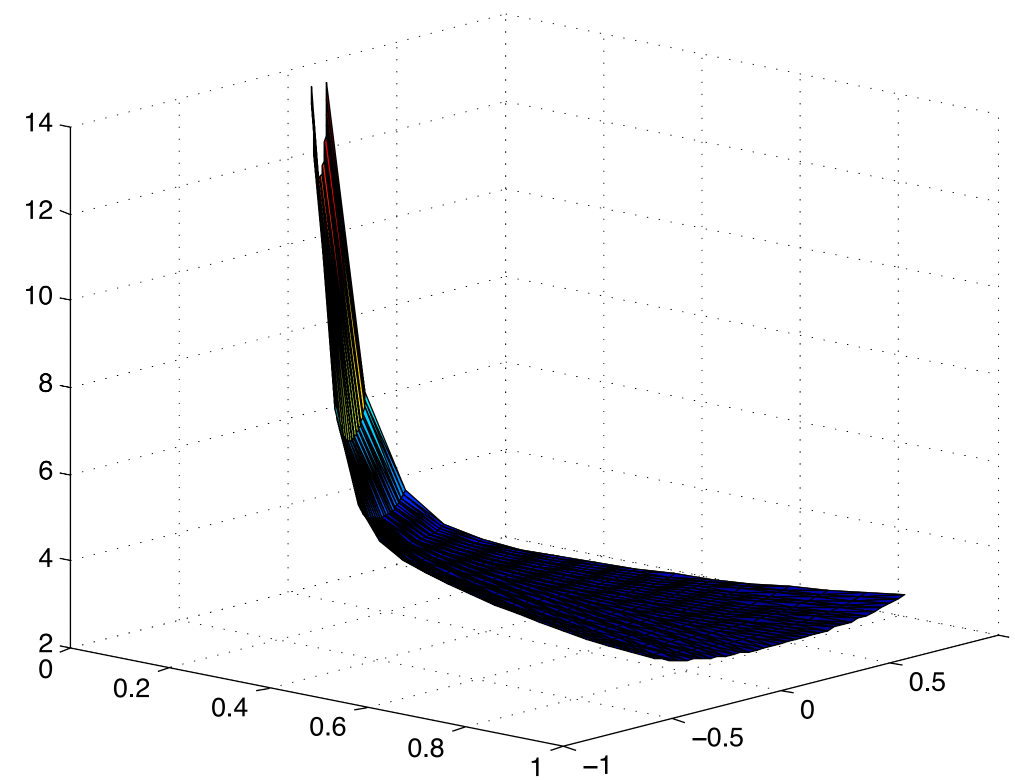
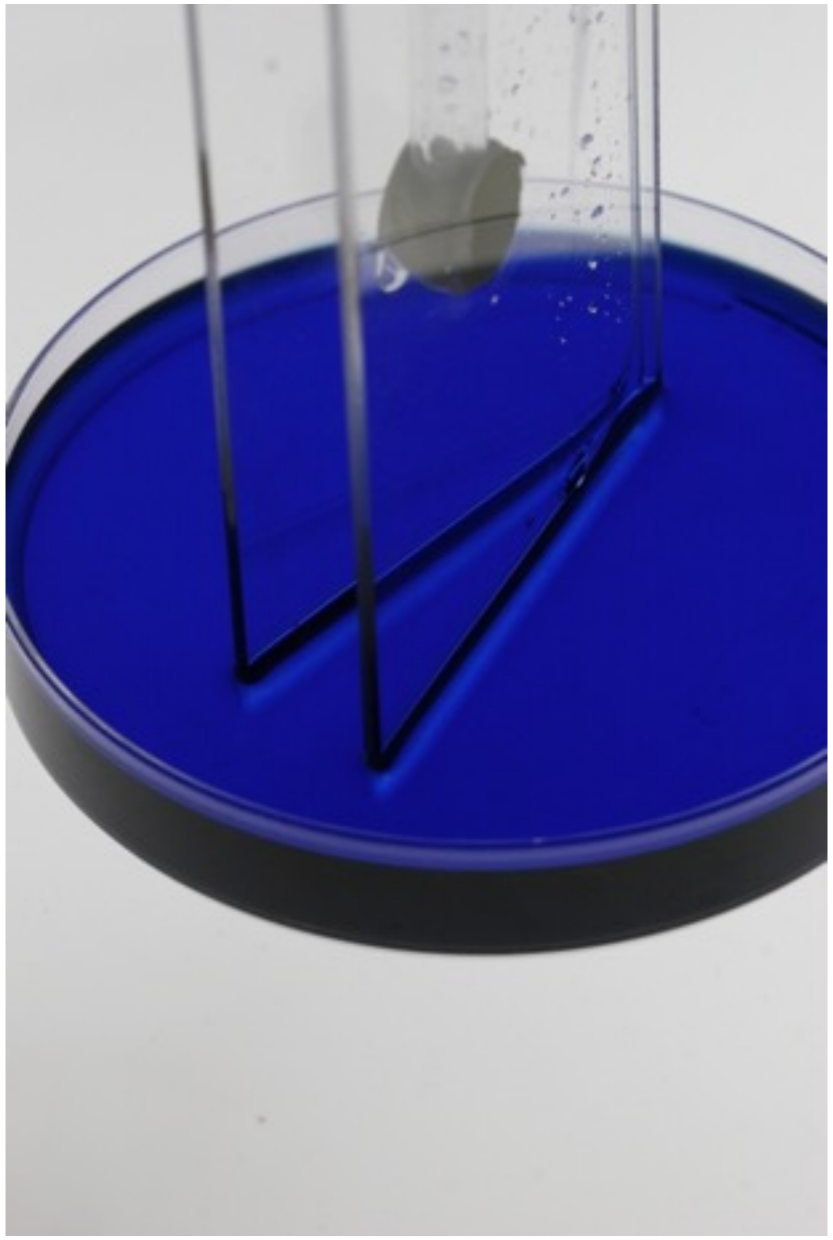
Global Approximation of Singular Capillary Surfaces: asymptotic analysis meets numerical analysis



Yasunori Aoki

Hans De Sterck

Liquid Surface at a corner



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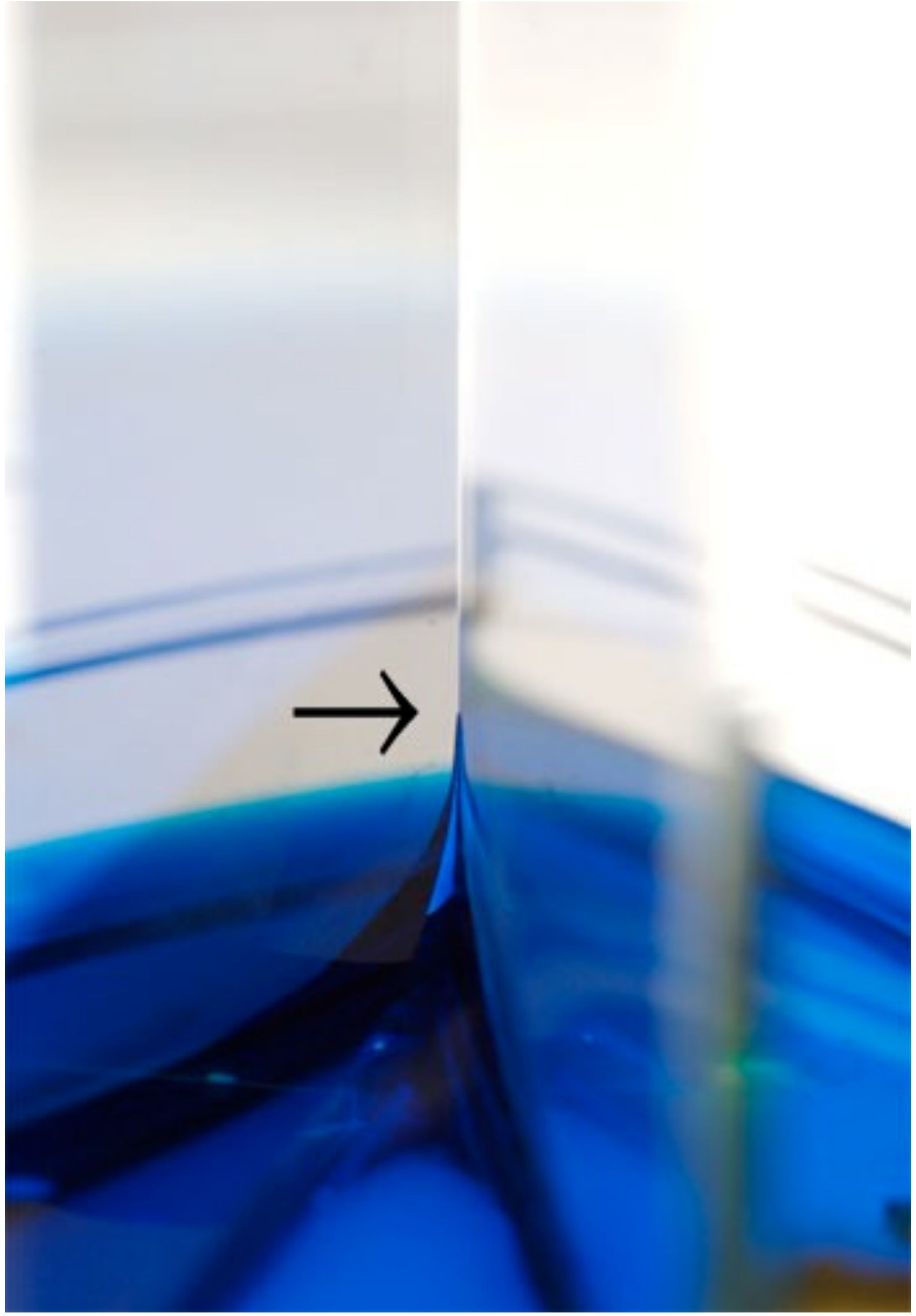
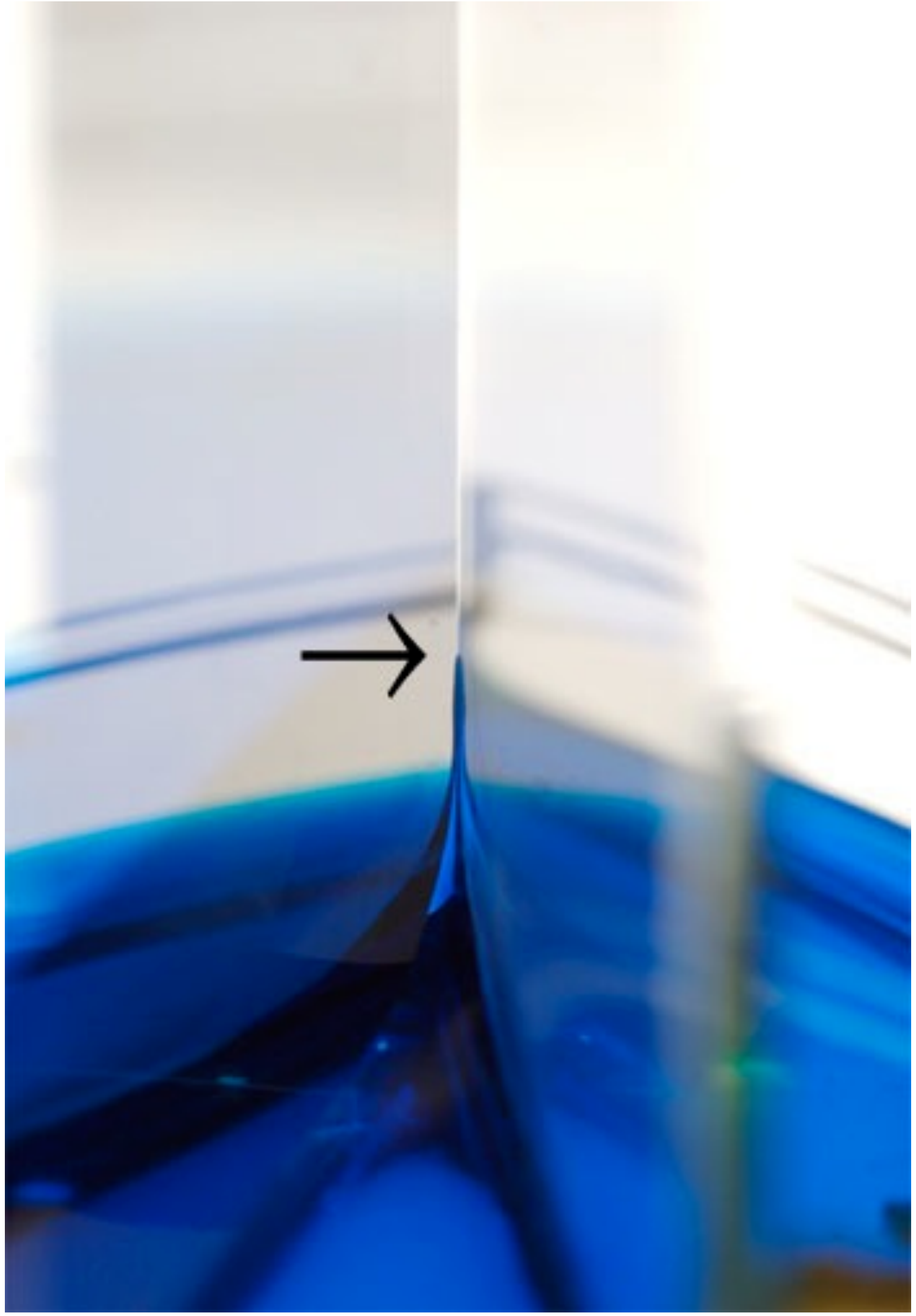
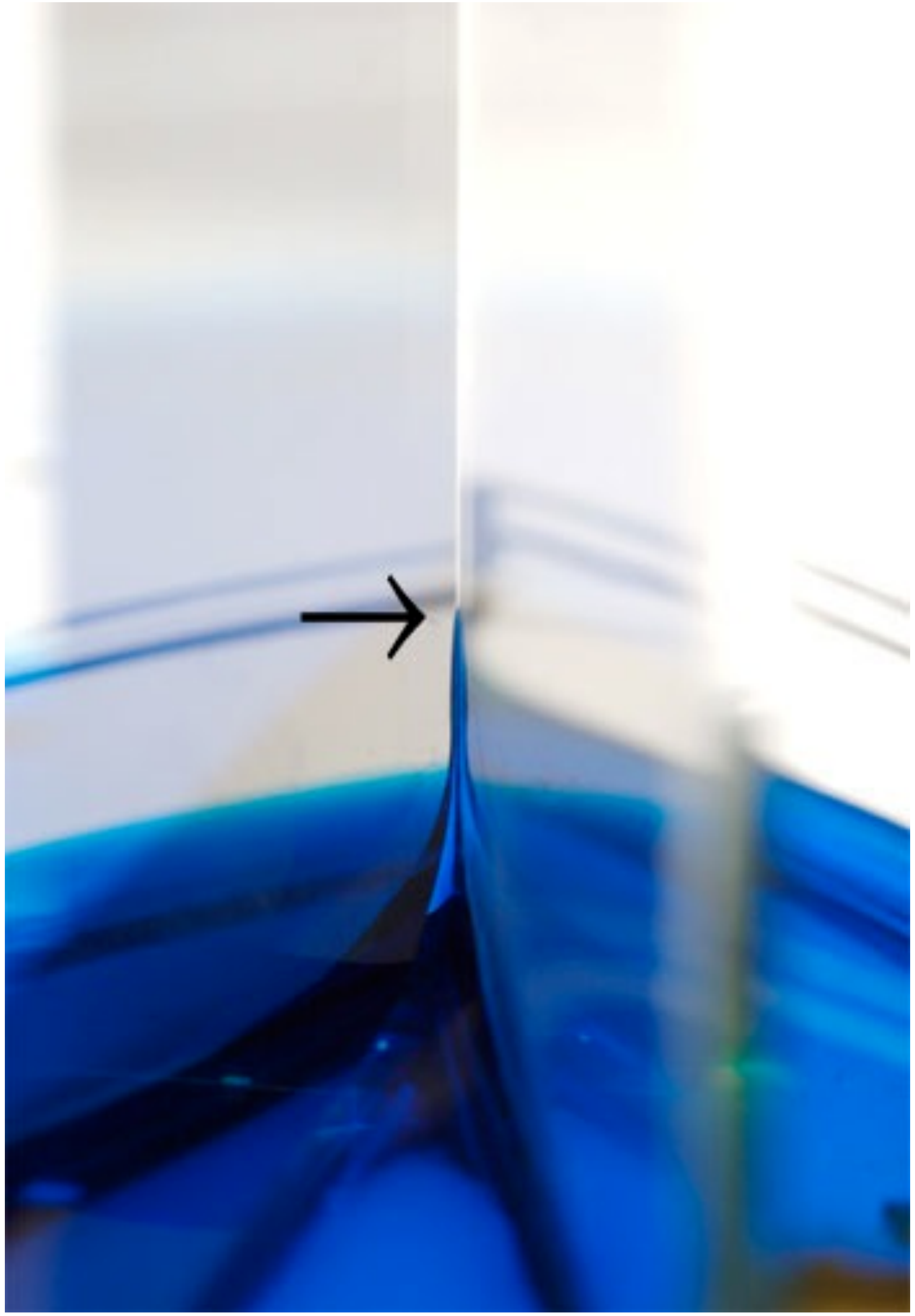


