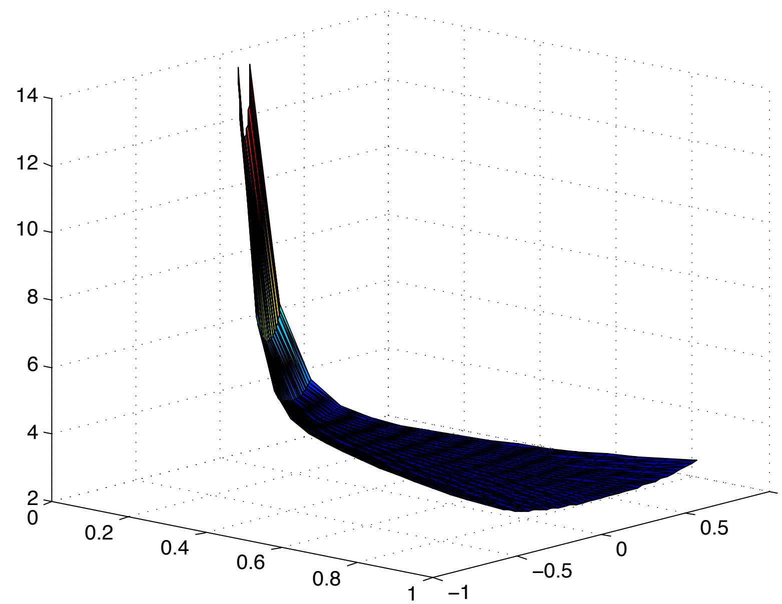
High-Order Finite Volume Element Methods for Elliptic PDEs with Singularities, and Applications to Capillarity



Yasunori Aoki

University of Waterloo
(now at Uppsala University)

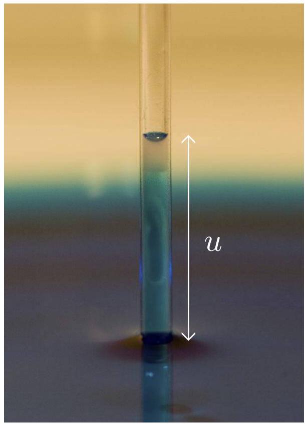
Hans De Sterck

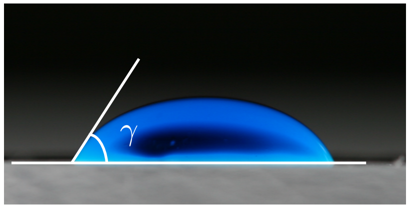
University of Waterloo

Laplace-Young Equation

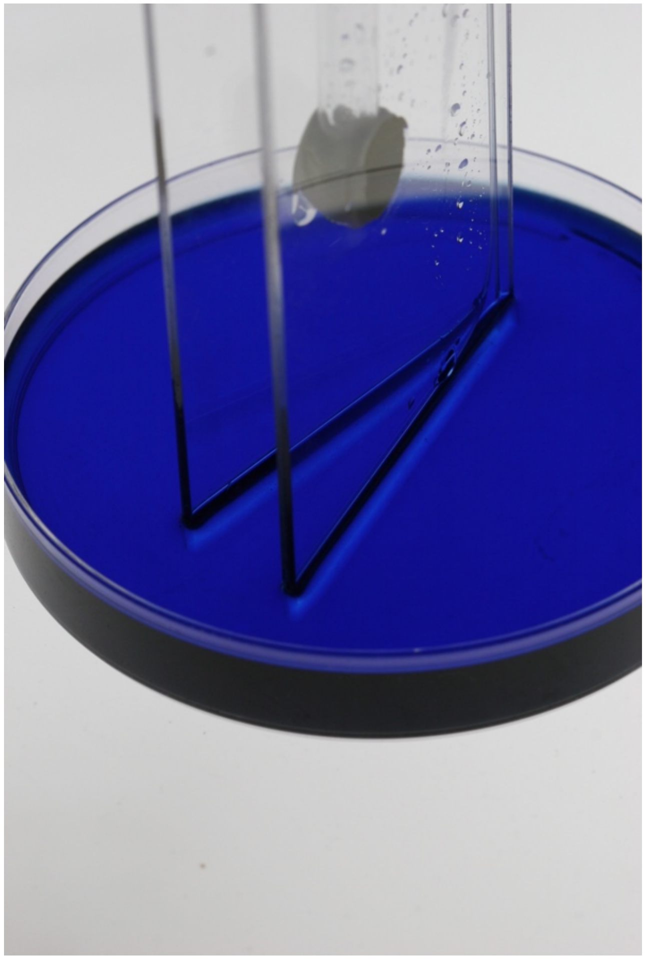
$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$

