	4
$s = 200,\ R = 2$	274	90.3	10	7	48.8	1.9	10
$s = 200,\ R = 3$	1830	682.3	10	12	61.7	11.2	10
$s = 200,\ R = 4$	2998	1249.5	8	79	178.0	11.6	9
$s = 200,\ R = 5$	5686	2611.4	3	220	440.7	2.6	4

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adaptive AMG numerical results

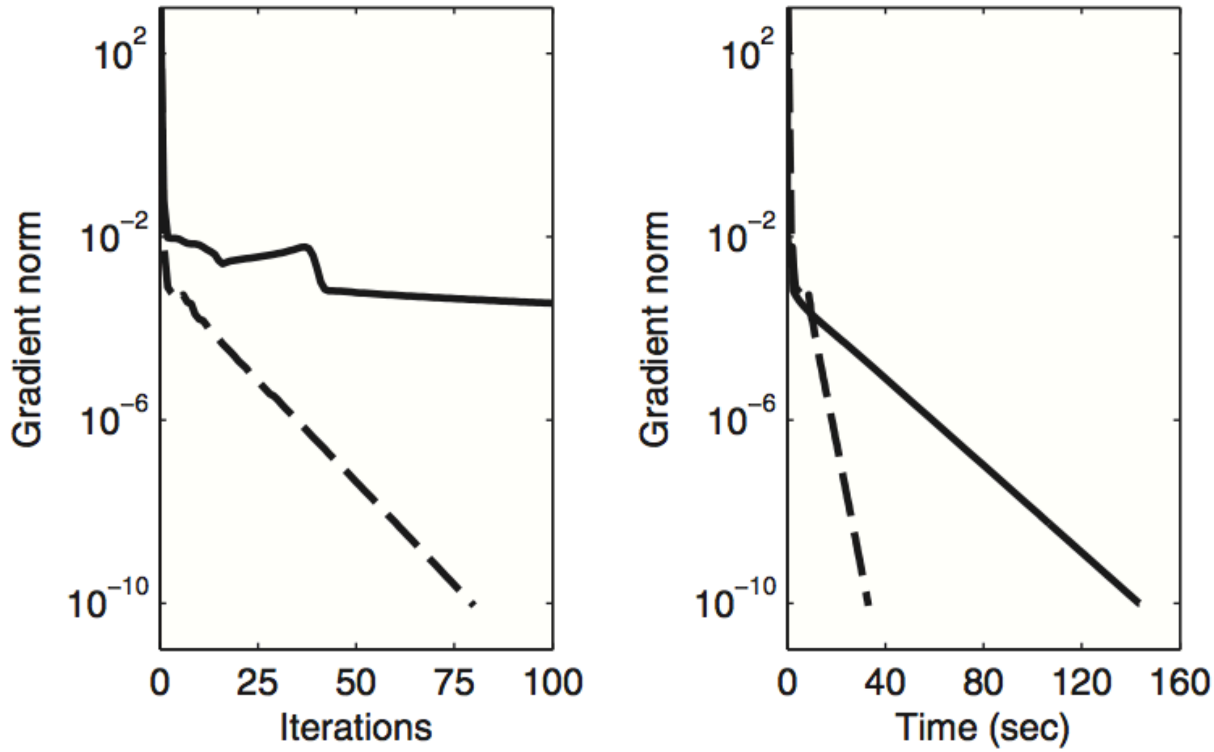

FIG. 6.6. *Random data problem. Convergence plot for test 5 in Table 6.5* ($s = 100$, $R = 5$). *The solid black line is ALS and the dashed line is the multilevel method without FMG.*

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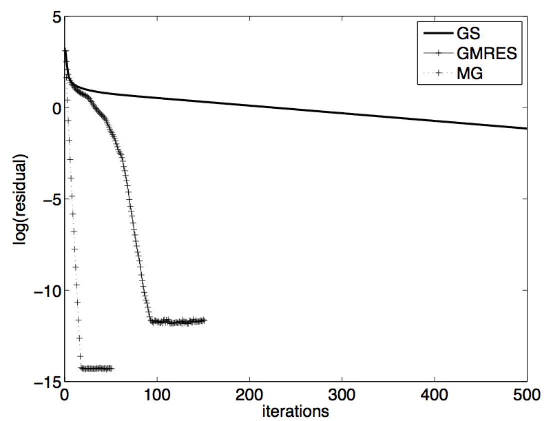
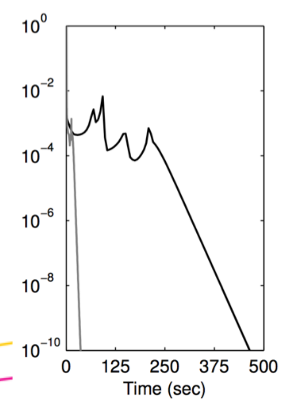
conclusions