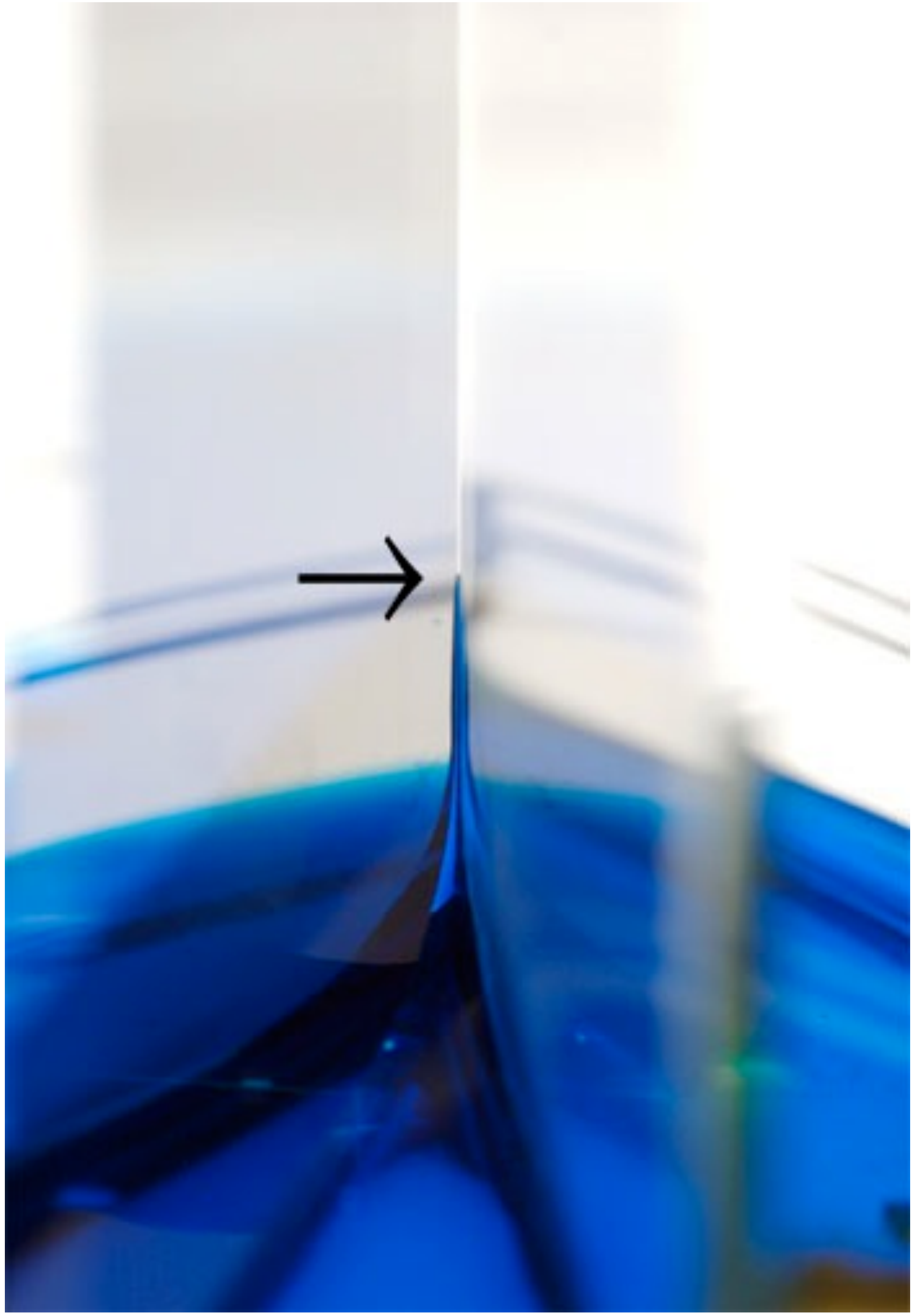
Liquid Surface at a corner



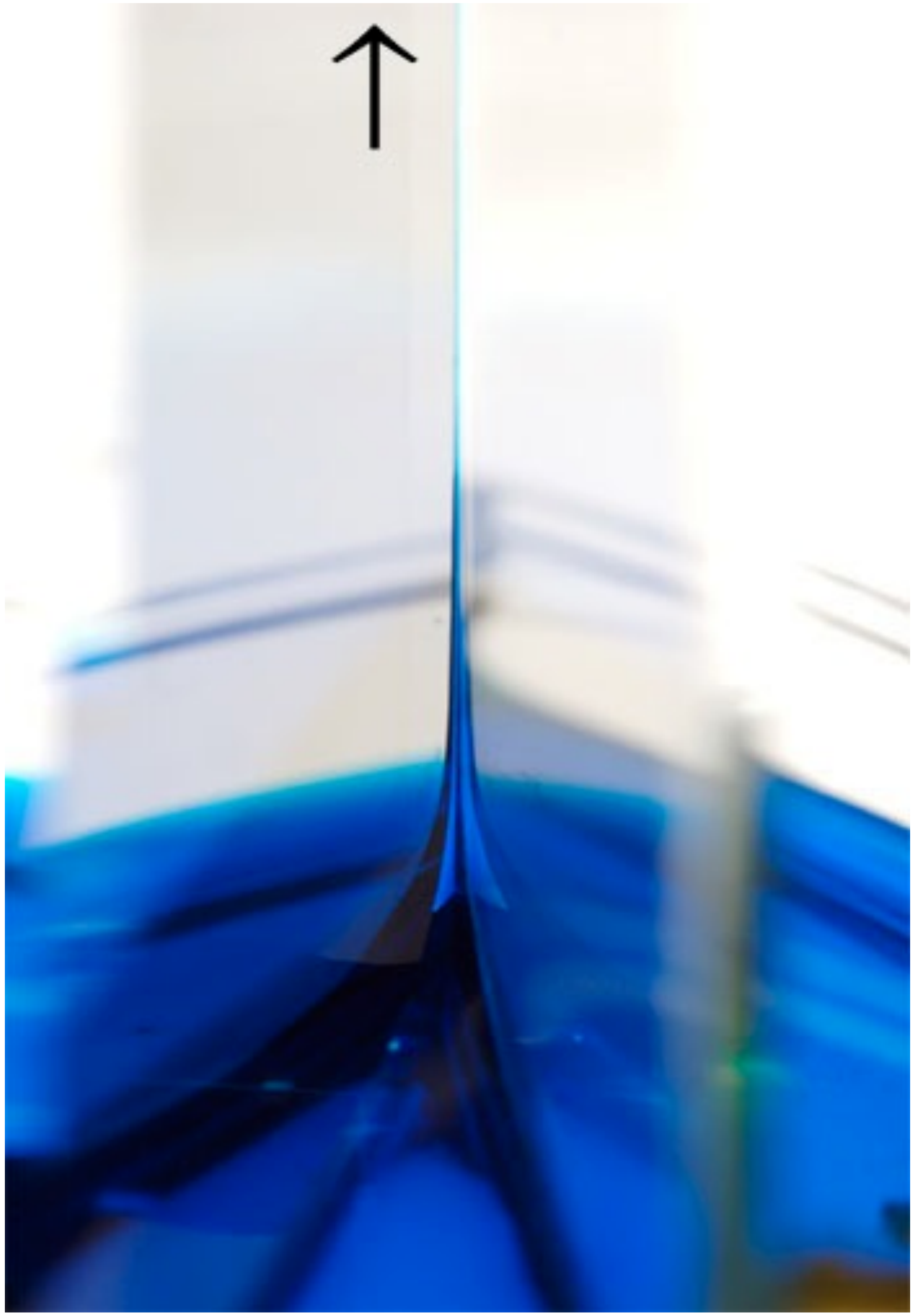
Liquid Surface at a corner



Liquid Surface at a corner



Liquid Surface at a corner



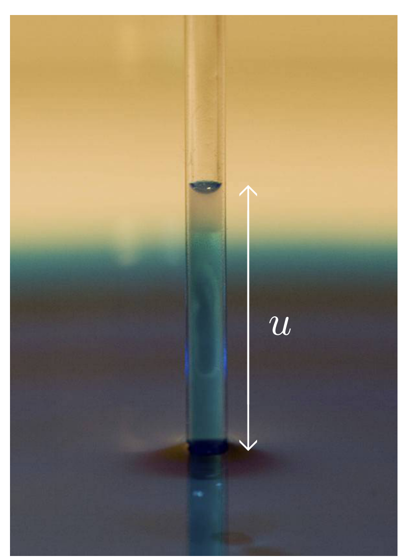
Height of the capillary surface depends discontinuously on the wedge angle.