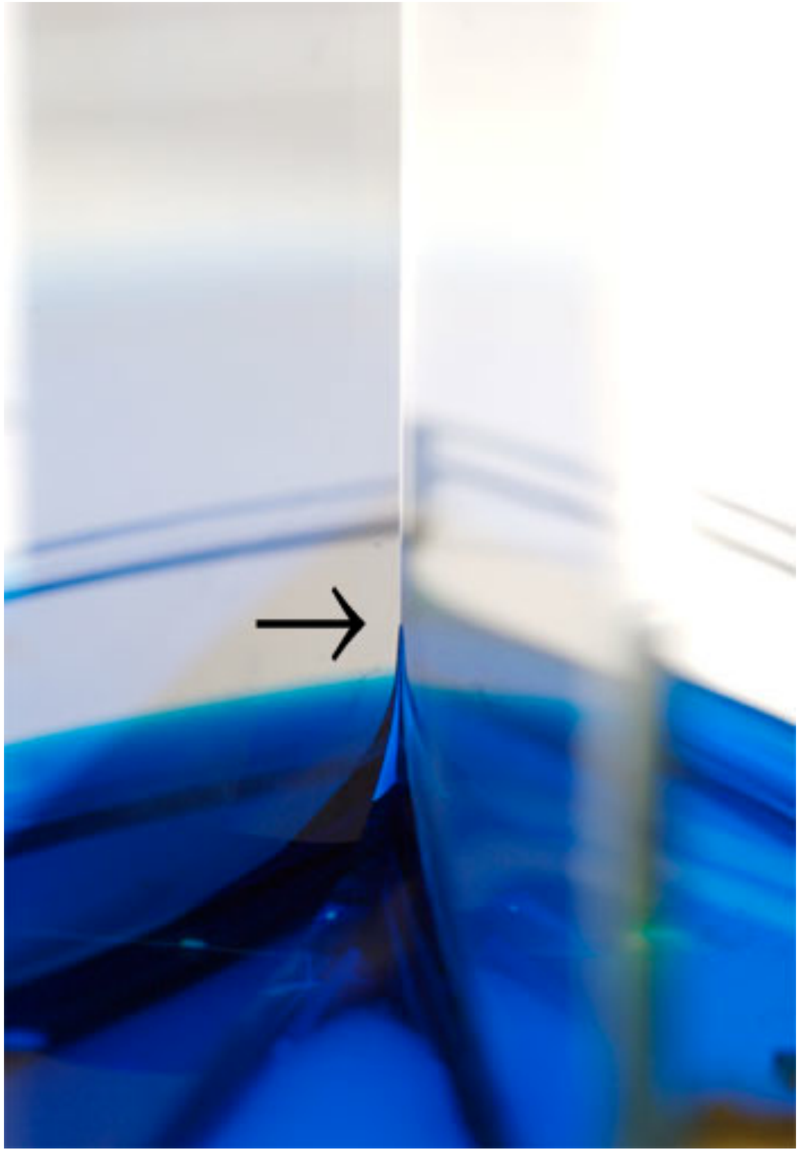




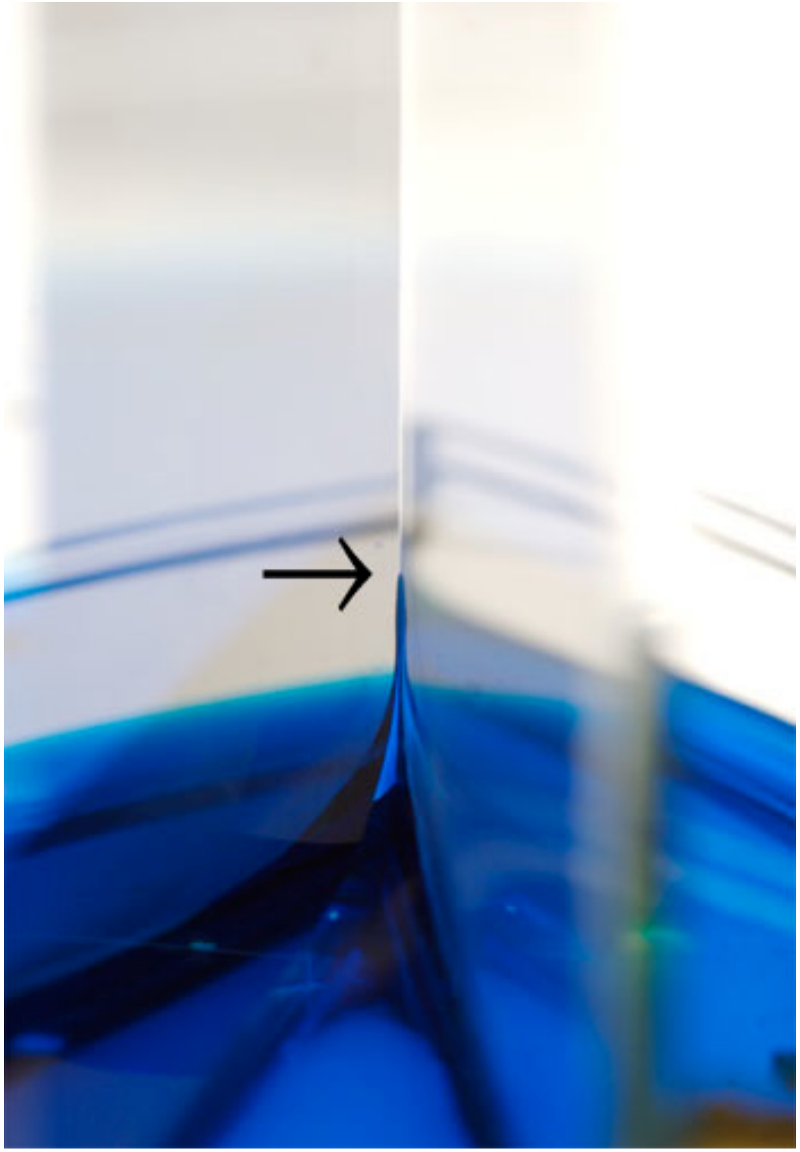
Liquid Surface at a corner



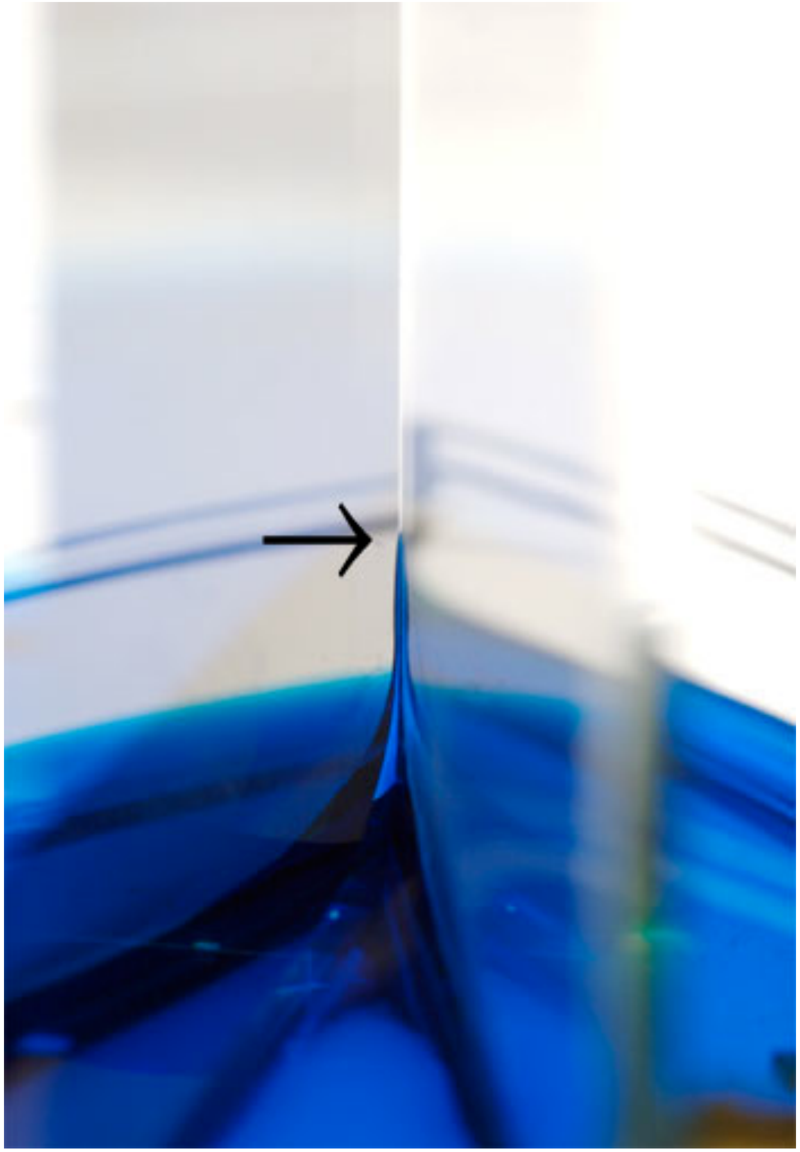
Liquid Surface at a corner



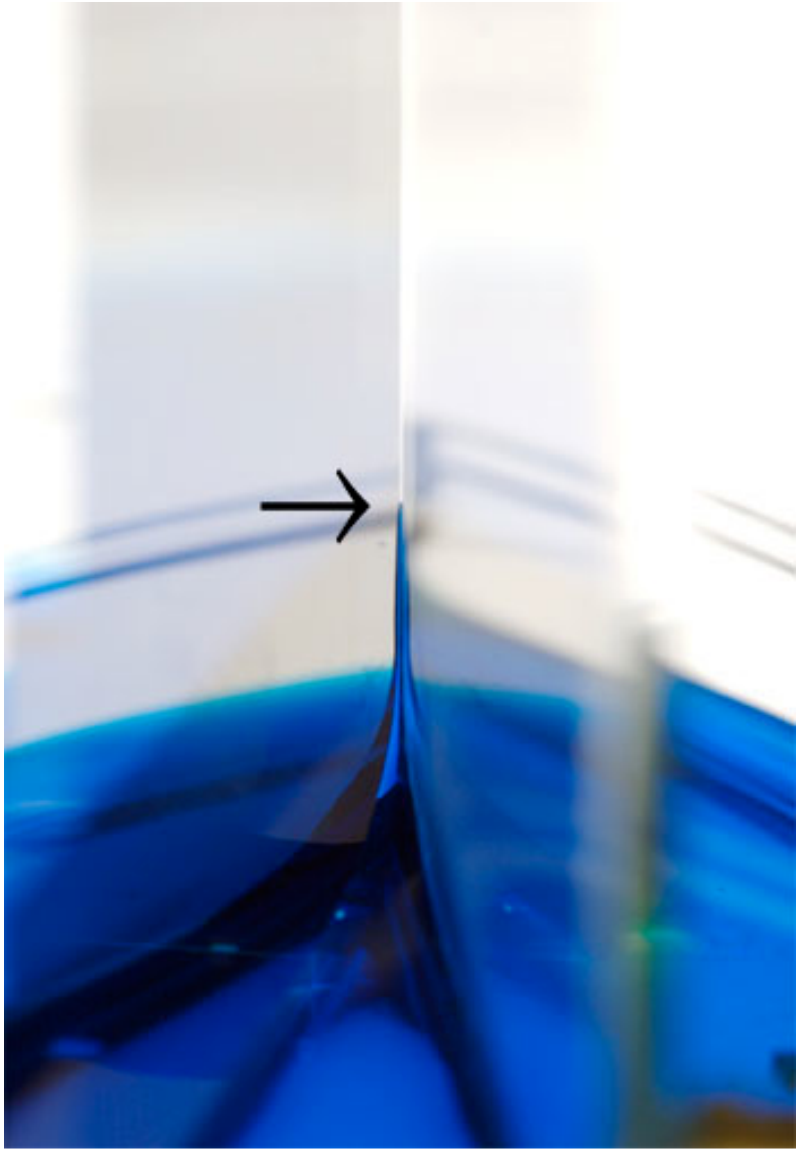
Liquid Surface at a corner



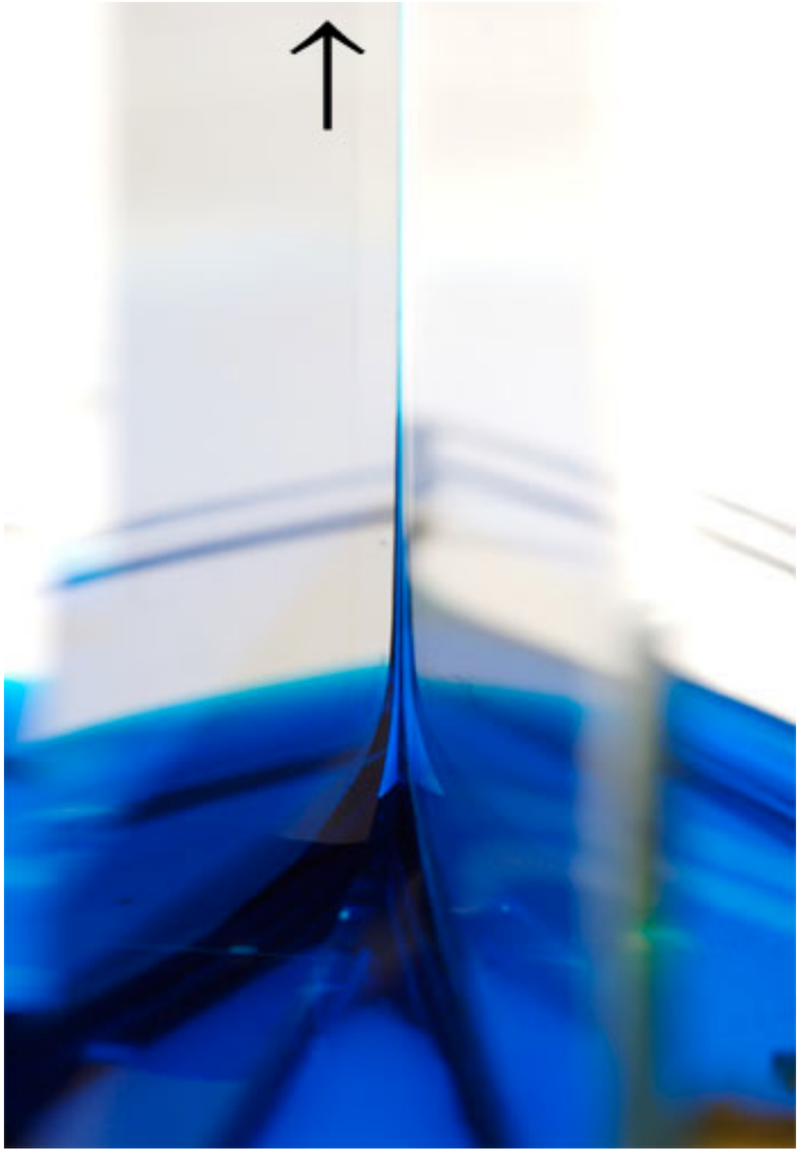
Liquid Surface at a corner



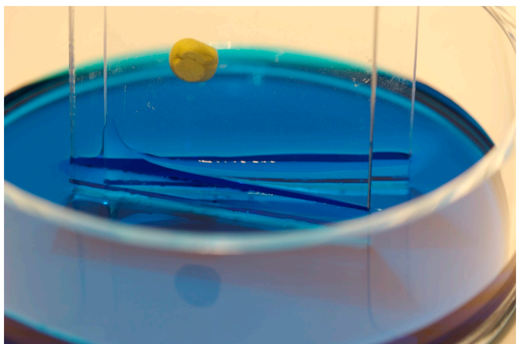
Liquid Surface at a corner

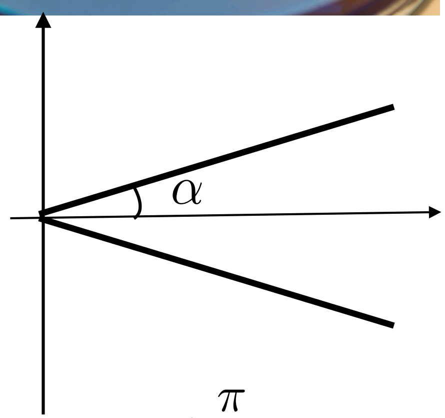


Liquid Surface at a corner



In a domain with a sharp corner, the solution becomes unbounded. (Concus and Finn)





$$\alpha < \frac{\pi}{2} - \gamma$$

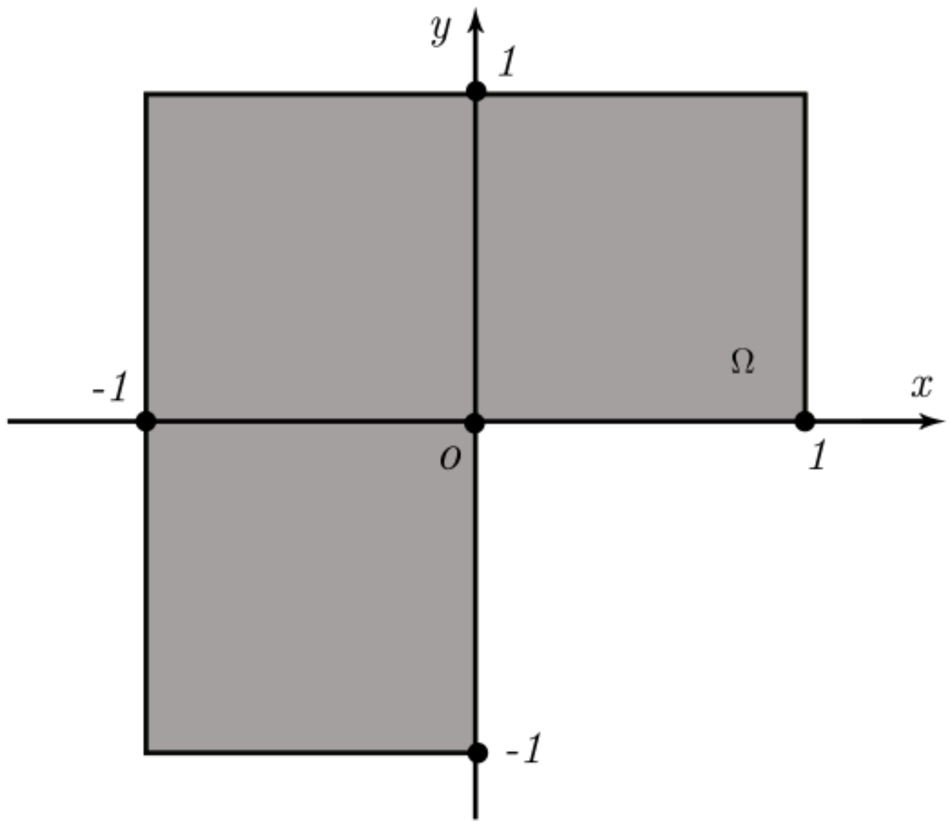
Overview

1. high-order finite volume element method (FVEM) for linear elliptic PDEs with singularities

2. FVEM for nonlinear capillary surfaces

1. high-order finite volume element method (FVEM) for linear elliptic PDEs with singularities

$$\Delta u = f(x, y) \quad \text{in } \Omega,$$
$$u = g(x, y) \quad \text{on } \partial\Omega,$$



FVEM idea

use FE trial functions integrated over FV control volumes (e.g., Bank and Rose 1987)

high-order FVEM: use high-order FE nodal trial functions... but which control volumes??