convergence acceleration for linear systems	convergence acceleration for nonlinear optimization	convergence acceleration for nonlinear systems	convergence acceleration for nonlinear optimization
P-CG	NCG, P?	NCG, P?	P-NCG
P-GMRES	?	P-NGMRES (Washio and Oosterlee, Anderson)	P-NGMRES for optimization
MG	?	FAS	adaptive AMG-FAS for optimization

- Hans De Sterck, 'A Self-learning Algebraic Multigrid Method for Extremal Singular Triplets and Eigenpairs', SIAM J. Sci. Comput. 34, 2012

- Hans De Sterck and Killian Miller, 'An adaptive algebraic multigrid algorithm for low-rank canonical tensor decomposition', SIAM J. Sci. Comput. 35, 2013





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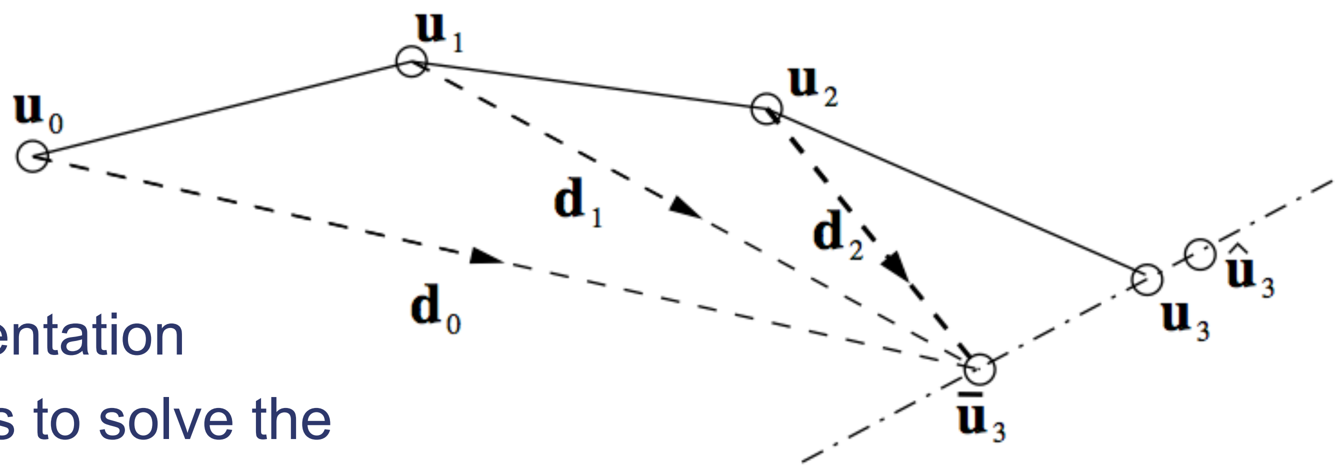
thank you

step II: N-GMRES acceleration: $\nabla f(\mathcal{A}_R) = \mathbf{g}(\mathcal{A}_R) = 0$



- windowed implementation
- several possibilities to solve the small least-squares problems:
  - normal equations (with reuse of scalar products; cost *2nw*) (Washio and Oosterlee, 1997)
  - QR with factor updating (Walker and Ni, 2011)
  - SVD and rank-revealing QR (Fang and Saad, 2009)

$$\hat{\mathbf{u}}_{i+1} = \bar{\mathbf{u}}_{i+1} + \sum_{j=0}^{i} \alpha_j \left(\bar{\mathbf{u}}_{i+1} - \mathbf{u}_j\right)$$

find coefficients $(\alpha_0, \ldots, \alpha_i)$ that minimize

$$\left\|\mathbf{g}(\bar{\mathbf{u}}_{i+1}) + \sum_{j=0}^{i} \alpha_j \left(\mathbf{g}(\bar{\mathbf{u}}_{i+1}) - \mathbf{g}(\mathbf{u}_j)\right)\right\|_2.$$