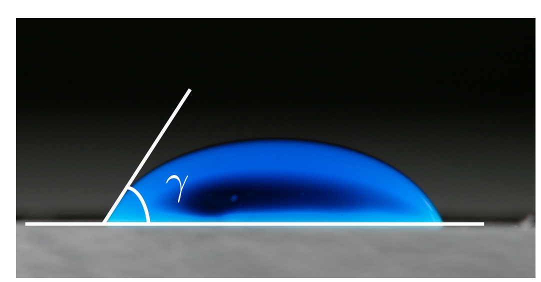
(Paul Concus and Robert Finn)

Laplace-Young Equation

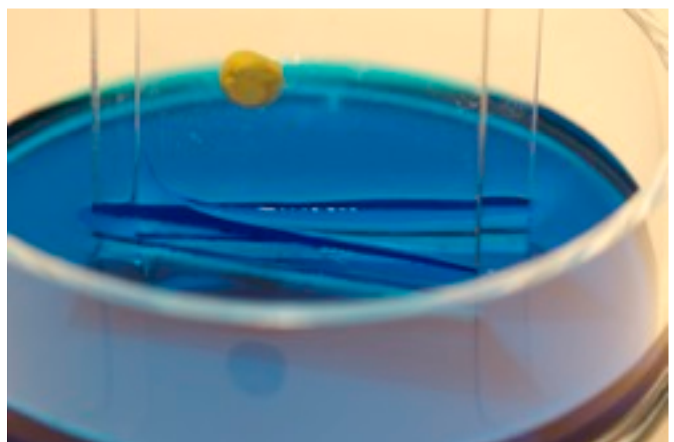
$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \quad u \qquad \text{in } \Omega$$

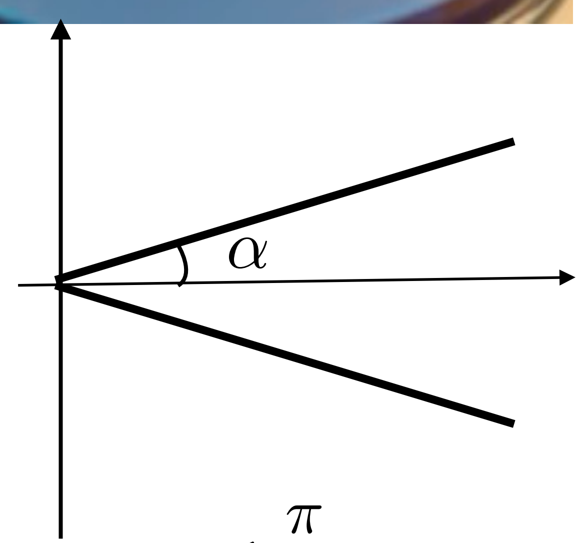
$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$





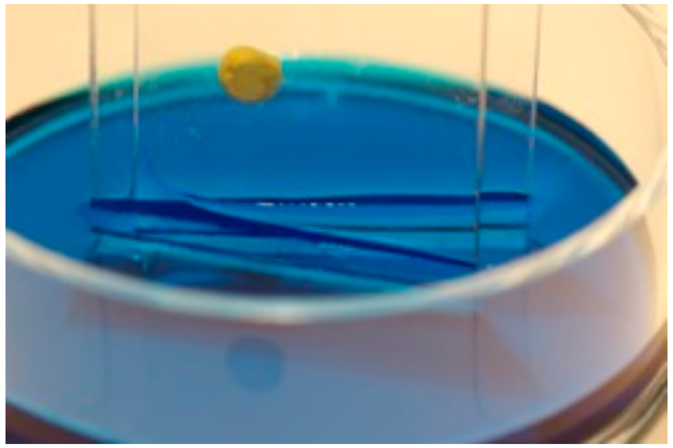
In a domain with a sharp corner, the solution becomes unbounded. (Concus and Finn)

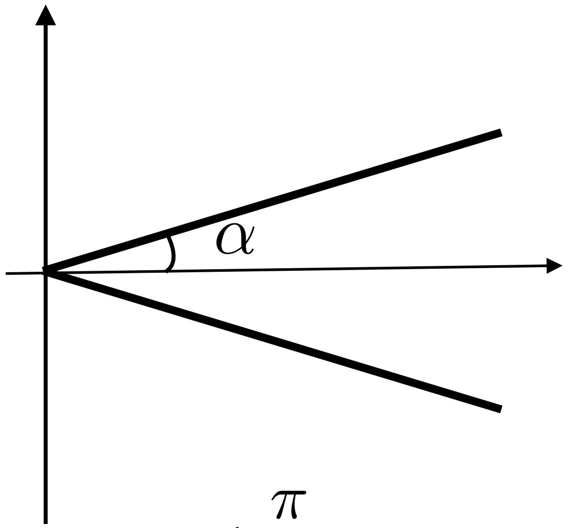




$$\alpha < \frac{\pi}{2} - \gamma$$

Wedge

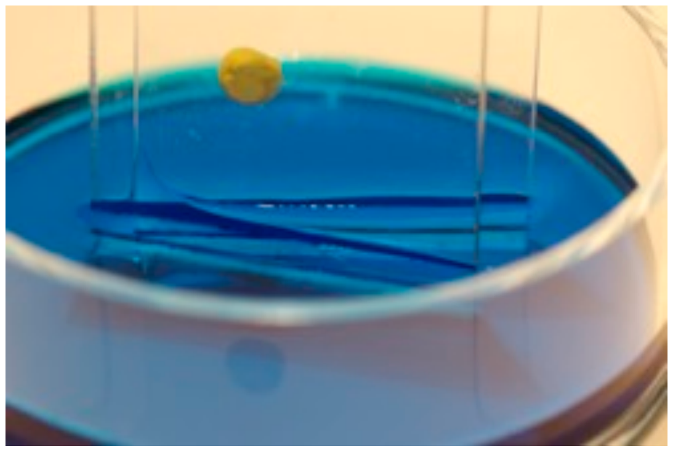


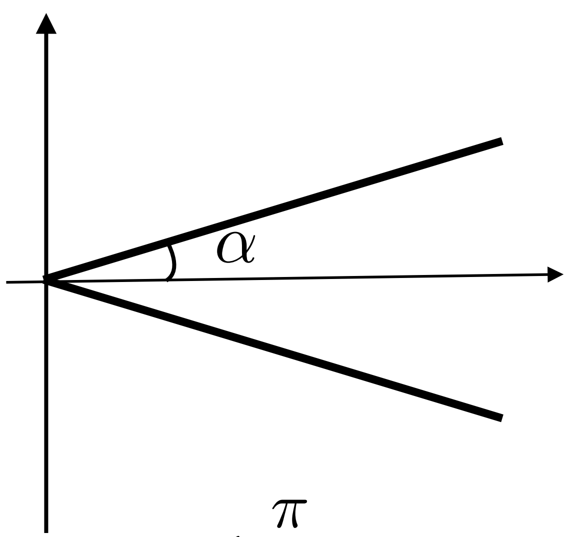


$$\alpha < \frac{\pi}{2} - \gamma$$

$$\Omega = \{(r, \theta) : 0 < r < R\,,\; -\alpha < \theta < \alpha\}$$

Wedge

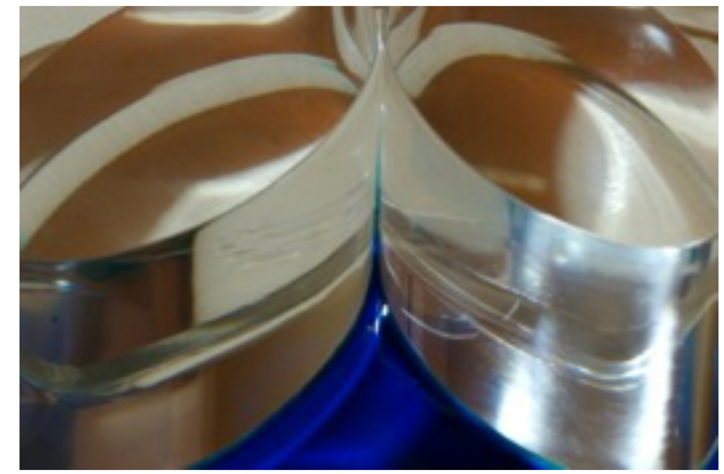


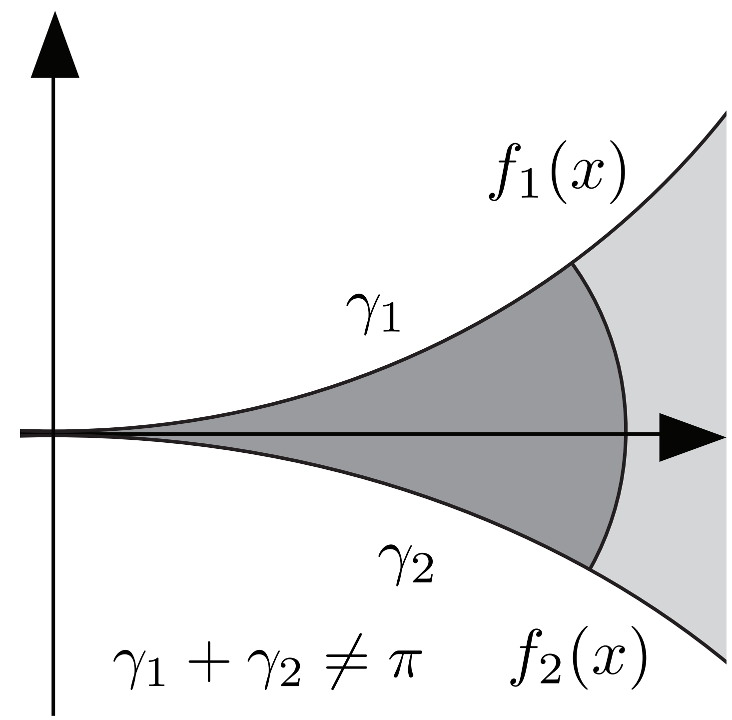


$$\alpha < \frac{\pi}{2} - \gamma$$

$$\Omega = \{(r,\theta) : 0 < r < R\,,\ -\alpha < \theta < \alpha\}$$

Cusp





$$\Omega = \{(x,y) : 0 < x,\ f_2(x) < y < f_1(x)\}$$

Asymptotic Analysis

Approximating Laplace-Young Equation

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

$$|\nabla u| >> 1$$

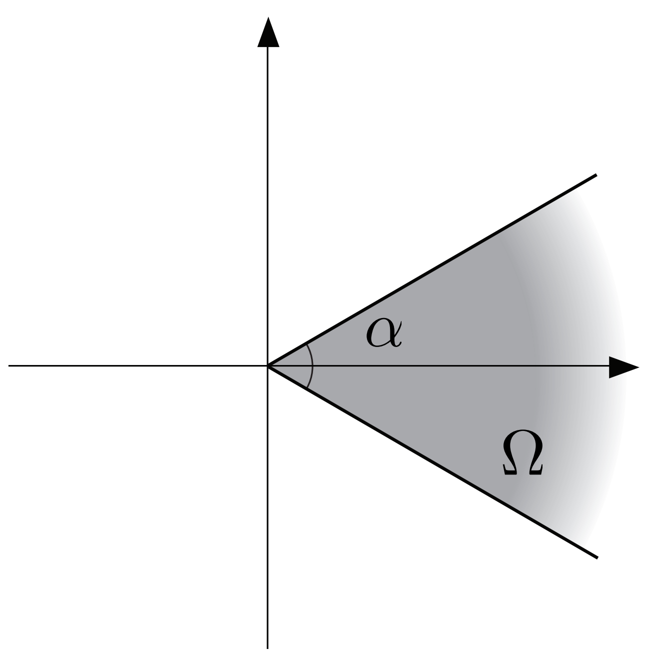
$$\nabla \cdot \frac{\nabla u}{|\nabla u|} \approx u$$

Asymptotic Laplace-Young Equation

(two lucky cases)

$$\nabla \cdot \frac{\nabla v}{|\nabla v|^2} = v \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla v}{|\nabla v|^2} = \cos\gamma \qquad \text{on } \partial\Omega$$



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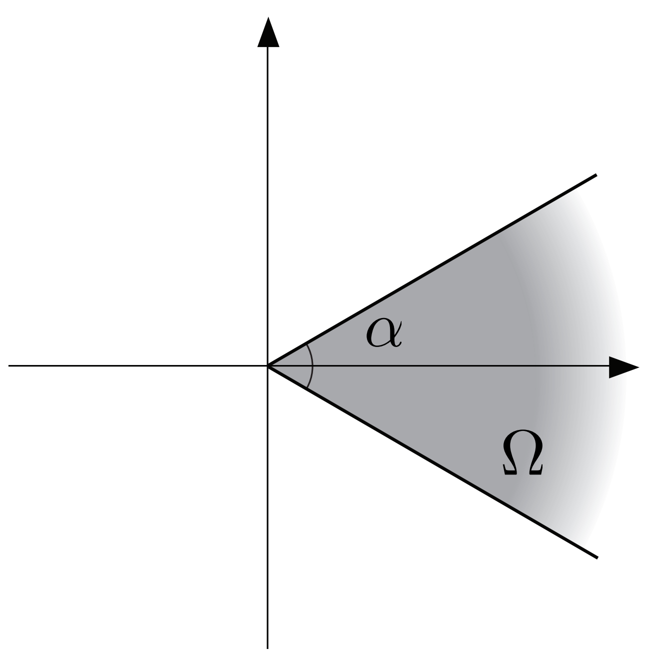

$$v(r,\theta) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{kr}$$

(Concus and Finn, Miersemann, and King et al.)