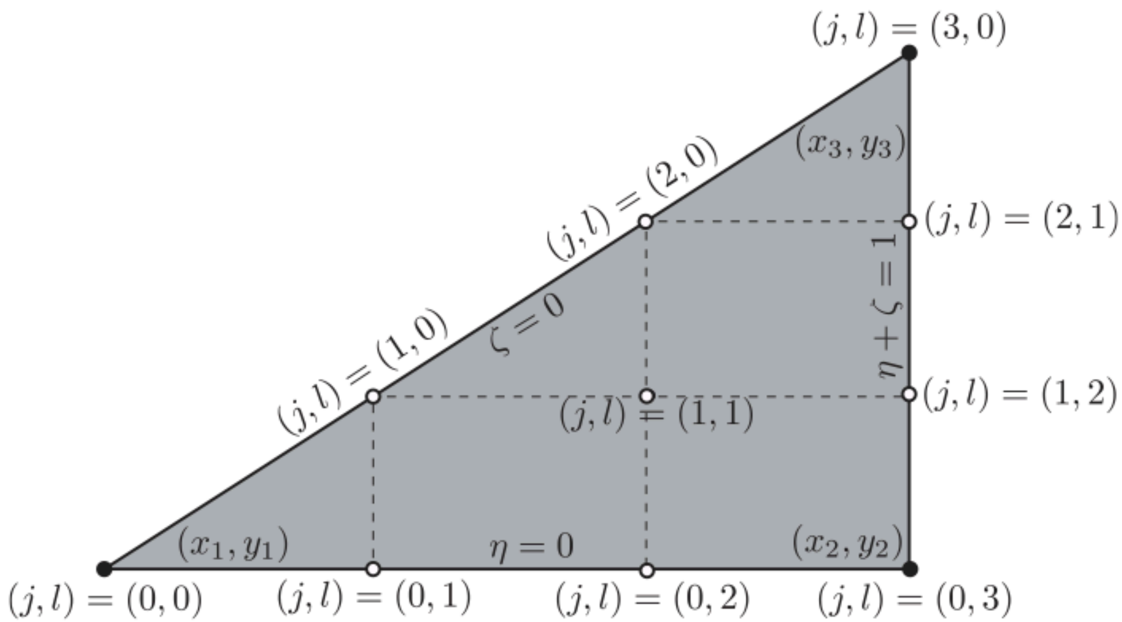
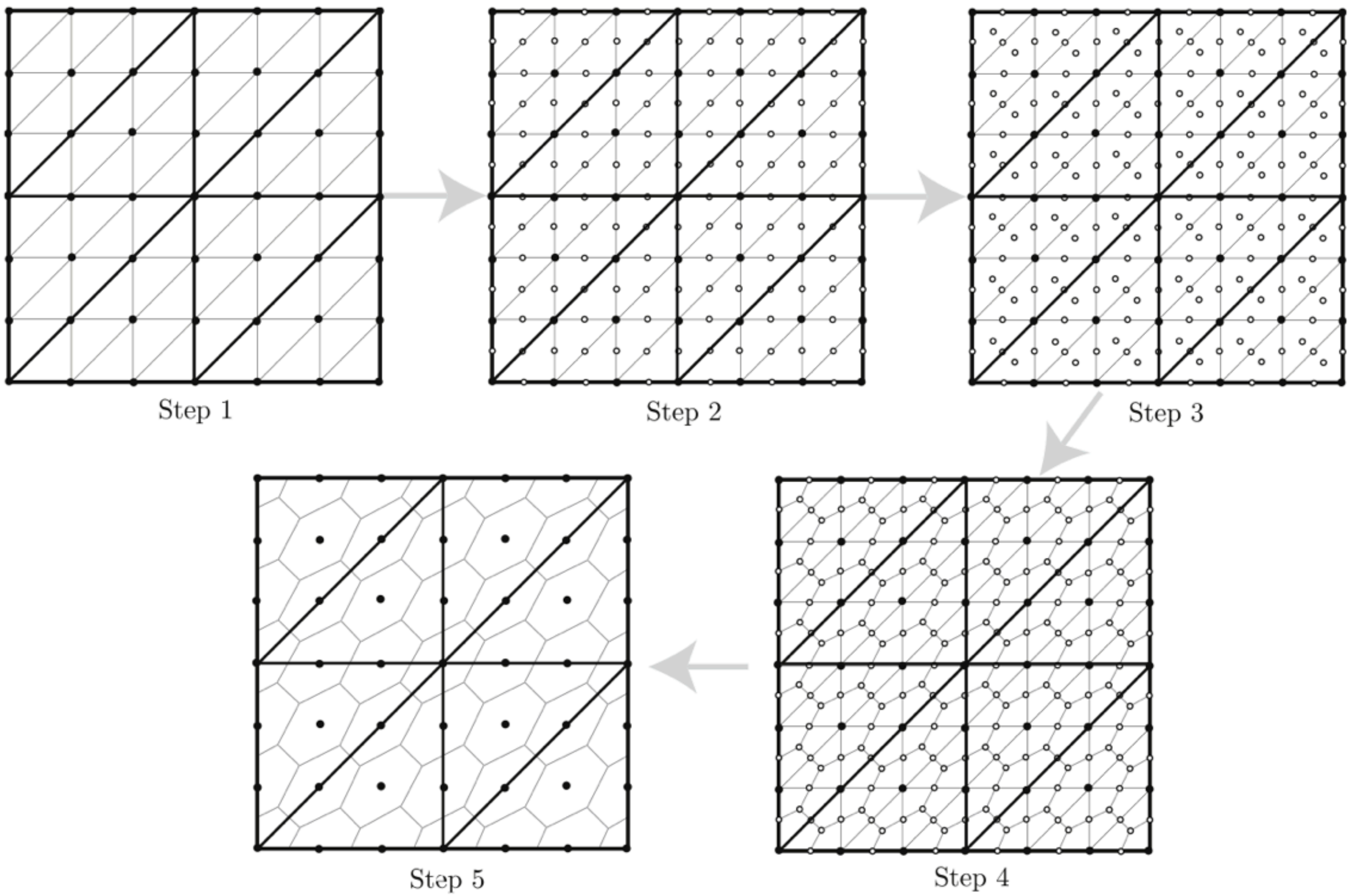


Fig. 3. Placement of nodes in an element triangle ($p = 3$).

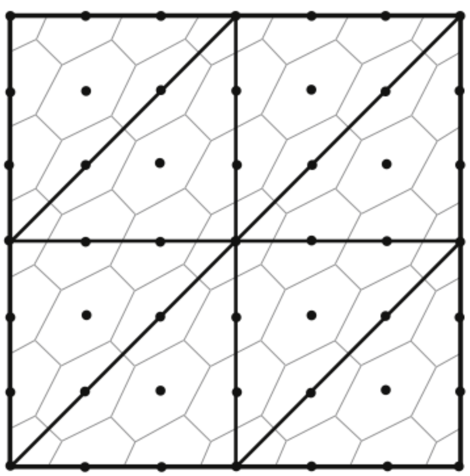
high-order FVEM (Vogel, Xu and Wittum, 2010)



construct control volumes in a systematic way

FVEM formulation

$$\Delta u = f(x, y) \quad \text{in } \Omega,$$
$$u = g(x, y) \quad \text{on } \partial\Omega,$$



$$\int_{\Omega_\alpha} \Delta u \, \mathrm{d}A = \int_{\Omega_\alpha} f \, \mathrm{d}A$$

$$\int_{\partial\Omega_\alpha} \nu \cdot \nabla u \, \mathrm{d}s = \int_{\Omega_\alpha} f \, \mathrm{d}A$$

$$\sum_{j=1}^{N_{node}} c_j \int_{\partial\Omega_i} \nu \cdot \nabla\phi_j \, \mathrm{d}s = \int_{\Omega_i} f \, \mathrm{d}A \quad \forall i \in \mathcal{N}_{int}$$

$$u^h(x_i, y_i) = \sum_{j=1}^{N_{node}} c_j \phi_j(x_i, y_i) = c_i = g(x_i, y_i) \quad \forall i \in \mathcal{N}_{bound}$$

sufficiently smooth solution (H_1)

we can use standard FE trial space

$$S_p^h := span\{\phi_1, \phi_2, \ldots, \phi_{N_{node}}\}$$

model problem 1: (smooth)

$$f(x,y) = 20x^3y^4 + 12x^5y^2 \quad \text{in } \Omega,$$

$$g(x,y) = x^5y^4 \quad \text{on } \partial\Omega,$$

where domain Ω is a unit square domain. The exact solution is

$$u(x,y) = x^5y^4 \quad \text{in } \Omega.$$

model problem I - standard trial space

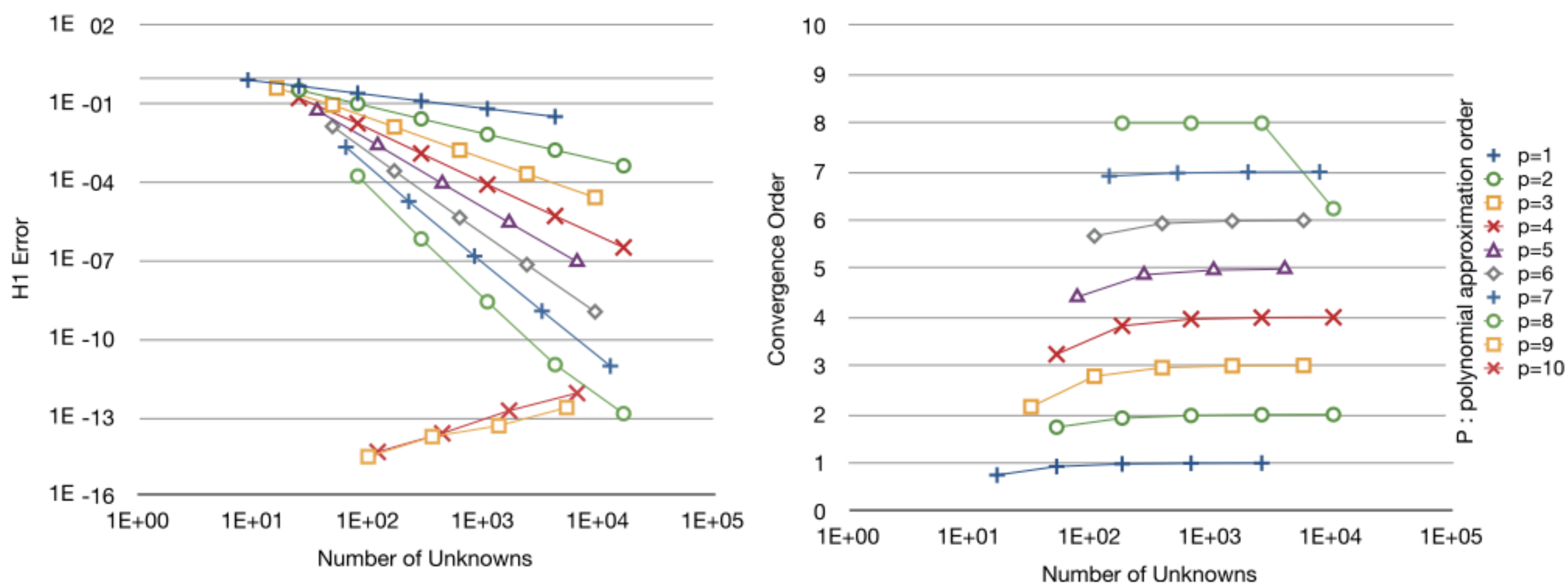


Fig. 6. H^1 error convergence for the Poisson problem with 9th order polynomial exact solution (Model Problem 1).

(note: no general convergence theory)

model problem I - standard trial space

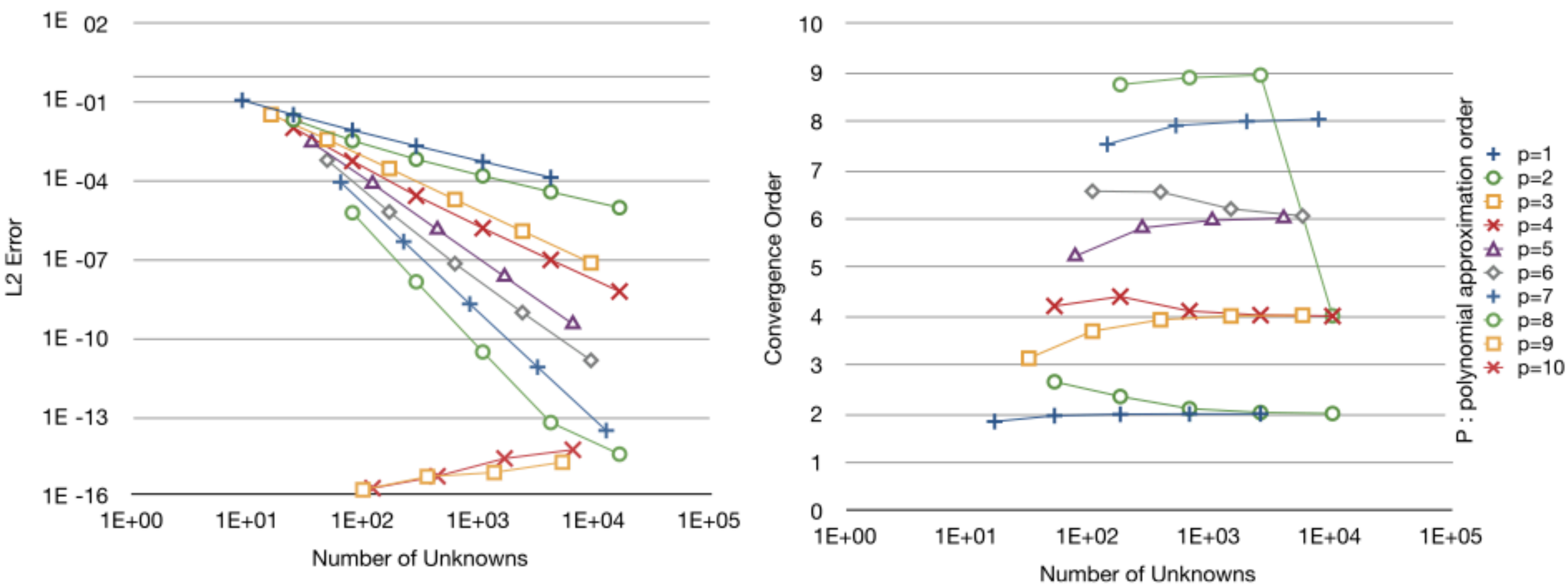


Fig. 7. L_2 error convergence for the Poisson problem with 9th order polynomial exact solution (Model Problem 1).

$$\|u - u^h\|_{H^1} = O(h^p),$$

$$\|u - u^h\|_{L_2} = \begin{cases} O(h^{p+1}) & \text{for } p = 1, 3, 5, 7, 8, \\ O(h^p) & \text{for } p = 2, 4, 6, \end{cases}$$

suboptimal convergence in L_2 for even p (also on irregular grids) (compare DG)

model problem II - singular solution

Consider Poisson problem (1)–(2) with the following right hand side and boundary data:

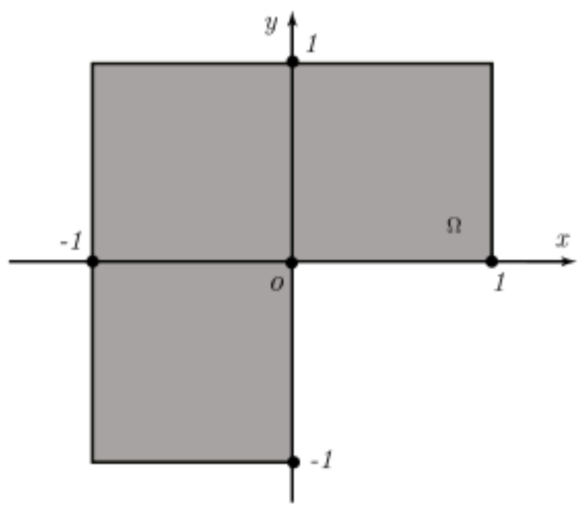
$$f(x,y) = 20x^3y^4 + 12x^5y^2 \quad \text{in } \Omega, \tag{9}$$

$$\begin{aligned} g(x,y) \;=\; & x^5y^4 + 2r^{\frac{2}{3}}\sin\left(\frac{2}{3}\theta\right) + 7r^{\frac{4}{3}}\sin\left(\frac{4}{3}\theta\right) + r^2\{\ln r\sin(2\theta) + \theta\cos(2\theta)\} \\ & + 8r^{\frac{8}{3}}\sin\left(\frac{8}{3}\theta\right) + 2r^{\frac{10}{3}}\sin\left(\frac{10}{3}\theta\right) + 8r^4\{\ln r\sin(4\theta) + \theta\cos(4\theta)\} \quad \text{on } \partial\Omega, \end{aligned} \tag{10}$$

where domain Ω is as illustrated in Fig. 2, and r and θ are polar coordinates centred at the origin. The exact solution is

$$\begin{aligned} u(x,y) = & x^5y^4 + 2r^{\frac{2}{3}}\sin\left(\frac{2}{3}\theta\right) + 7r^{\frac{4}{3}}\sin\left(\frac{4}{3}\theta\right) + r^2\{\ln r\sin(2\theta) + \theta\cos(2\theta)\} \\ & + 8r^{\frac{8}{3}}\sin\left(\frac{8}{3}\theta\right) + 2r^{\frac{10}{3}}\sin\left(\frac{10}{3}\theta\right) + 8r^4\{\ln r\sin(4\theta) + \theta\cos(4\theta)\} \quad \text{in } \Omega. \end{aligned} \tag{11}$$

Note that the r-directional derivative of $u(x,y)$ blows up at the origin, but $g(x,y)$ is analytic on $\partial\Omega$ since $g(x,0) = 0$ and $g(0,y) = (-3/2\,y^2 + 12\,y^4)\pi$.



model problem II - standard trial space

Fig. 8. L_2 error convergence for the Poisson problem with 9th order polynomial exact solution on a randomly perturbed grid (Model Problem 1).

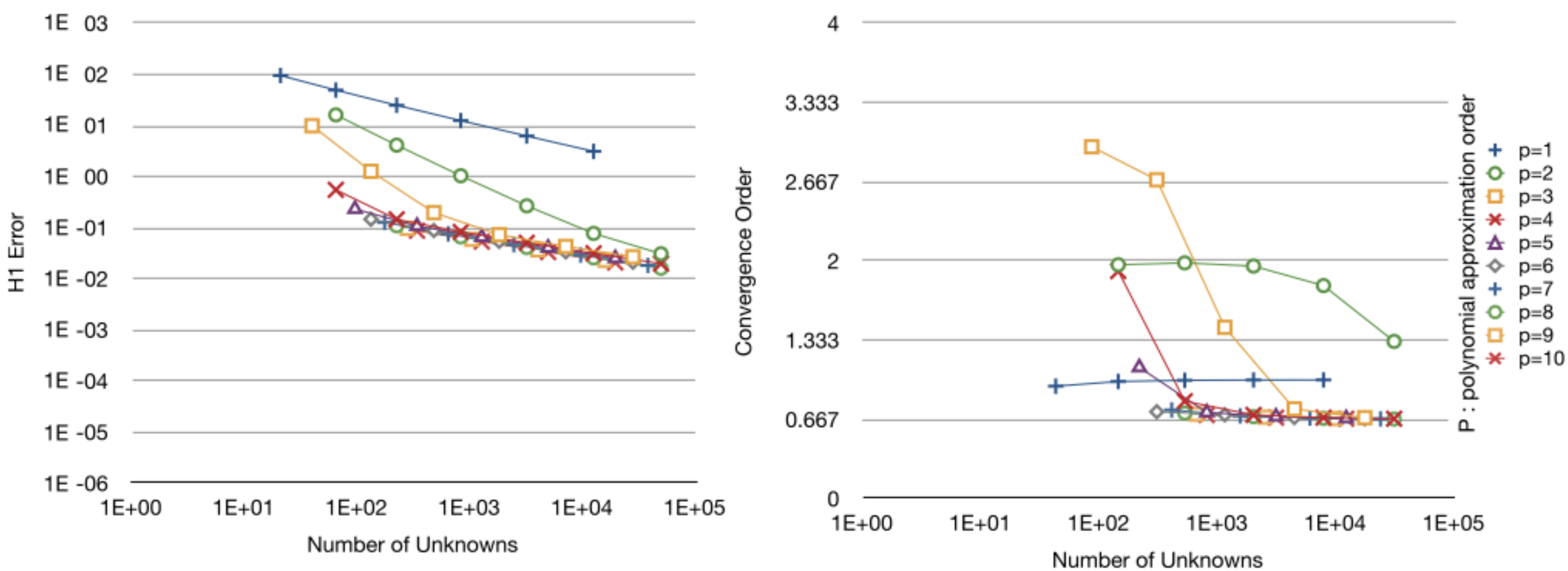


Fig. 9. H^1 error convergence for derivative blow-up singular solution (Model Problem 2).

model problem II - augmented trial space

$$\hat{S}_p^h := span\{\phi_1, \phi_2, \ldots, \phi_{N_{node}}, \psi_{1,1}, \psi_{1,2}, \ldots, \psi_{1,N_s}\}$$

$$u \approx \hat{u}^h := \sum_{i=1}^{N_{node}} c_i \phi_i + \sum_{i=1}^{N_s} k_i \psi_{1,i}$$

$$\psi_{1,1} = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\theta\right),$$

$$\psi_{1,2} = r^{\frac{4}{3}} \sin\left(\frac{4}{3}\theta\right),$$

$$\psi_{1,3} = r^2\{\ln r \sin(2\theta) + \theta \cos(2\theta)\},$$

$$\vdots$$

$$\sum_{j=1}^{N_{node}} c_j \int_{\partial\Omega_i} \nu \cdot \nabla\phi_j \, ds + \sum_{j=1}^{N_s} k_j \int_{\partial\Omega_i} \nu \cdot \nabla\psi_{1,j} \, ds = \int_{\Omega_i} f \, dA \quad \forall i \in \mathcal{N}_{int}$$

note: $\int_{\Omega_i} \Delta\psi_{1,j} \, dA = \int_{\partial\Omega_i} \nu \cdot \nabla\psi_{1,j} \, ds = 0 \quad \forall i \in \mathcal{N}_{int}$ since $\Delta\psi_{1,j} = 0$

$$\sum_{j=1}^{N_{node}} c_j \int_{\partial\Omega_i} \nu \cdot \nabla\phi_j \, ds = \int_{\Omega_i} f \, dA \quad \forall i \in \mathcal{N}_{int}$$

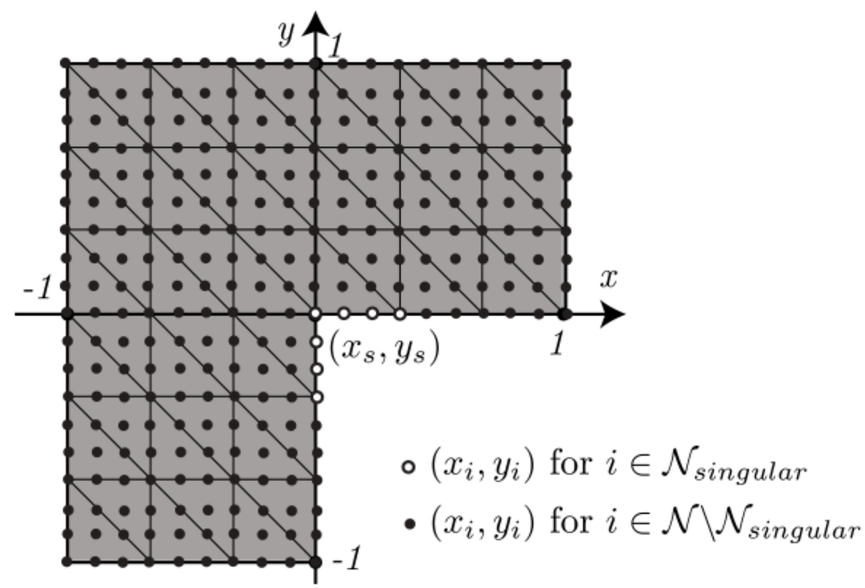
model problem II - augmented trial space

$$\sum_{j=1}^{N_{node}} c_j \int_{\partial\Omega_i} \nu \cdot \nabla\phi_j \, ds = \int_{\Omega_i} f \, dA \quad \forall i \in \mathcal{N}_{int}$$

$$u \approx \hat{u}^h := \sum_{i=1}^{N_{node}} c_i \, \phi_i + \sum_{i=1}^{N_s} k_i \, \psi_{1,i}$$

$$c_i + \sum_{j=1}^{N_s} k_j \psi_{1,j}(x_i, y_i) = g(x_i, y_i) \quad \forall i \in \mathcal{N}_{bound}$$

N_s extra control volumes are needed; chosen near the singularity



model problem II - augmented trial space

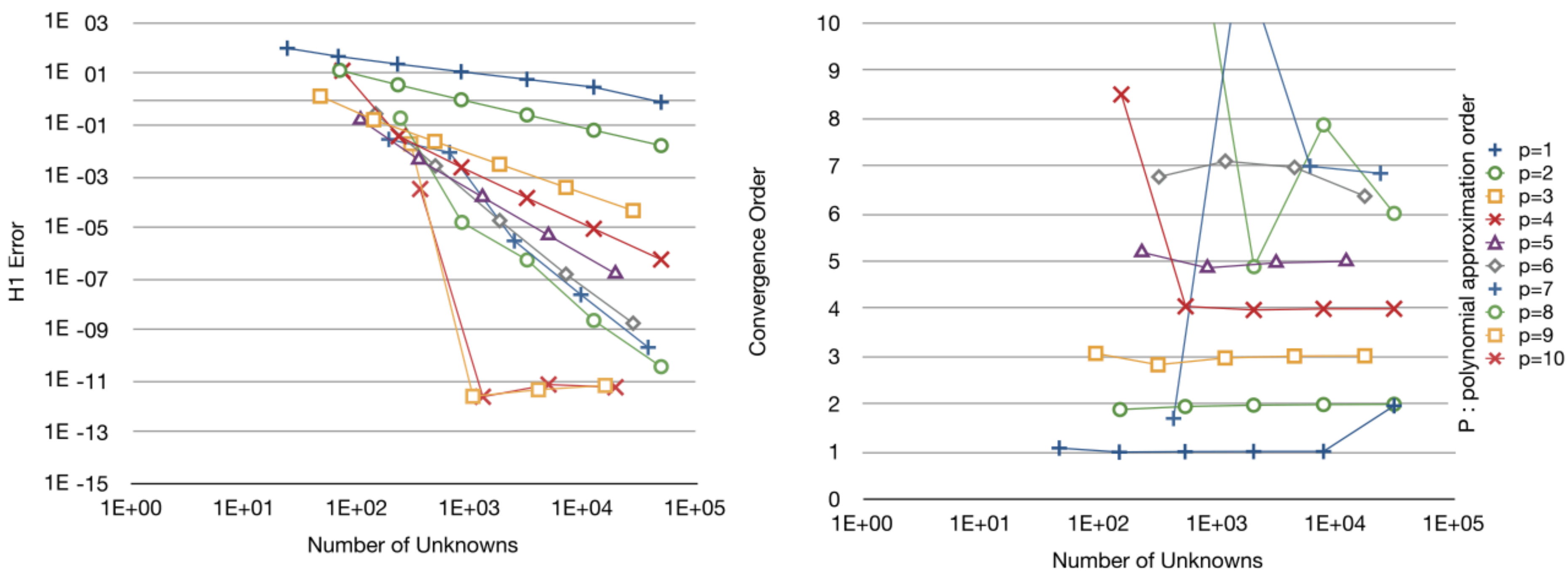
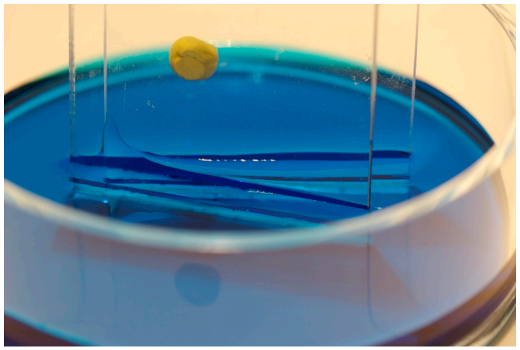
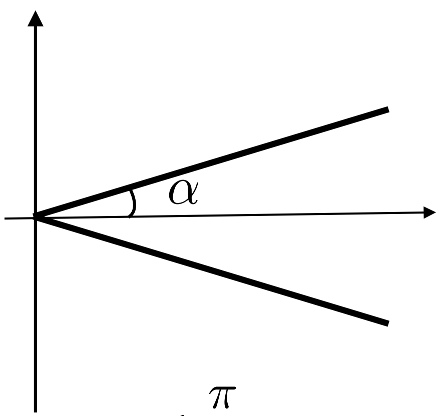


Fig. 10. H^1 error convergence for derivative blow-up singular solution with augmented trial function space (Model Problem 2).