Asymptotic Laplace-Young Equation

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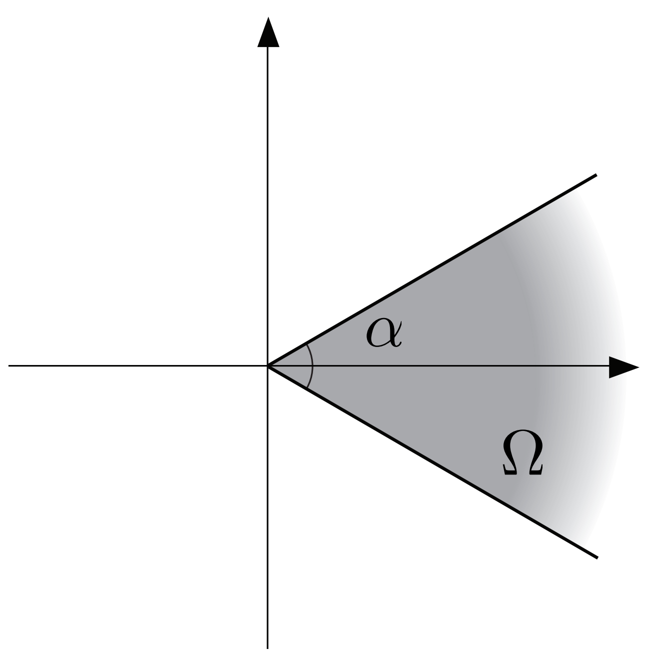


$$v(r,\theta) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{kr}$$

(Concus and Finn, Miersemann, and King et al.)

$$u(r,\theta) = v(r,\theta) + O(r^3) \qquad \text{as } r \to 0$$

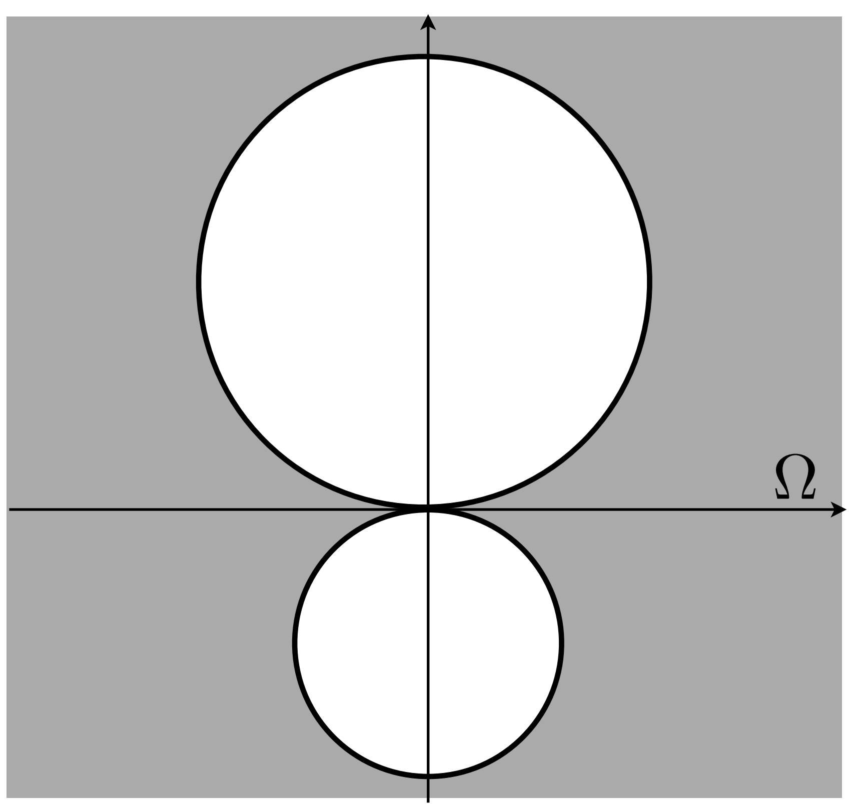
(Miersemann)

Asymptotic Laplace-Young Equation

(two lucky cases)

$$\nabla \cdot \frac{\nabla v}{|\nabla v|^2} = v \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla v}{|\nabla v|^2} = \cos\gamma \qquad \text{on } \partial\Omega$$



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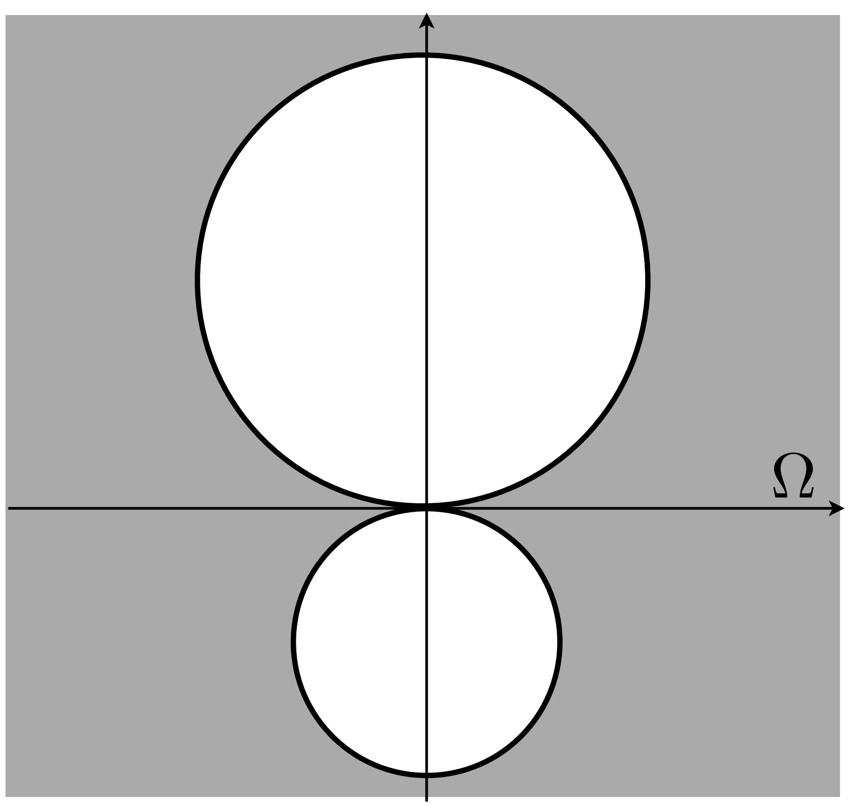

$$v(p,q) = Ap^2 - 2\sqrt{1 - A^2(q-q_0)^2}\; p - A(q-q_0)^2 + Aq_0^2$$

Asymptotic Laplace-Young Equation

(two lucky cases)

$$\nabla \cdot \frac{\nabla v}{|\nabla v|^2} = v \qquad \text{in } \Omega$$

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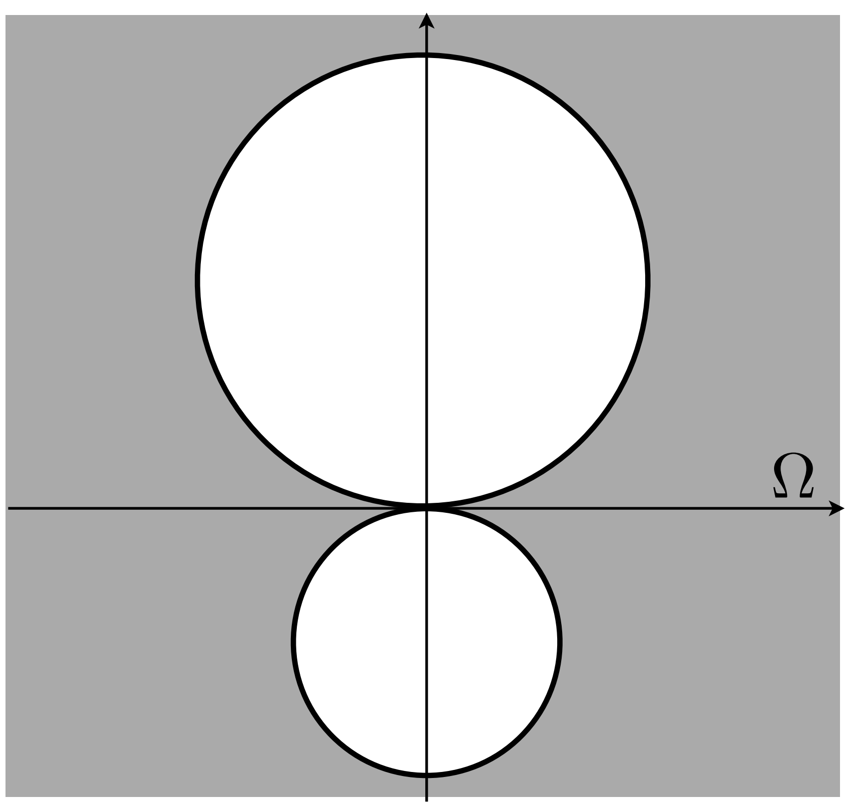


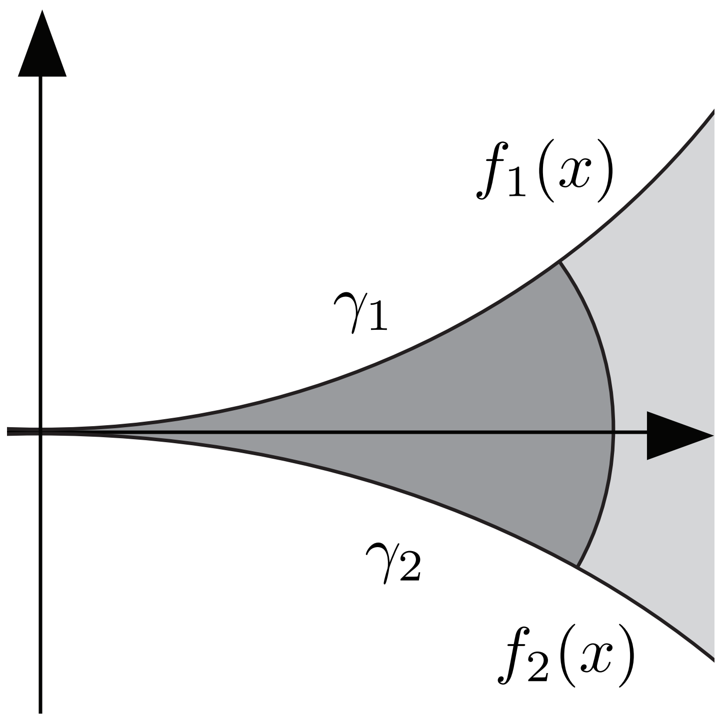
$$v(p,q) = Ap^2 - 2\sqrt{1 - A^2(q-q_0)^2}\; p - A(q-q_0)^2 + Aq_0^2$$

$$u(p,q) = v(p,q) + O(p^{-5}) \qquad \text{as } p \to \infty$$

(Aoki M.Math thesis)

Asymptotic Analysis

(general cases)



$\gamma_1 + \gamma_2 \neq \pi$

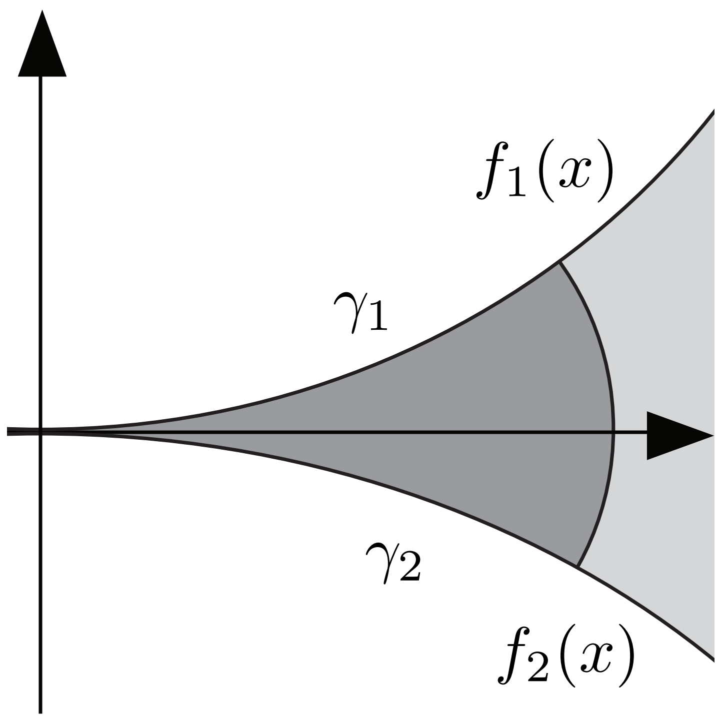
$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$

after some calculation ...

Asymptotic Analysis

(general cases)



$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$

after some calculation ...

$\gamma_1 + \gamma_2 \neq \pi$

$$u(x,y) = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + O\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right) \qquad \text{as } x \to 0^+$$

* some restrictions on f_1 and f_2 apply

(Aoki and Siegel)

Asymptotic Analysis

(summary)

Corner: $$u(r, \theta) \approx \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{kr}$$

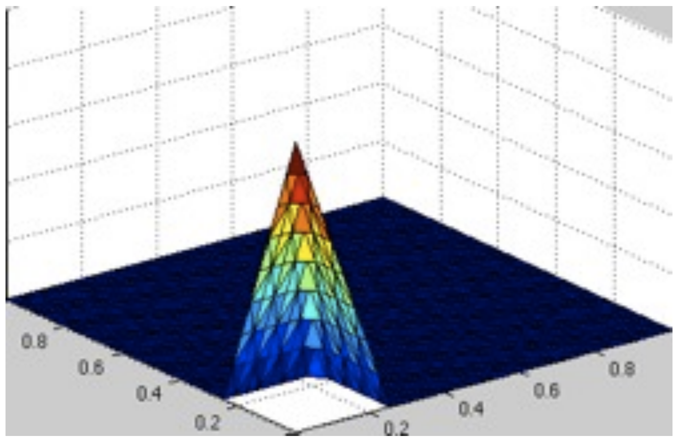
Cusp: $$u(x, y) \approx \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)}$$

Approximation only accurate near the singularity!

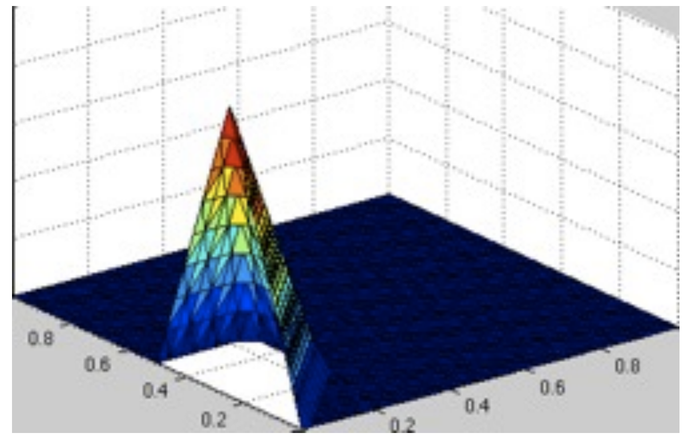
Numerical Analysis

Finite Element Approximation

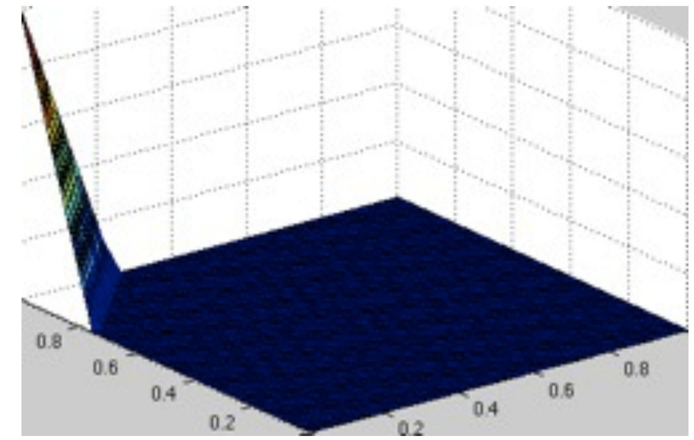
Basis Functions



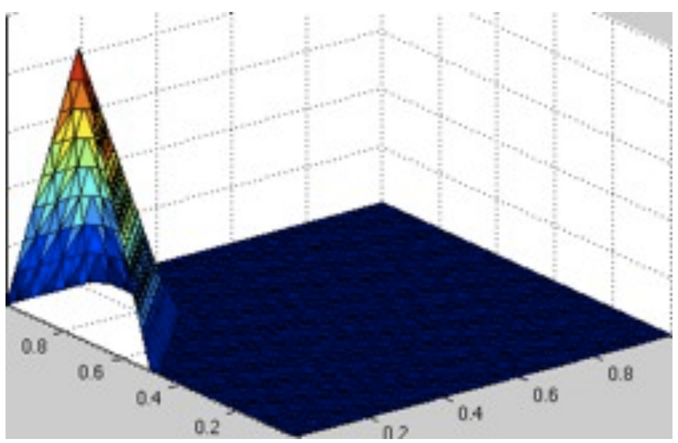
ϕ_1



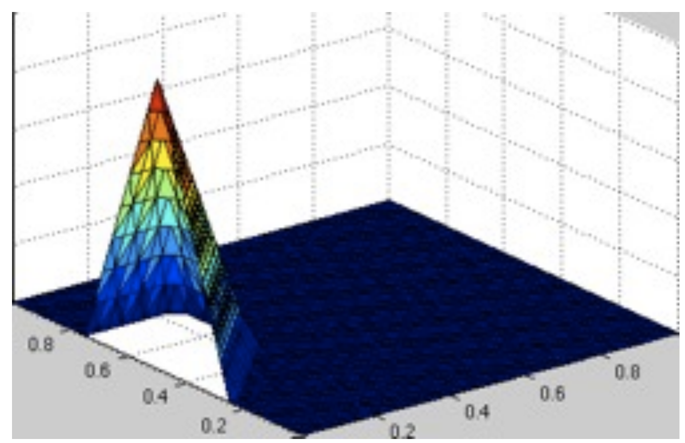
ϕ_2



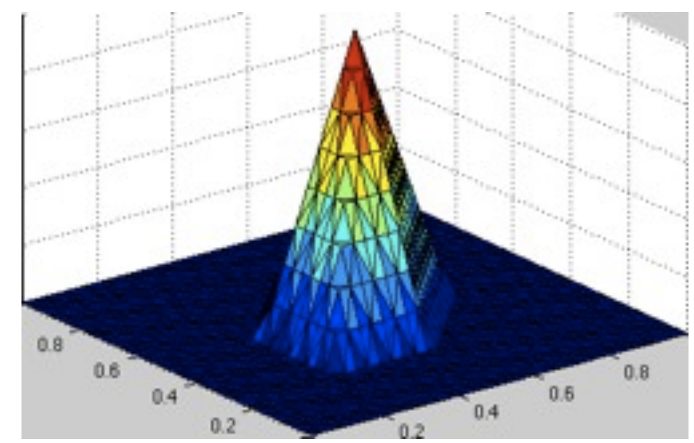
ϕ_5



ϕ_4



ϕ_3



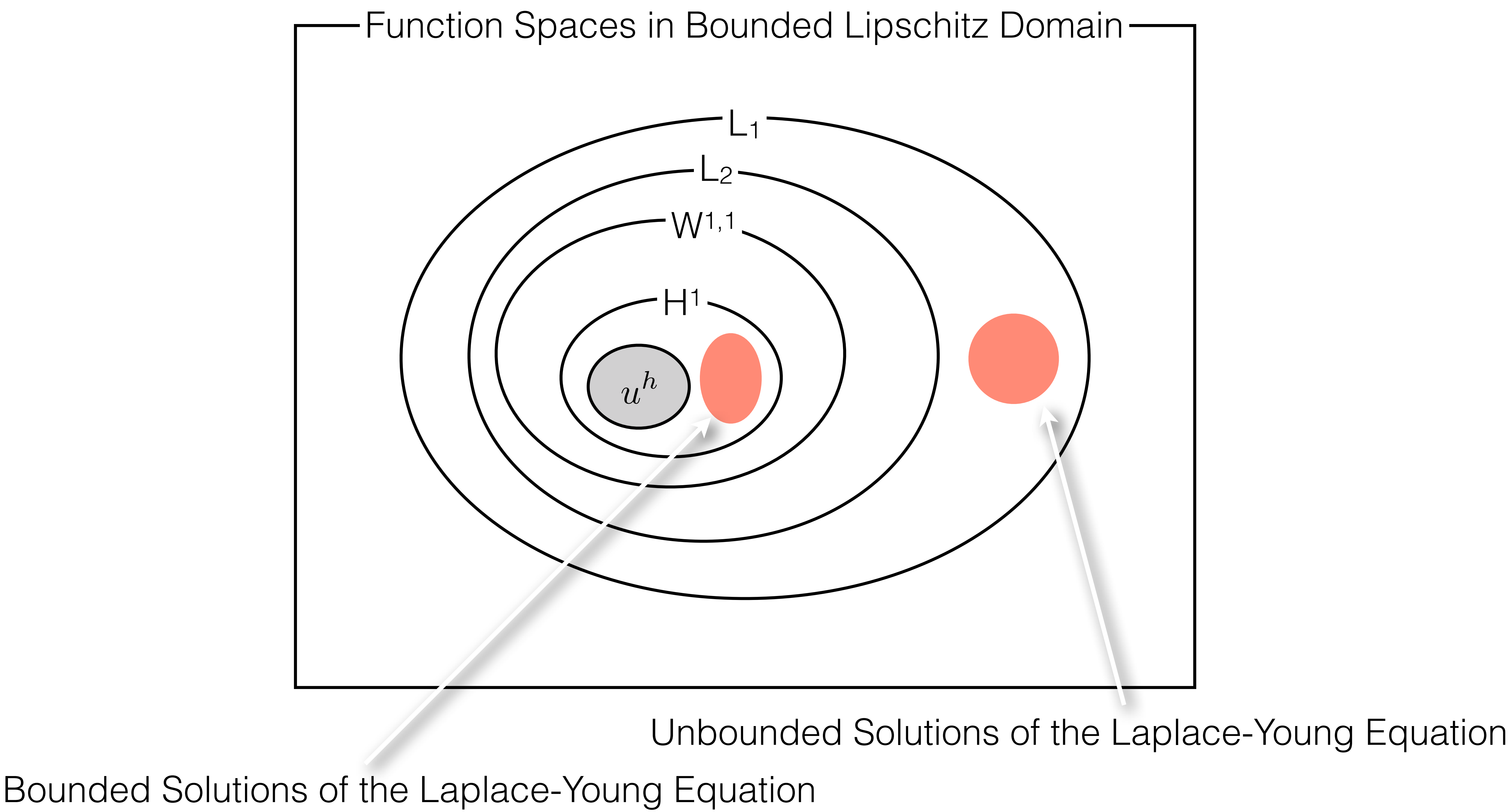
ϕ_i

Standard Trial Function Expansion

$$u \approx u^h := \sum_{i=1}^{N_{\text{node}}} c_i \phi_i$$

Finite Element Approximation

(Function Spaces)

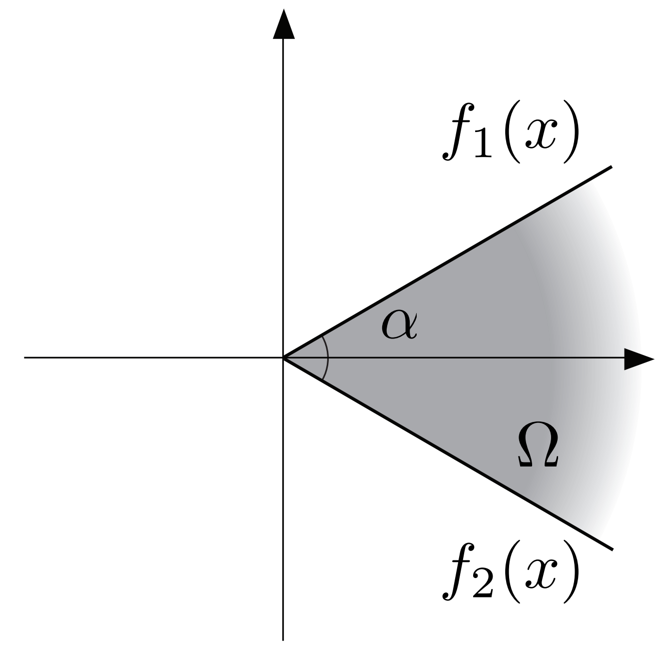


Asymptotic Analysis

Corner:

$$u = O\left(\frac{1}{r}\right) \; = O\left(\frac{1}{f_1(x) - f_2(x)}\right)$$

$$= \frac{O(1)}{f_1(x) - f_2(x)}$$



Asymptotic Analysis

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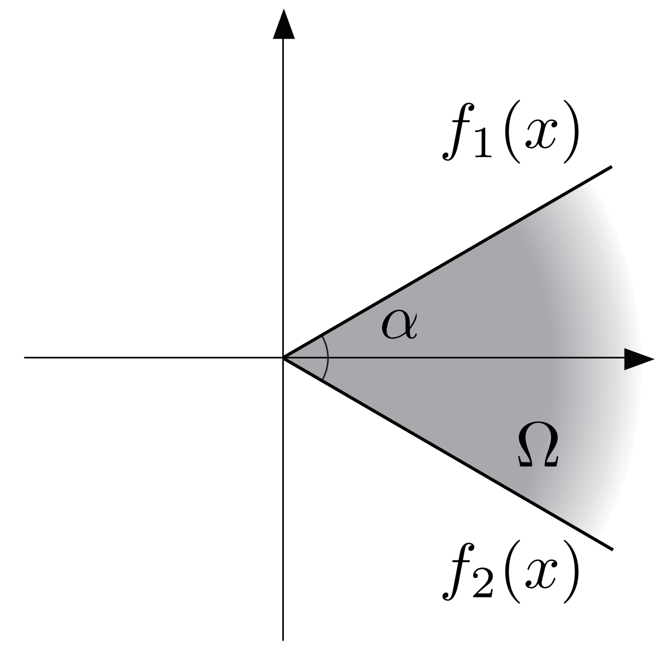
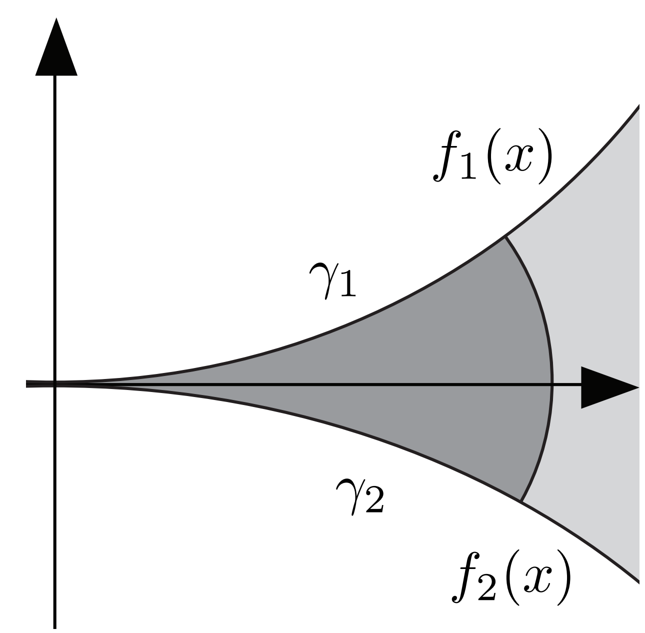