2. FVEM for nonlinear capillary surfaces

Wedge

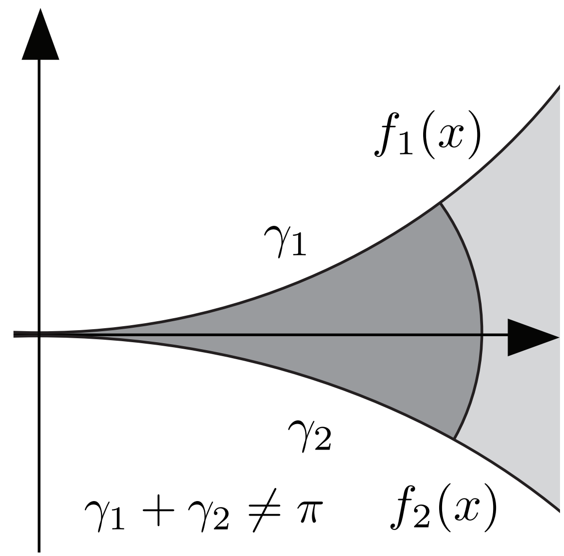




$$\alpha < \frac{\pi}{2} - \gamma$$

$$\Omega = \{(r, \theta) : 0 < r < R\,,\ -\alpha < \theta < \alpha\}$$

Cusp



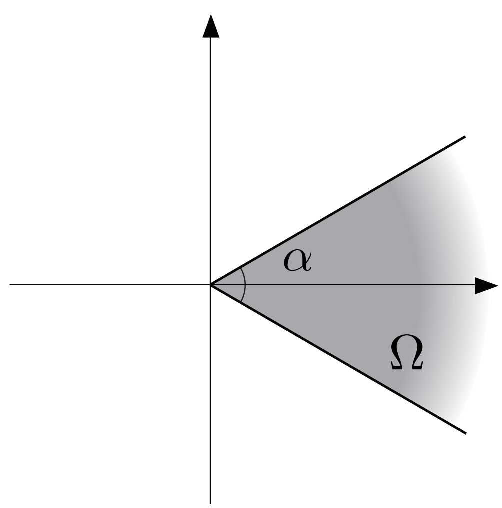
$$\gamma_1 + \gamma_2 \neq \pi$$

$$\Omega = \{(x, y) : 0 < x,\ f_2(x) < y < f_1(x)\}$$

Asymptotic Laplace-Young Equation

$$\nabla \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = u \qquad\qquad \nabla \cdot \frac{\nabla v}{|\nabla v|^2} = v \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla v}{|\nabla v|^2} = \cos\gamma \qquad \text{on } \partial\Omega$$



$$v(r,\theta) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{kr}$$

(Concus and Finn, Miersemann, and King et al.)

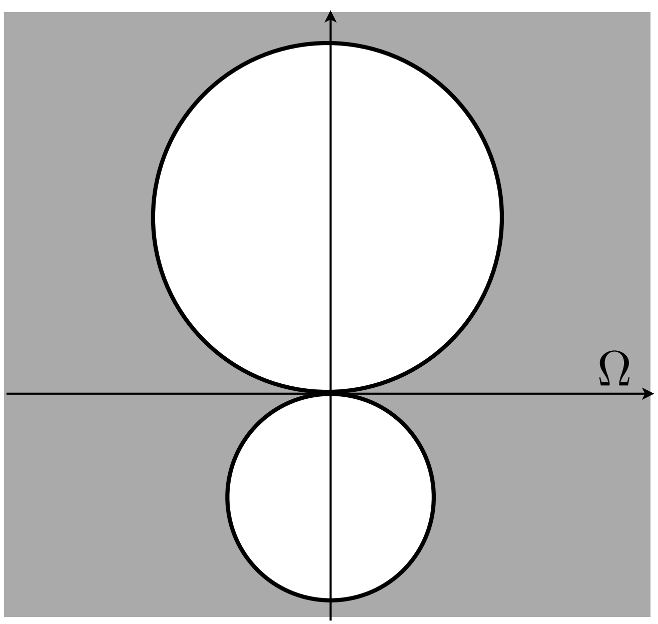
$$u(r,\theta) = v(r,\theta) + O(r^3) \qquad \text{as } r \to 0$$

(Miersemann)

Asymptotic Laplace-Young Equation

$$\nabla \cdot \frac{\nabla v}{|\nabla v|^2} = v \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla v}{|\nabla v|^2} = \cos\gamma \qquad \text{on } \partial\Omega$$

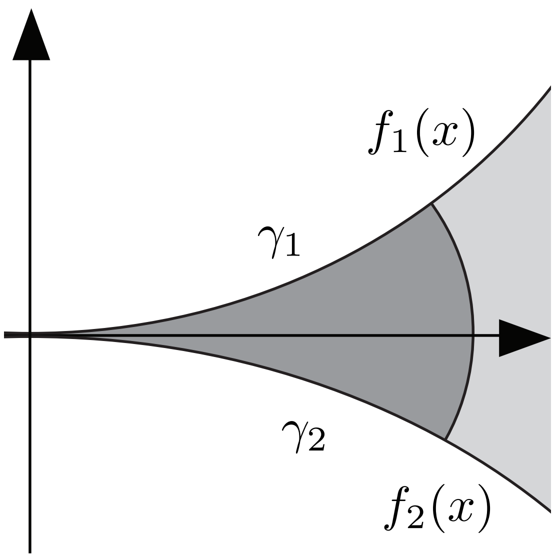


$$v(p,q) = Ap^2 - 2\sqrt{1 - A^2(q-q_0)^2}\; p - A(q-q_0)^2 + Aq_0^2$$

$$u(p,q) = v(p,q) + O(p^{-5}) \qquad \text{as } p \to \infty$$

(Aoki M.Math thesis)

Asymptotic Analysis
(general cases)



$$\gamma_1 + \gamma_2 \neq \pi$$

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$

after some calculation ...

$$u(x,y) = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + O\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right) \qquad \text{as } x \to 0^+$$

* some restrictions on f_1 and f_2 apply

(Aoki and Siegel)

Asymptotic Analysis

(summary)

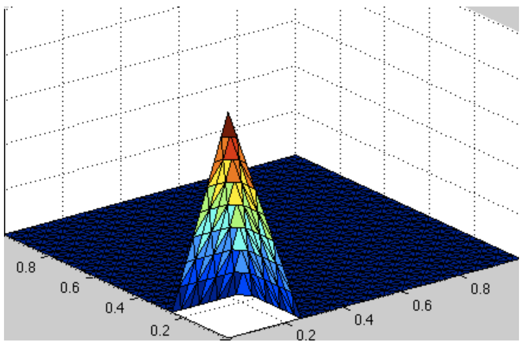
Corner: $$u(r, \theta) \approx \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{kr}$$

Cusp: $$u(x, y) \approx \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)}$$

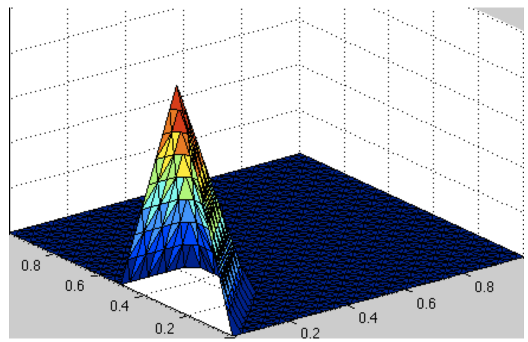
Approximation only accurate near the singularity!

Finite Element Approximation

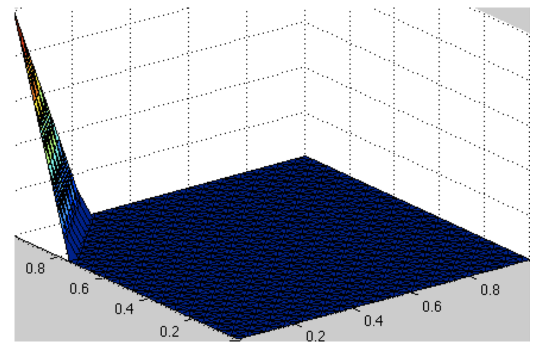
Basis Functions (p=1)