Cusp:

$$u = O\left(\frac{1}{f_1(x) - f_2(x)}\right)$$

$$= \frac{O(1)}{f_1(x) - f_2(x)}$$



Asymptotic Analysis

$$u = \frac{O(1)}{f_1(x) - f_2(x)}$$

Asymptotic Analysis

$$u = \frac{O(1)}{f_1(x) - f_2(x)}$$

Change of Variable

Bounded function

$$u = \frac{v}{f_1(x) - f_2(x)}$$

Unbounded function

Asymptotic Analysis

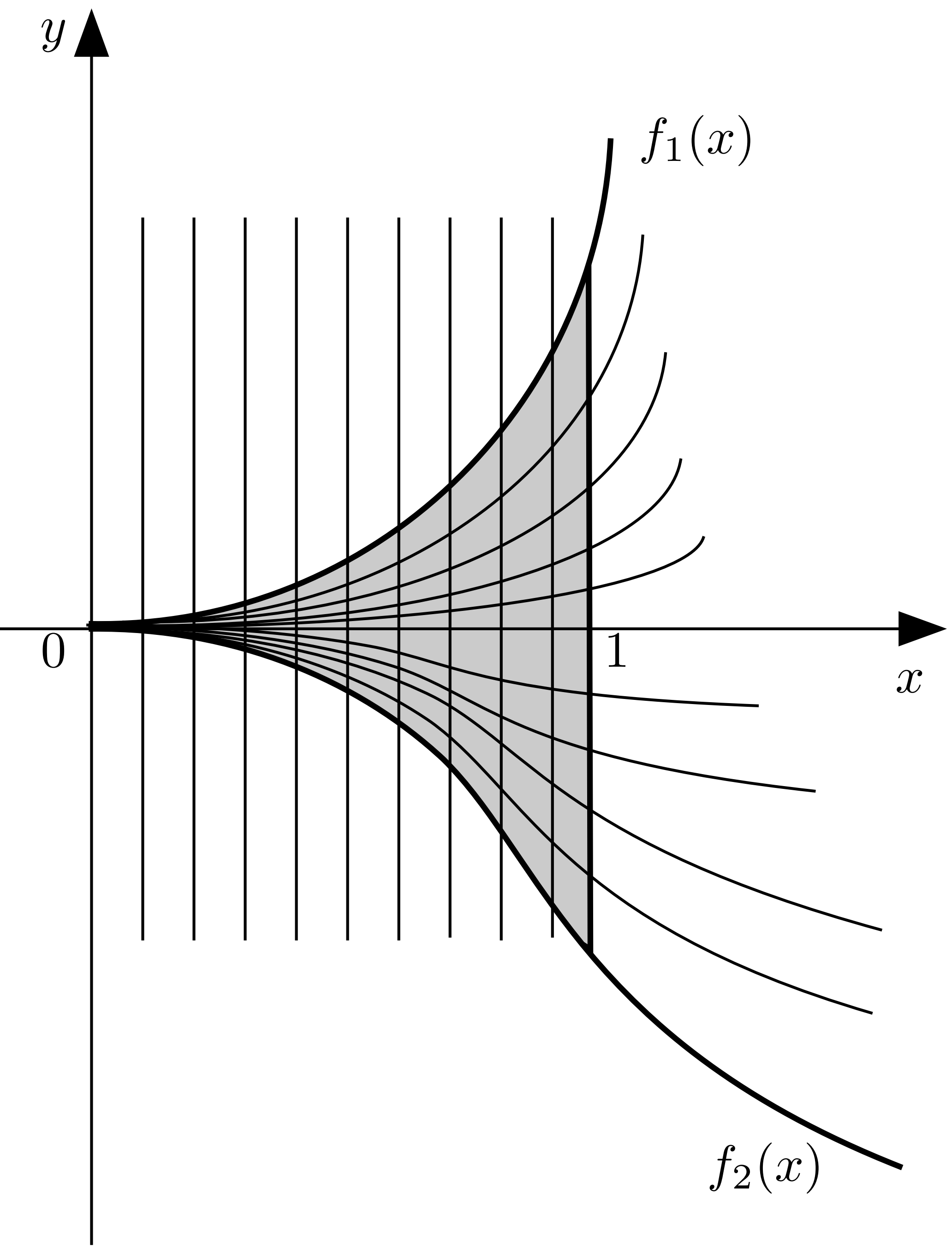
$$u = \frac{O(1)}{f_1(x) - f_2(x)}$$

Change of Variable

$$u = \frac{v}{f_1(x) - f_2(x)}$$

$$u \approx \frac{\sum_{i=1}^{N_{\text{node}}} c_i \phi_i}{f_1(x) - f_2(x)}$$

Change of Coordinates

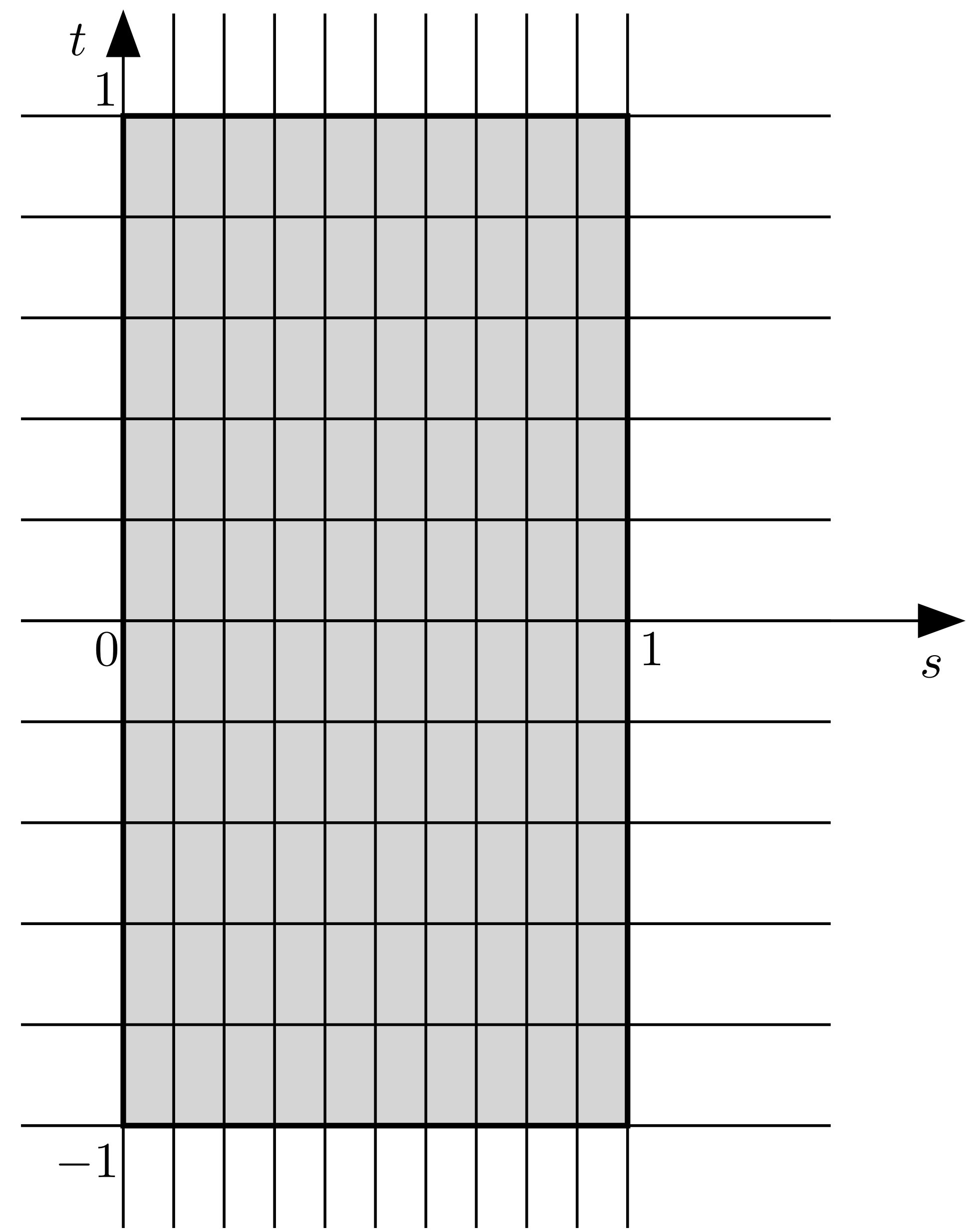


$$t = \frac{2y - (f_1 + f_2)}{f_1 - f_2}$$

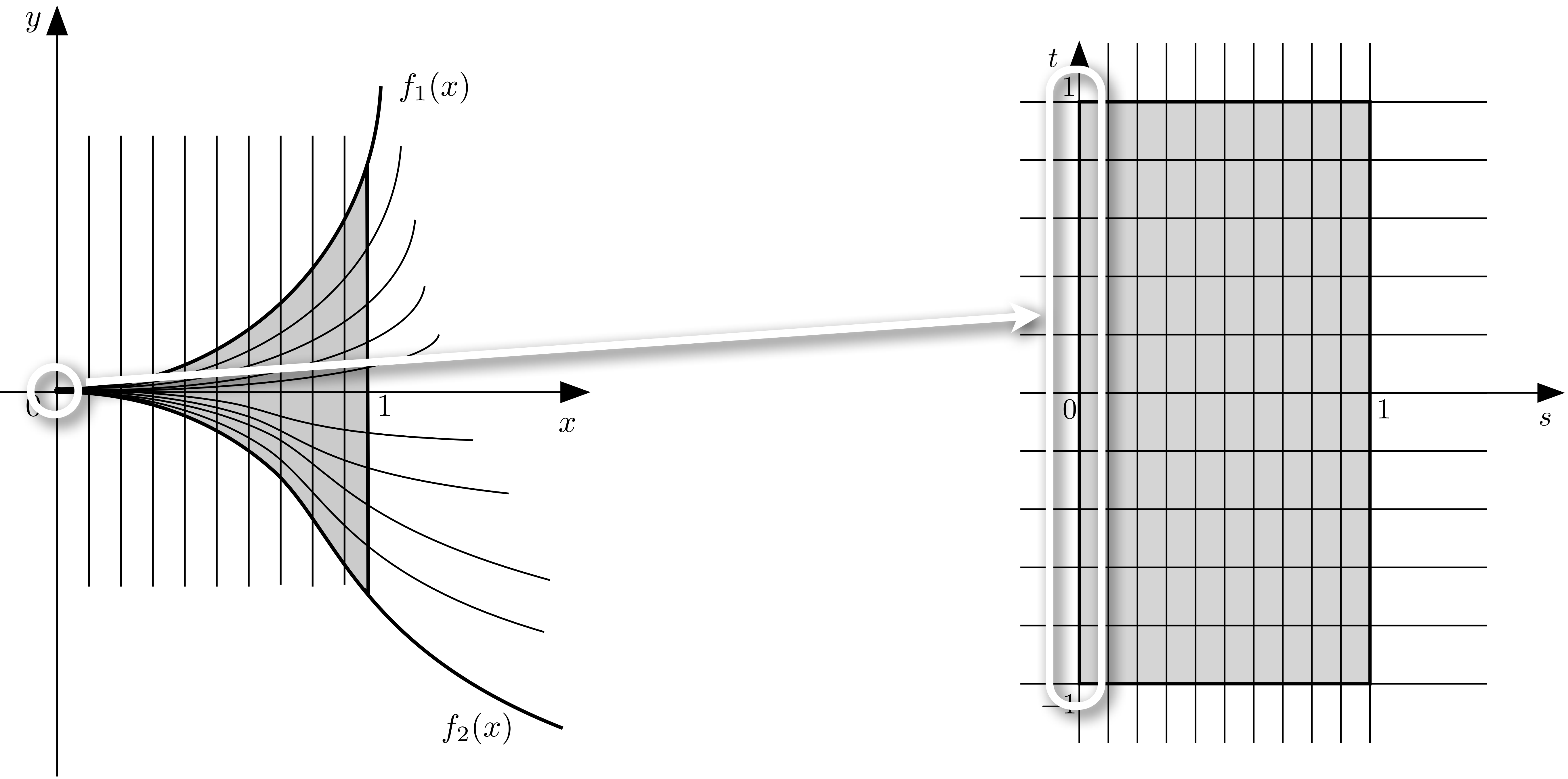
$$s = x$$

$$x = s$$

$$y = \frac{1+t}{2} f_1(s) + \frac{1-t}{2} f_2(s)$$



Change of Coordinates



Finite Volume Element method

Let $u \in C^2(\Omega)$ be the solution of the following PDE:

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u$$

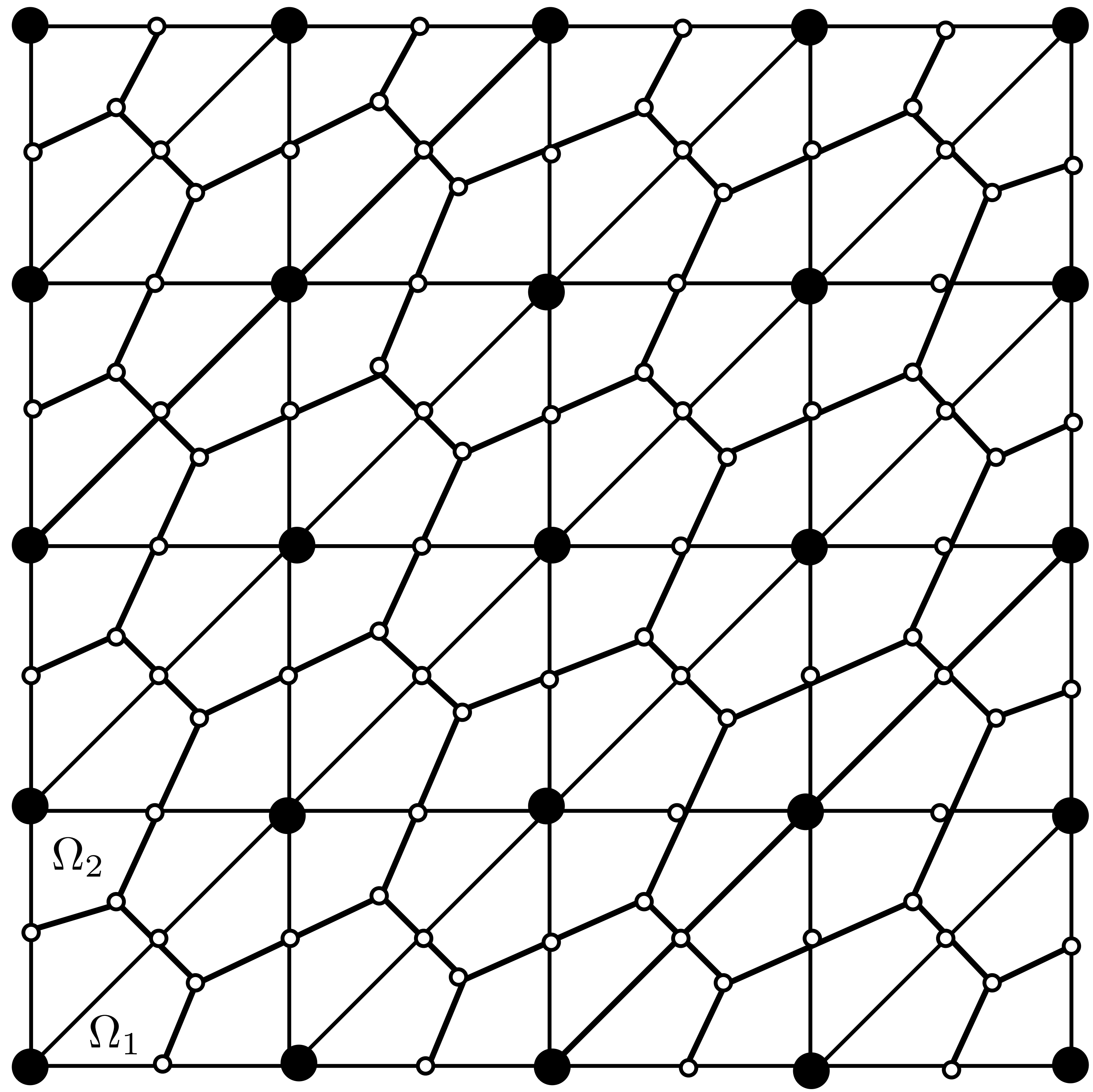
$$\Updownarrow$$

$$\int_{\Omega_\alpha} \nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} dA = \int_{\Omega_\alpha} u \, dA \qquad \text{for all } \Omega_\alpha \subset \Omega$$

$$\Updownarrow$$

$$\int_{\partial\Omega_\alpha} \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ds = \int_{\Omega_\alpha} u \, dA \qquad \text{for all } \Omega_\alpha \subset \Omega$$

Finite Volume Element method



Finite Volume Element method

$$\int_{\partial\Omega_\alpha} \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ds = \int_{\Omega_\alpha} u \, dA$$

+

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$

Finite Volume Element method

$$\int_{\partial\Omega_\alpha} \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ds = \int_{\Omega_\alpha} u \, dA$$

+

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$

$$\int_{\partial\Omega_\alpha \setminus \partial\Omega} \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ds + \int_{\partial\Omega_\alpha \cap \partial\Omega} \cos\gamma ds = \int_{\Omega_\alpha} u \, dA$$

for all $\Omega_\alpha \subset \Omega$

Finite Volume Element method

$$\int_{\partial\Omega_\alpha} \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ds = \int_{\Omega_\alpha} u\, dA$$

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$$\text{for all } \Omega_\alpha \subset \Omega$$

We now approximate the solution with a finite element approximation

$$u \approx \frac{\sum_{i=1}^{N_{\text{node}}} c_i \phi_i}{f_1(x) - f_2(x)} = u^h$$

Finite Volume Element method

$$\int_{\partial\Omega_\alpha \setminus \partial\Omega} \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ds + \int_{\partial\Omega_\alpha \cap \partial\Omega} \cos\gamma ds = \int_{\Omega_\alpha} u \, dA$$

for all $\Omega_\alpha \subset \Omega$

Finite Volume Element method

$$\int_{\partial\Omega_\alpha \setminus \partial\Omega} \nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ds + \int_{\partial\Omega_\alpha \cap \partial\Omega} \cos\gamma ds = \int_{\Omega_\alpha} u \, dA$$

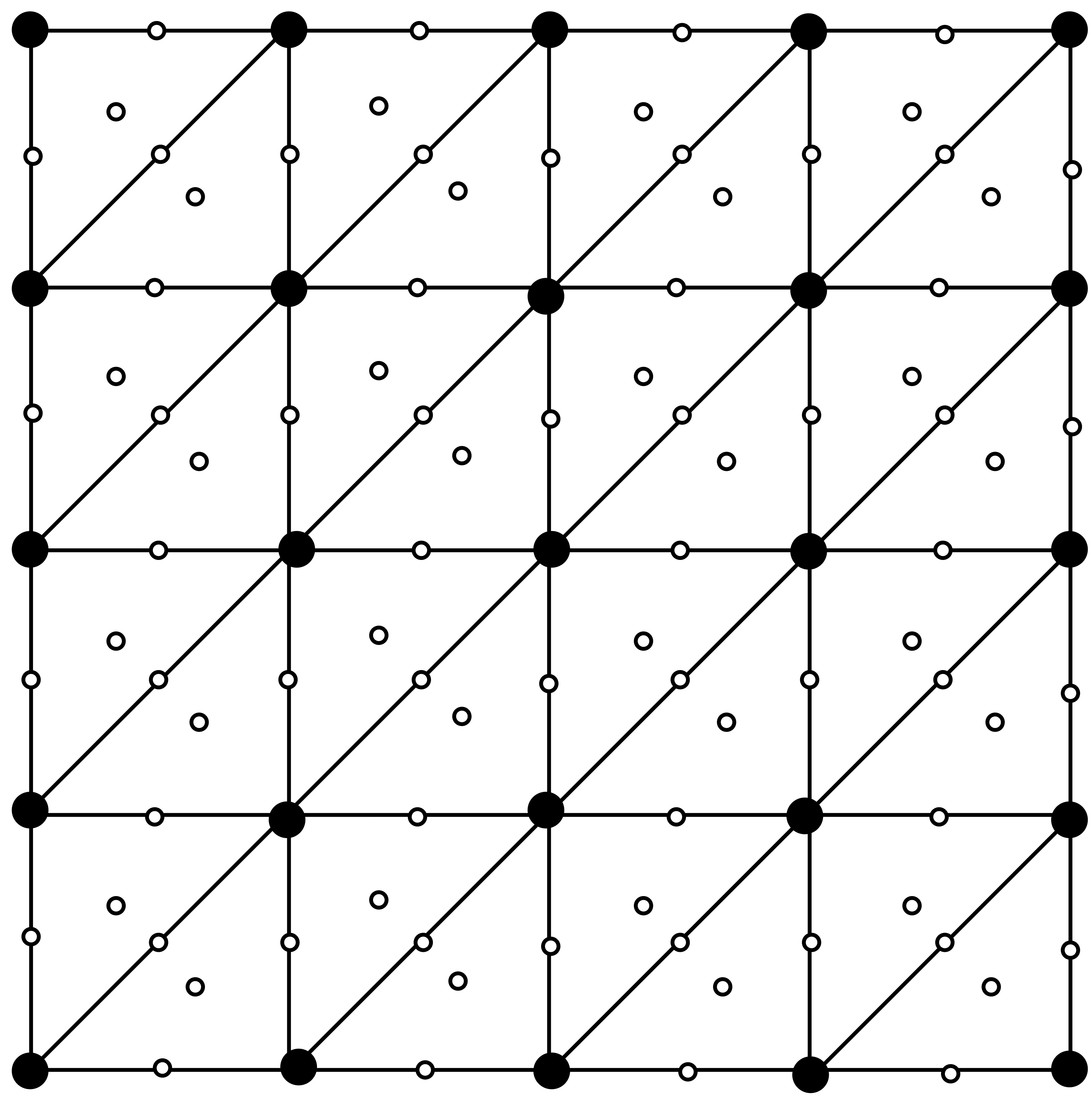
for all $\Omega_\alpha \subset \Omega$

$$\int_{\partial\Omega_j \setminus \partial\Omega} \nu \cdot \frac{\sum_{i=1}^{N_{\text{node}}} c_i \nabla \left(\frac{\phi_i}{f_1(x)-f_2(x)}\right)}{\sqrt{1+|\sum_{i=1}^{N_{\text{node}}} c_i \nabla \left(\frac{\phi_i}{f_1(x)-f_2(x)}\right)|^2}} ds + \int_{\partial\Omega_j \cap \partial\Omega} \cos\gamma ds$$

$$= \int_{\Omega_j} \sum_{i=1}^{N_{\text{node}}} c_i \left(\frac{\phi_i}{f_1(x)-f_2(x)}\right) dA$$