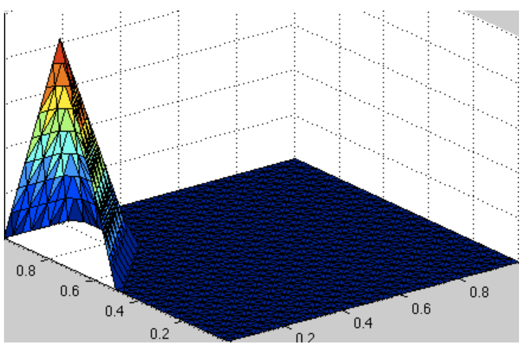
ϕ_1



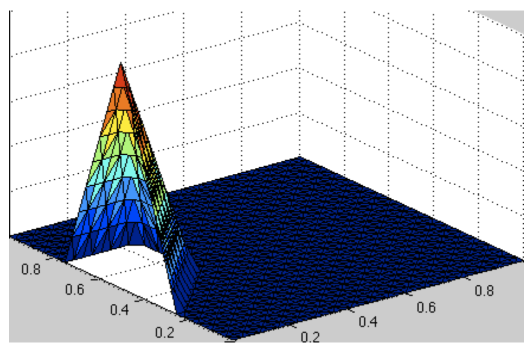
ϕ_2



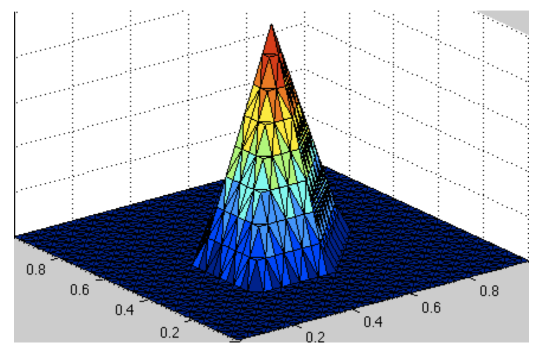
ϕ_5



ϕ_4



ϕ_3



ϕ_i

Standard Trial Function Expansion

$$u \approx u^h := \sum_{i=1}^{N_{\text{node}}} c_i \phi_i$$

Asymptotic Analysis

$$u = \frac{O(1)}{f_1(x) - f_2(x)}$$

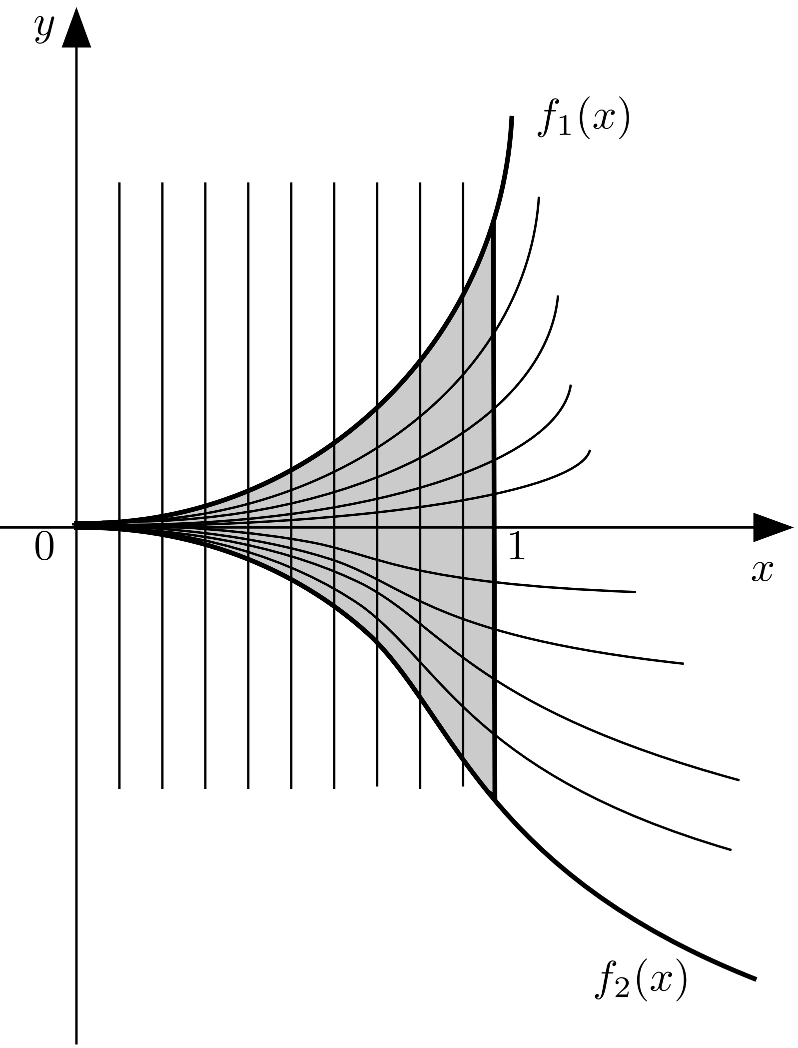
(1) Change of Variable

Bounded function

$$u = \frac{v}{f_1(x) - f_2(x)}$$

Unbounded function

(2) Change of Coordinates

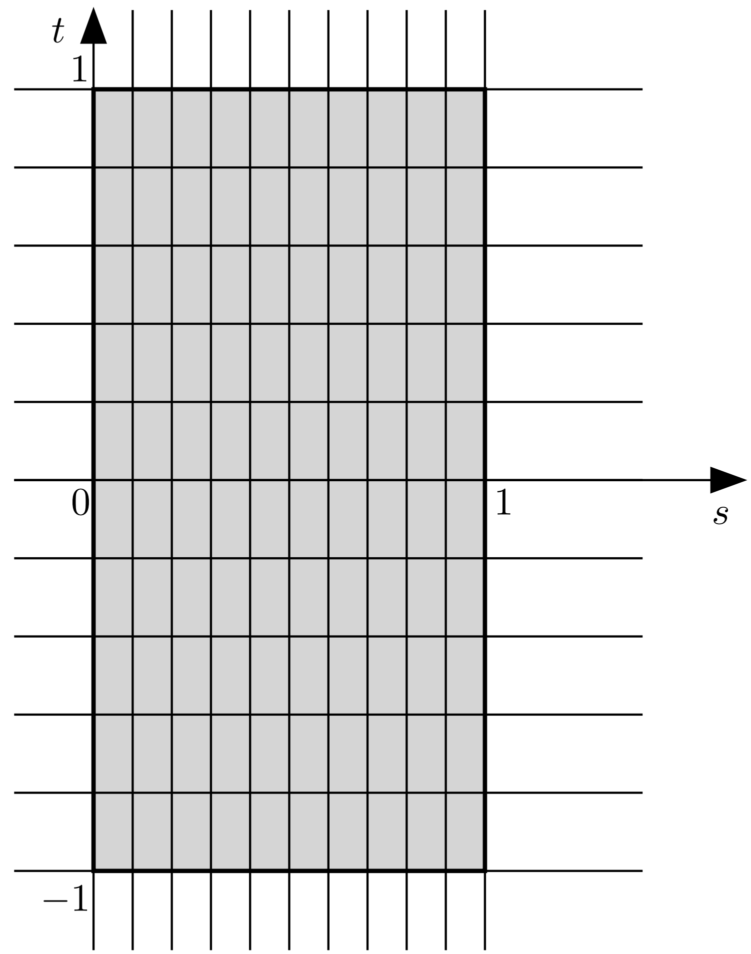


$$t = \frac{2y - (f_1 + f_2)}{f_1 - f_2}$$

$$s = x$$

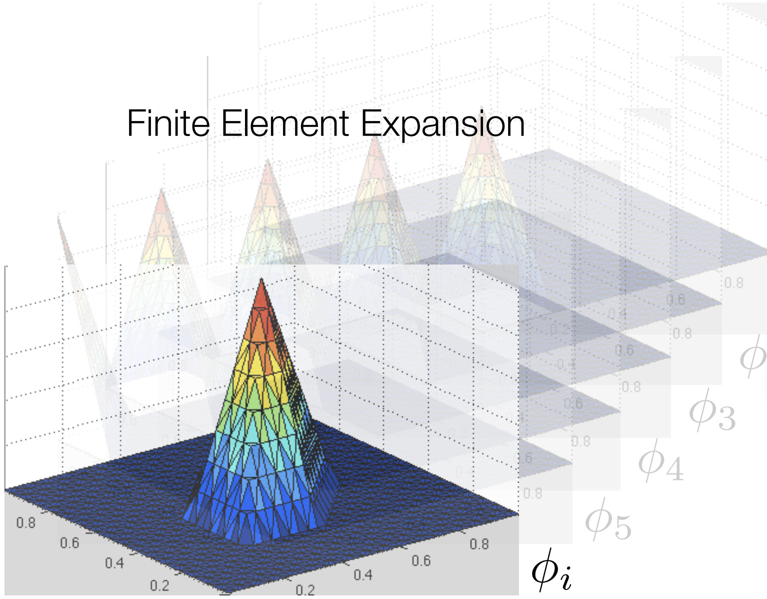
$$x = s$$

$$y = \frac{1+t}{2} f_1(s) + \frac{1-t}{2} f_2(s)$$

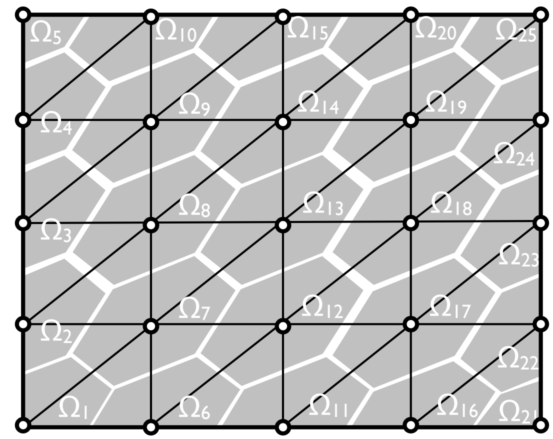


Finite Volume Element method

Finite Element Expansion



Finite Volume Method Control Volumes

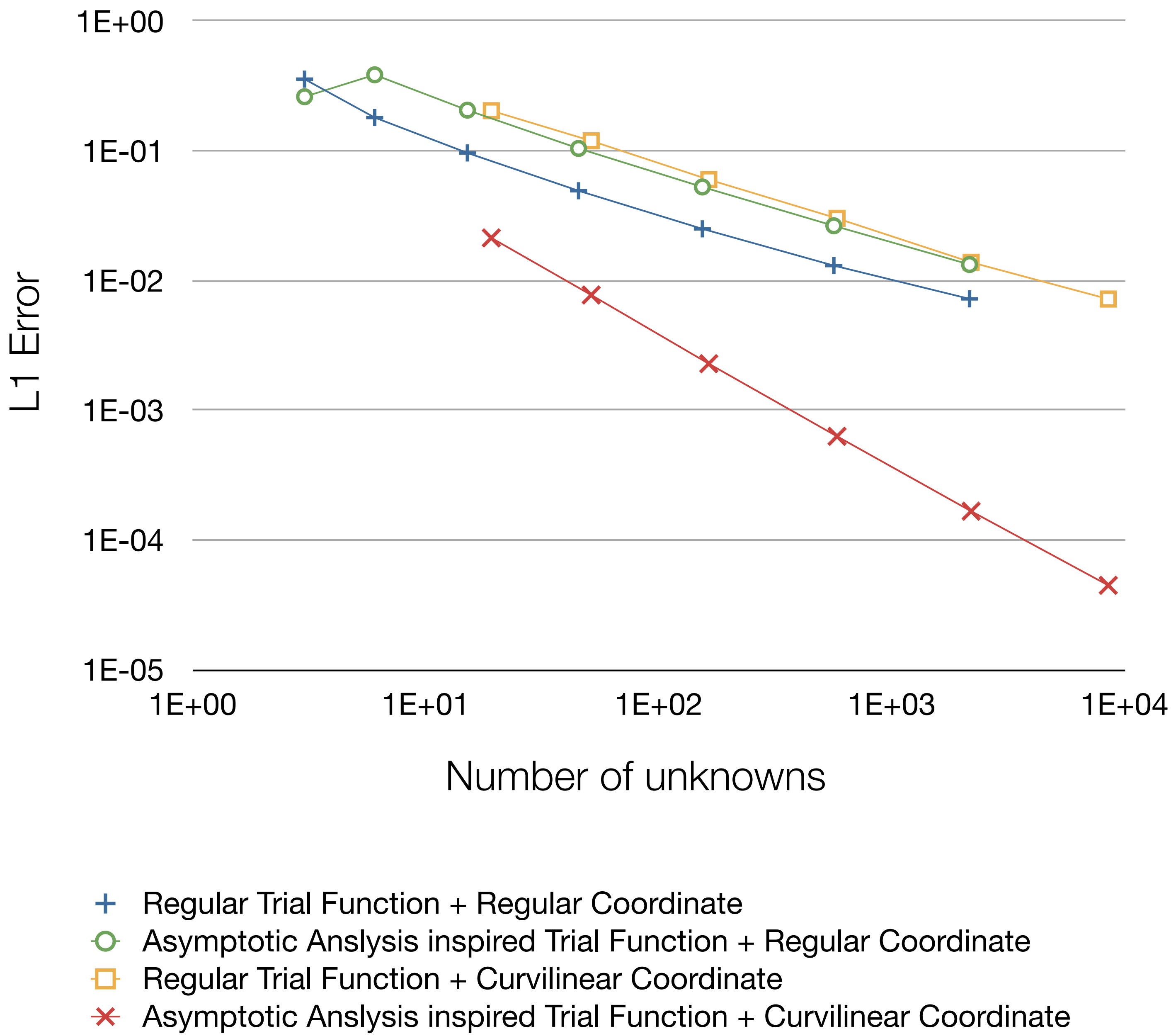


$$\int_{\partial\Omega_j \setminus \partial\Omega} \nu \cdot \frac{\sum_{i=1}^{N_{\text{node}}} c_i \nabla \left(\frac{\phi_i}{f_1(x) - f_2(x)} \right)}{\sqrt{1 + \left| \sum_{i=1}^{N_{\text{node}}} c_i \nabla \left(\frac{\phi_i}{f_1(x) - f_2(x)} \right) \right|^2}} ds + \int_{\partial\Omega_j \cap \partial\Omega} \cos\gamma \, ds$$

$$= \int_{\Omega_j} \sum_{i=1}^{N_{\text{node}}} c_i \left(\frac{\phi_i}{f_1(x) - f_2(x)} \right) dA \qquad \text{for } j = 1, 2, \ldots, N_{\text{node}}$$

Convergence Study

(Asymptotic Laplace-Young Equation in a Corner domain)



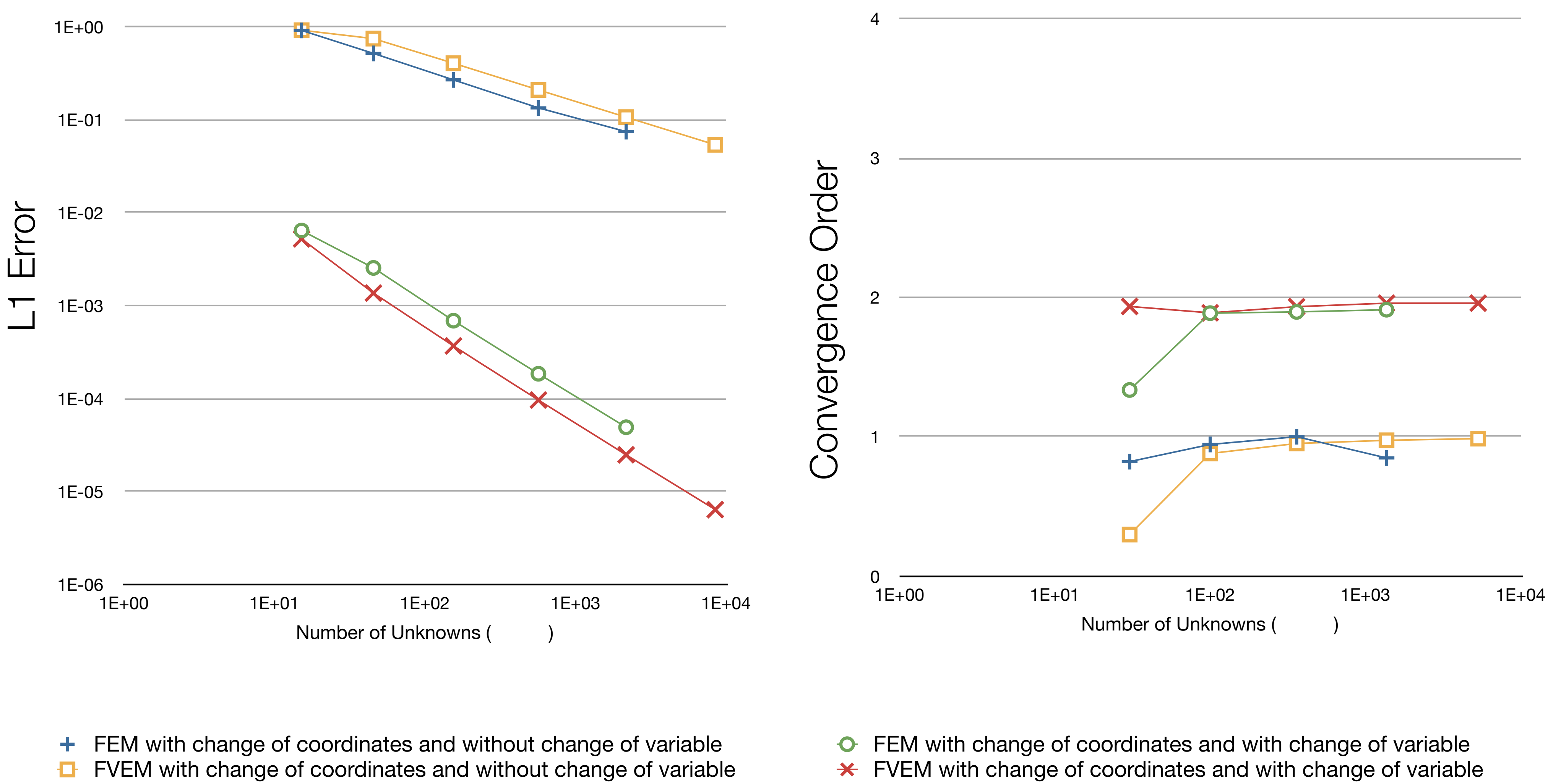
Convergence Study

(Asymptotic Laplace-Young Equation in a Corner domain)

	Without Change of Variable	With Change of Variable
Regular Coordinates	Linear	Linear
Curvilinear Coordinates	Linear	Quadratic

Convergence Study

(Asymptotic Laplace-Young Equation in a Circular Cusp domain)

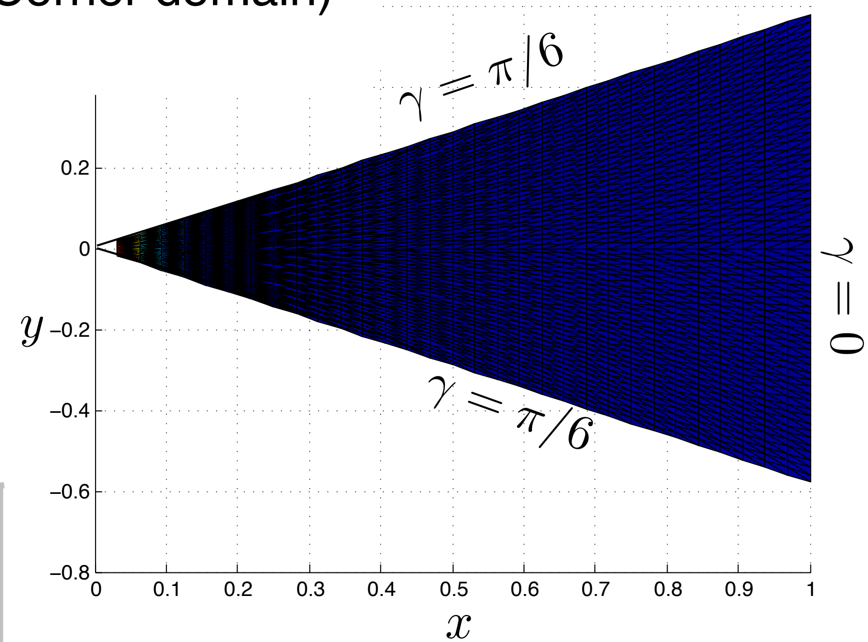


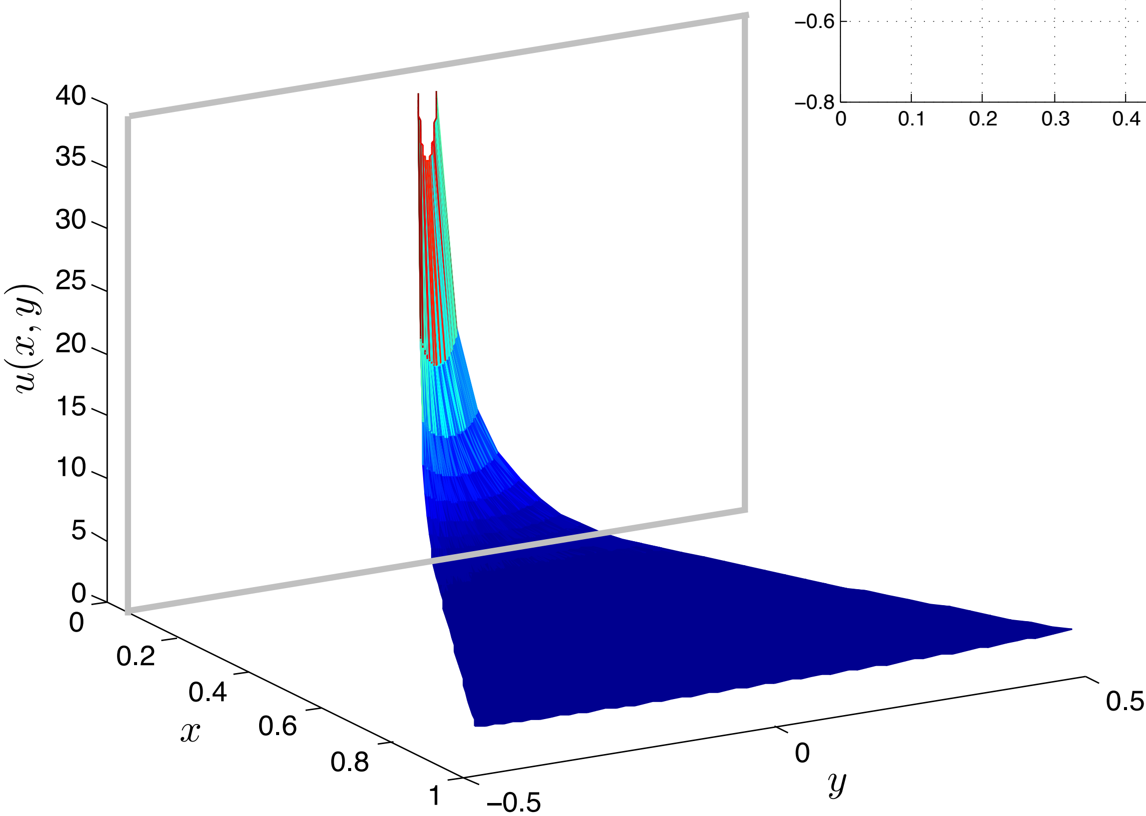
Numerical Experiment

(Laplace Young Equation in a Corner domain)

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

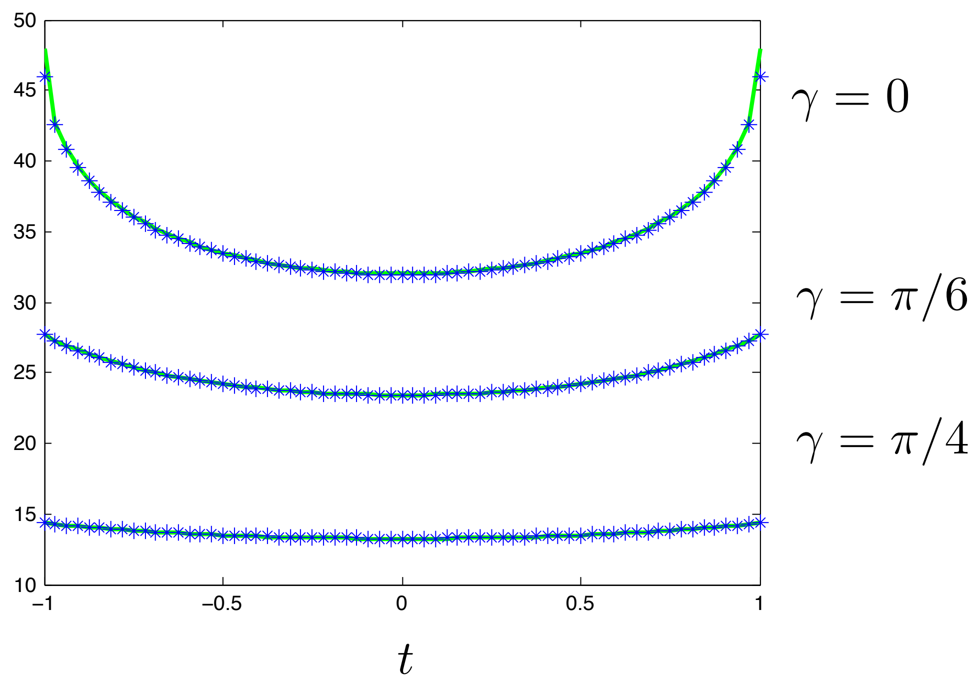
$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$





Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)

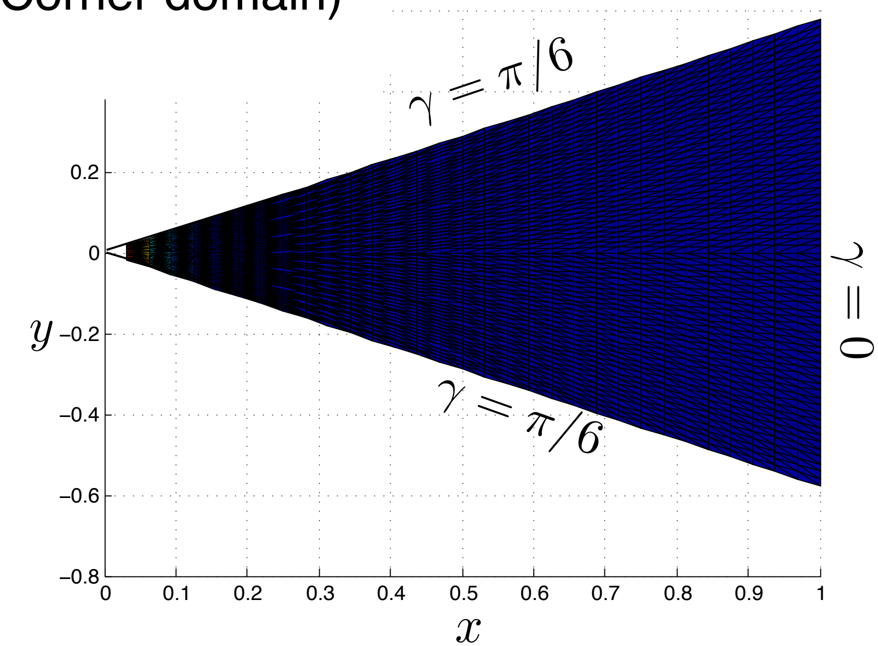


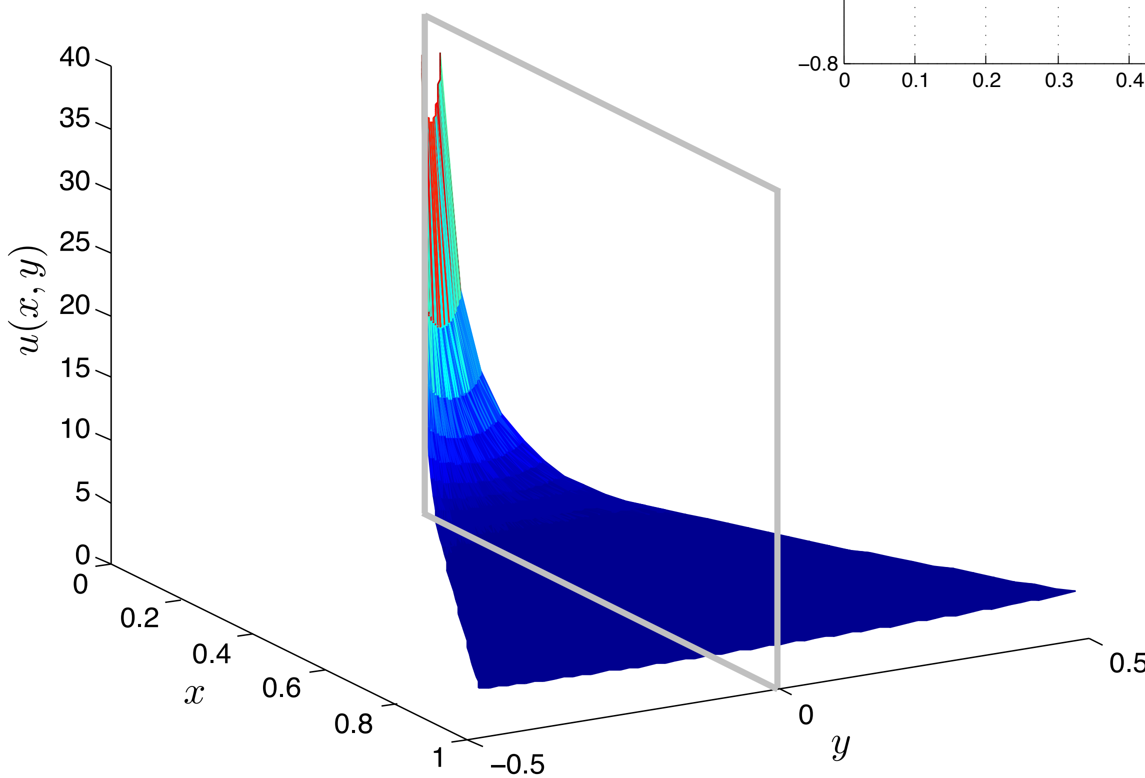
Numerical Experiment

(Laplace Young Equation in a Corner domain)

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

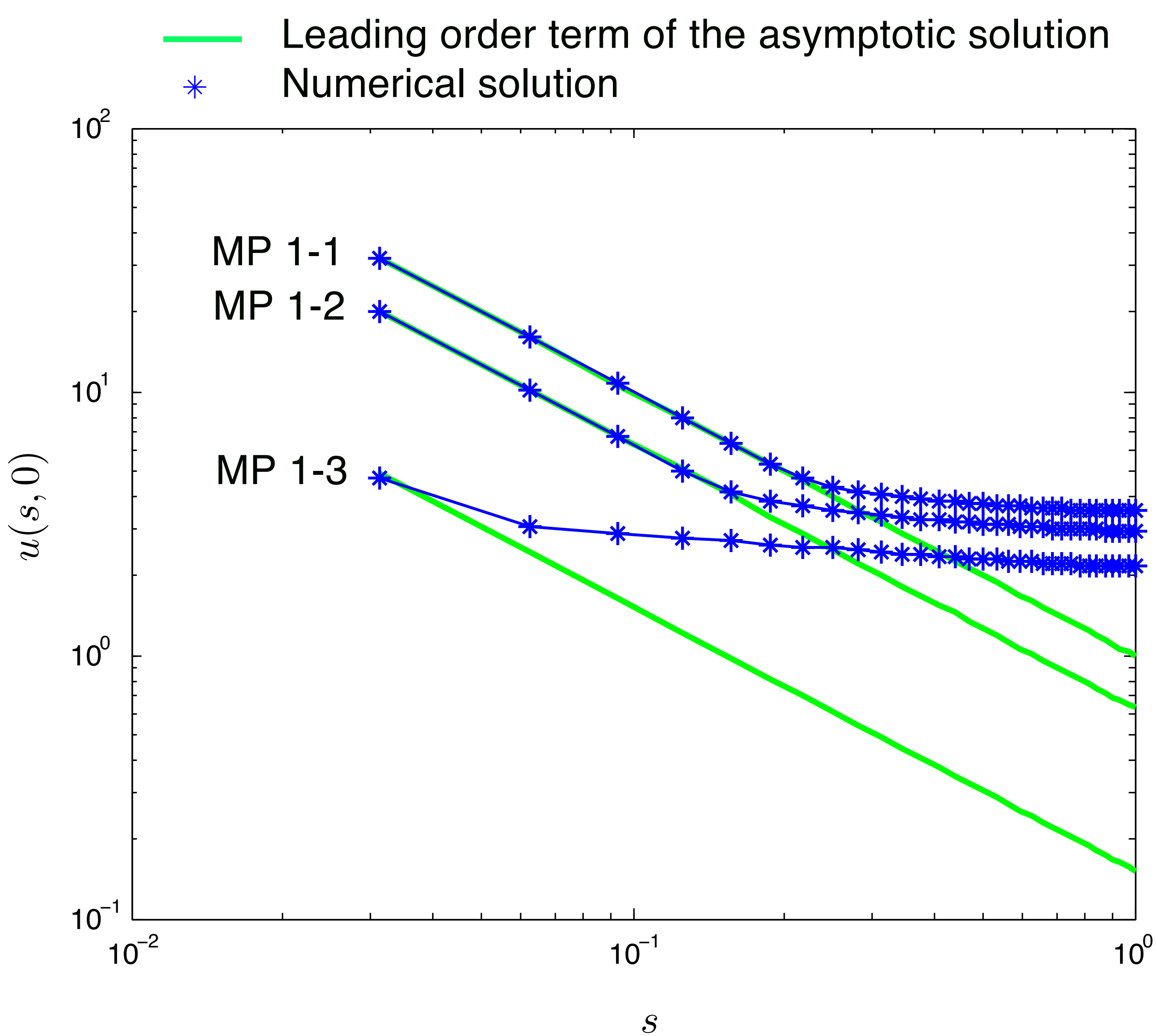
$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$





Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)

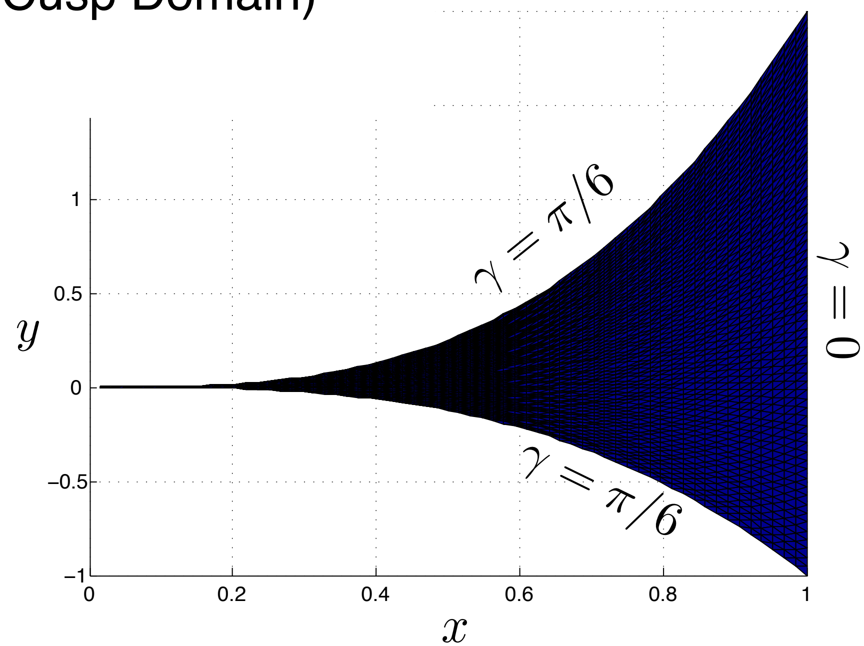


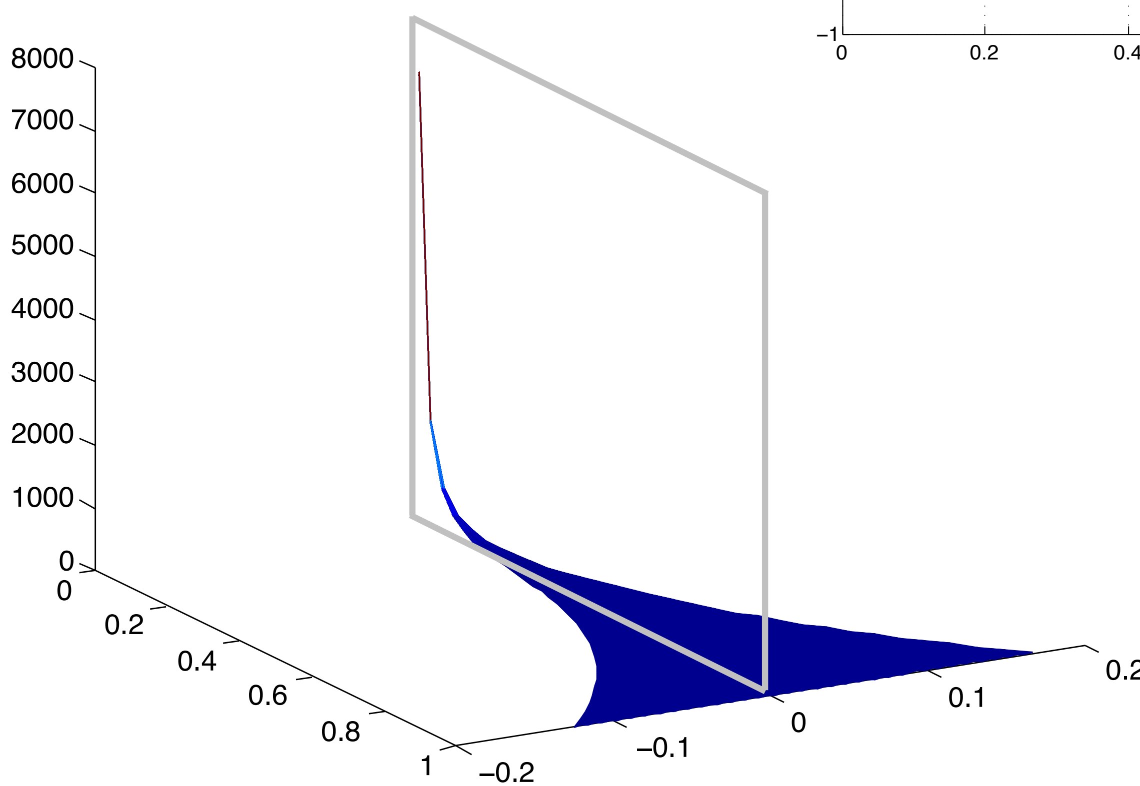
Numerical Experiment

(Laplace-Young Equation in a Cusp Domain)

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

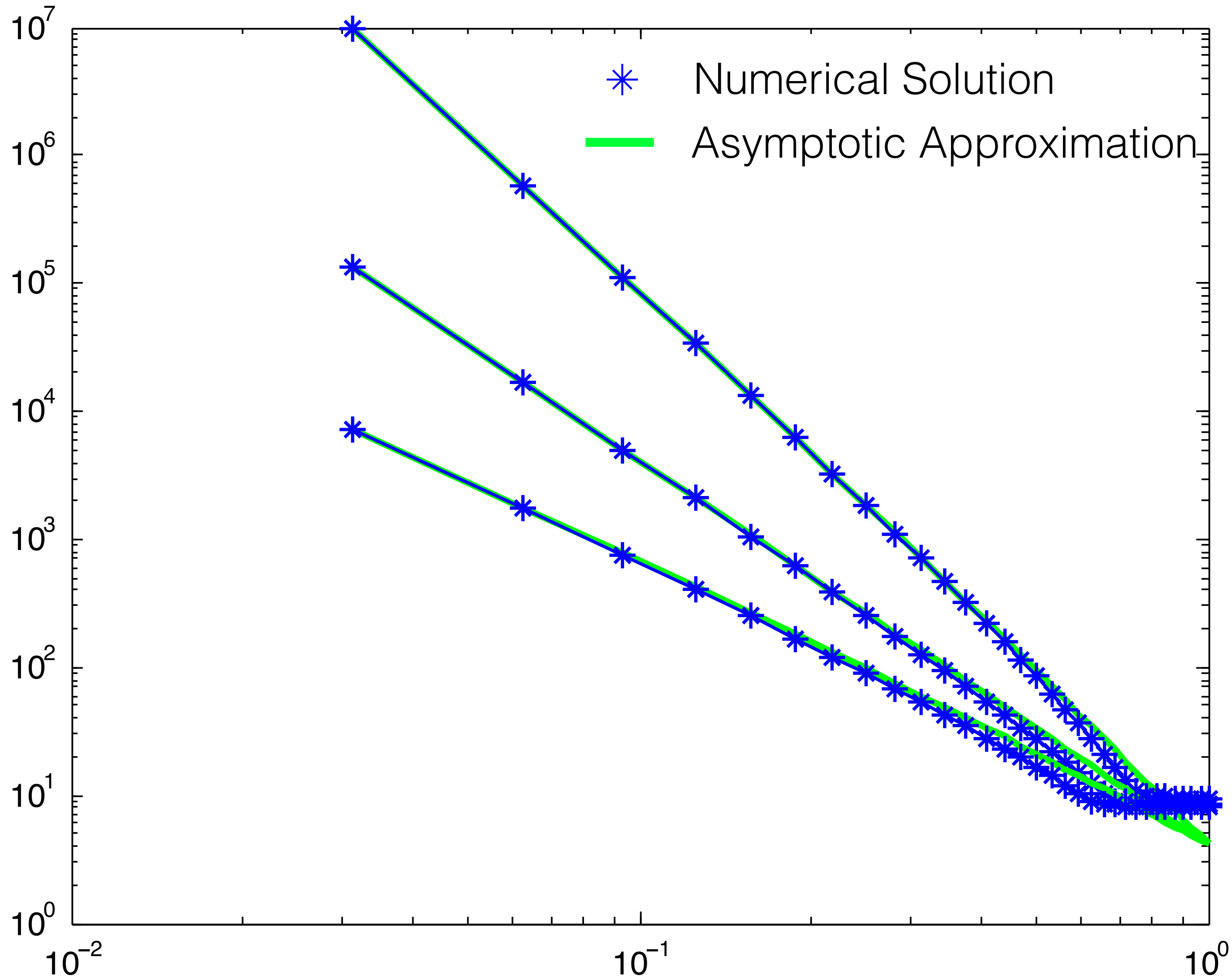
$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$





Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)



Asymptotic Analysis

Change of Variable + Curvilinear Coordinate System

Finite Volume Element method
or
Finite Element method

Numerical Approximation valid for the **entire** domain