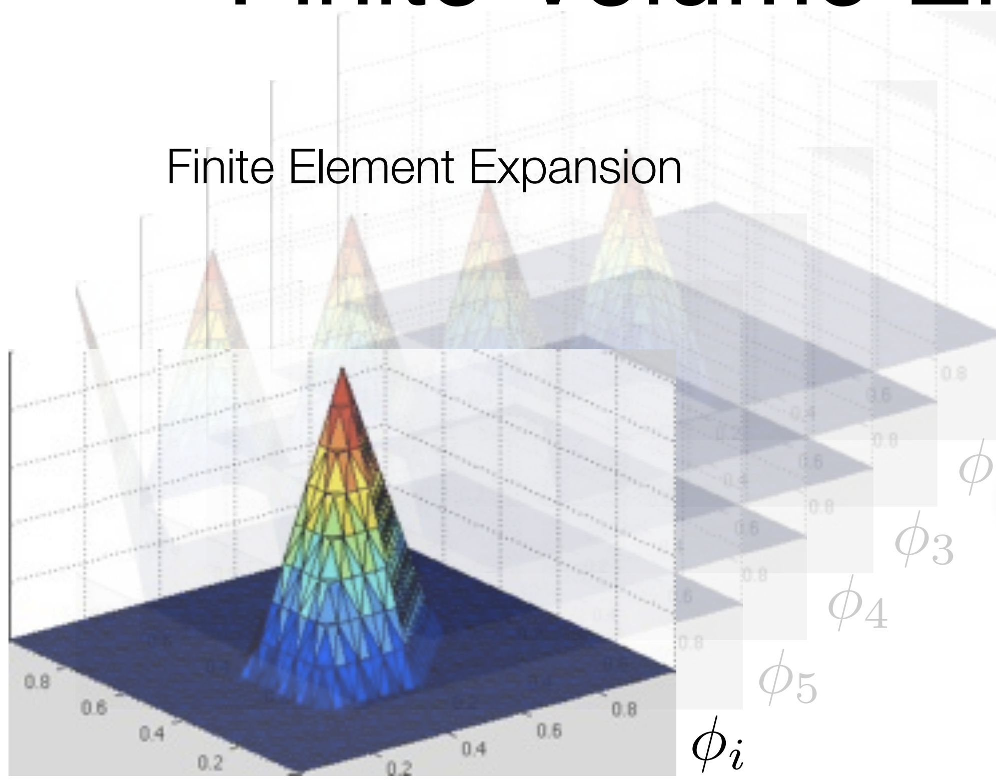
for $j = 1, 2, \ldots, N_{\text{node}}$

Finite Volume Element method



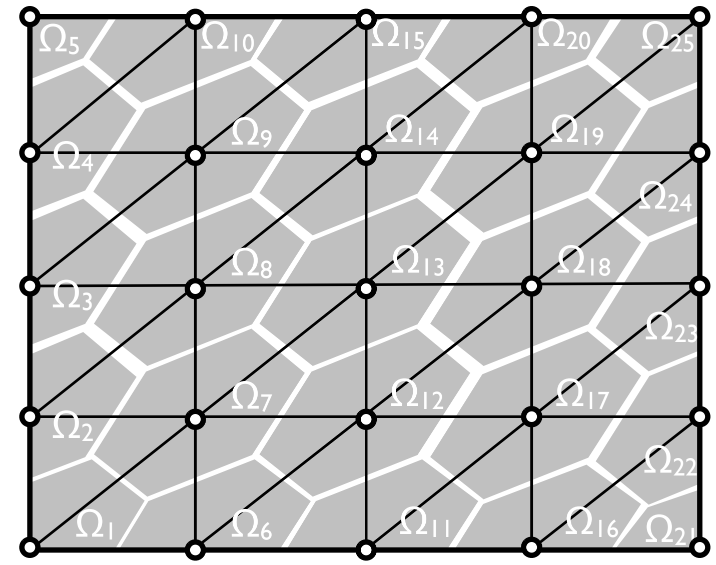
Finite Volume Element method

Finite Element Expansion



+

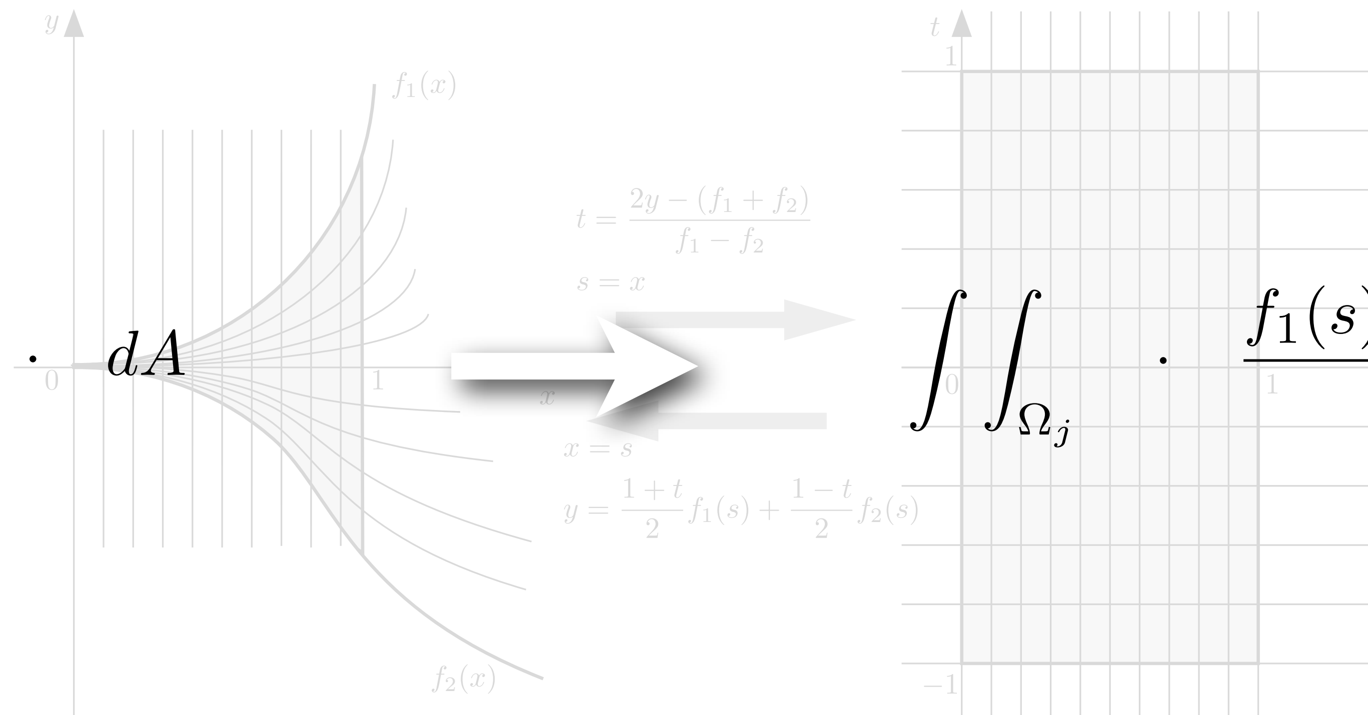
Finite Volume Method Control Volumes



$$\int_{\partial\Omega_j \setminus \partial\Omega} \nu \cdot \frac{\sum_{i=1}^{N_{\text{node}}} c_i \nabla \left(\frac{\phi_i}{f_1(x) - f_2(x)} \right)}{\sqrt{1 + |\sum_{i=1}^{N_{\text{node}}} c_i \nabla \left(\frac{\phi_i}{f_1(x) - f_2(x)} \right)|^2}} ds + \int_{\partial\Omega_j \cap \partial\Omega} \cos\gamma ds$$

$$= \int_{\Omega_j} \sum_{i=1}^{N_{\text{node}}} c_i \left(\frac{\phi_i}{f_1(x) - f_2(x)} \right) dA \qquad \text{for } j = 1, 2, \ldots, N_{\text{node}}$$

Finite Volume Element method

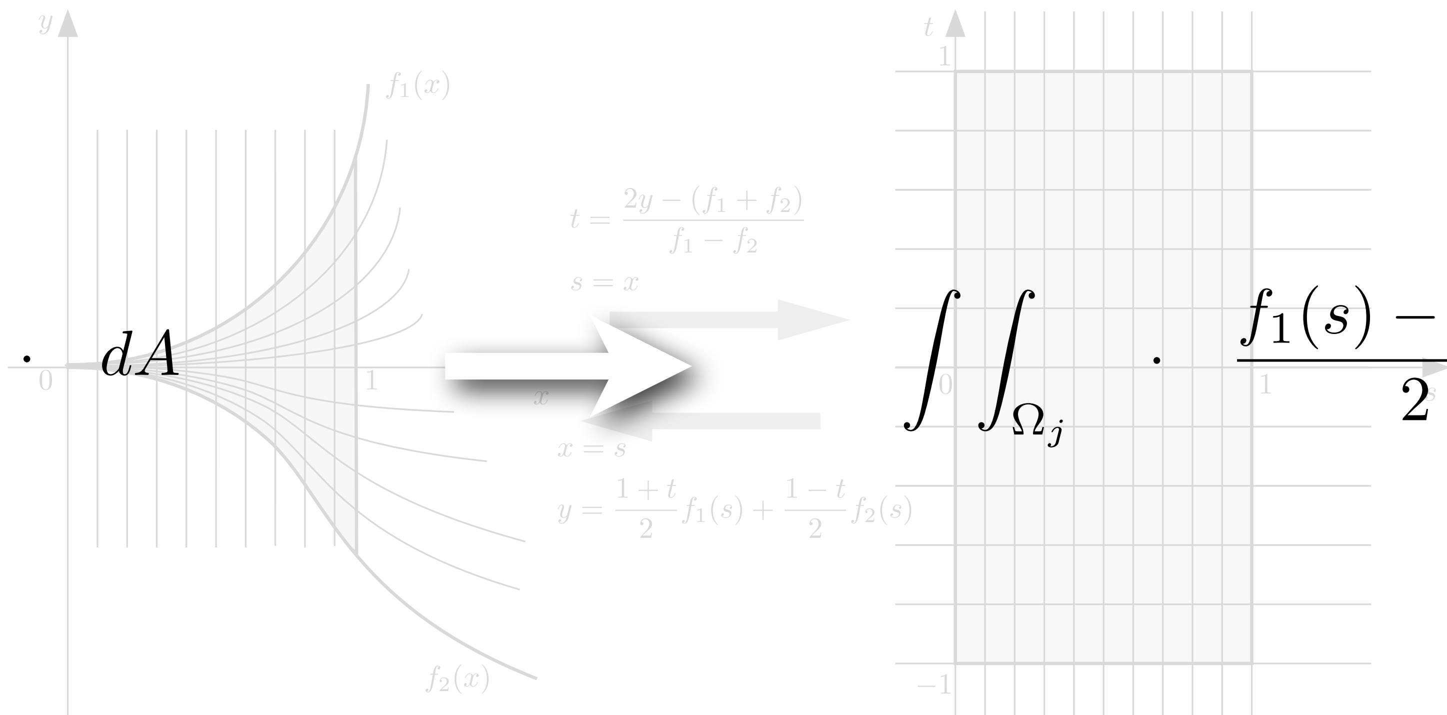


$$\int_{\Omega_j} \cdot \; dA \quad \Rightarrow \quad \int\int_{\Omega_j} \cdot \; \frac{f_1(s)-f_2(s)}{2}\, ds\, dt$$

$$\int_{\partial\Omega_j \setminus \partial\Omega} \nu \cdot \frac{\sum_{i=1}^{N_{\mathrm{node}}} c_i \nabla \left(\frac{\phi_i}{f_1(x)-f_2(x)}\right)}{\sqrt{1+|\sum_{i=1}^{N_{\mathrm{node}}} c_i \nabla \left(\frac{\phi_i}{f_1(x)-f_2(x)}\right)|^2}} ds + \int_{\partial\Omega_j \cap \partial\Omega} \cos\gamma ds$$

$$= \int_{\Omega_j} \sum_{i=1}^{N_{\mathrm{node}}} c_i \left(\frac{\phi_i}{f_1(x)-f_2(x)}\right) dA \qquad \text{for } j = 1, 2, \ldots, N_{\mathrm{node}}$$

Finite Volume Element method



$$\int_{\Omega_j} \cdot \, dA \quad \longrightarrow \quad \int \int_{\Omega_j} \cdot \, \frac{f_1(s) - f_2(s)}{2} \, ds \, dt$$

$$\int_{\partial\Omega_j \setminus \partial\Omega} \nu \cdot \frac{\sum_{i=1}^{N_{\rm node}} c_i \nabla \left(\frac{\phi_i}{f_1(x) - f_2(x)} \right)}{\sqrt{1 + |\sum_{i=1}^{N_{\rm node}} c_i \nabla \left(\frac{\phi_i}{f_1(x) - f_2(x)} \right) |^2}} ds + \int_{\partial\Omega_j \cap \partial\Omega} \cos\gamma ds$$

$$= \frac{1}{2} \int \int_{\Omega_j} \sum_{i=1}^{N_{\rm node}} c_i \phi_i \, ds \, dt \qquad \text{for } j = 1, 2, \ldots, N_{\rm node}$$

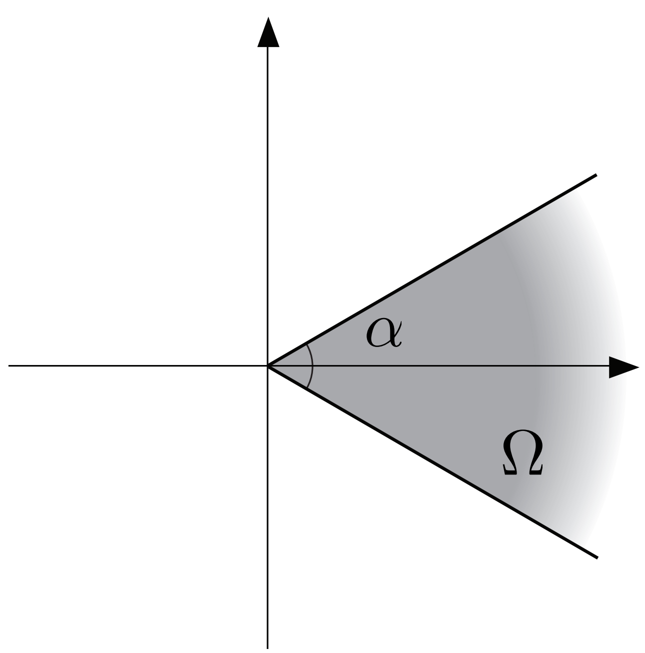
Numerical Experiments

Asymptotic Laplace-Young Equation

(in a Corner domain)

$$\nabla \cdot \frac{\nabla v}{|\nabla v|^2} = v \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla v}{|\nabla v|^2} = \cos\gamma \qquad \text{on } \partial\Omega$$



$$v(r,\theta) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{kr}$$

$$u(r,\theta) = v(r,\theta) + O(r^3) \qquad \text{as } r \to 0$$

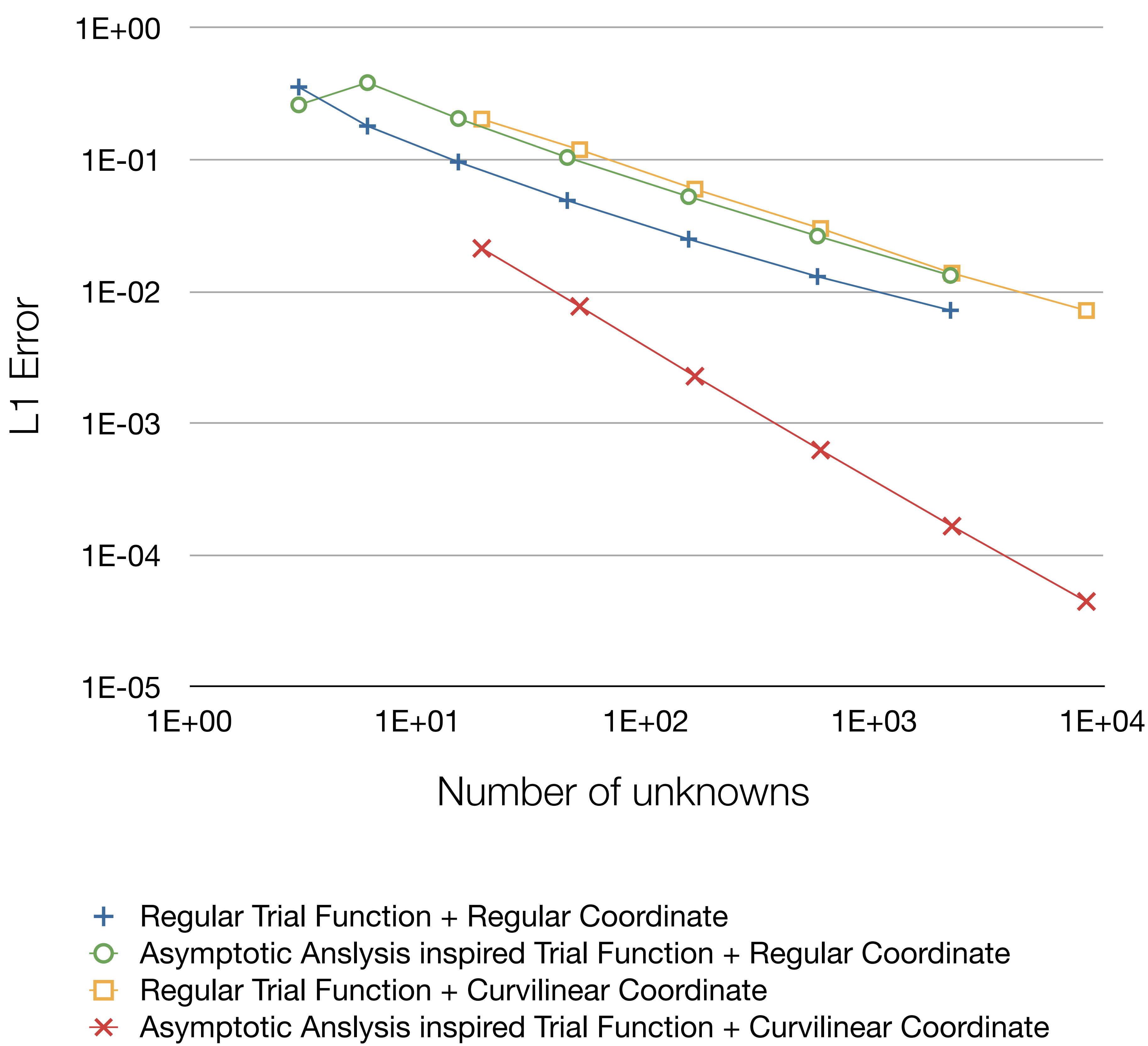
Convergence Study

(Asymptotic Laplace-Young Equation in a Corner domain)

	Without Change of Variable	With Change of Variable
Regular Coordinates	(Scott et al.)	
Curvilinear Coordinates		

Convergence Study

(Asymptotic Laplace-Young Equation in a Corner domain)



Convergence Study

(Asymptotic Laplace-Young Equation in a Corner domain)

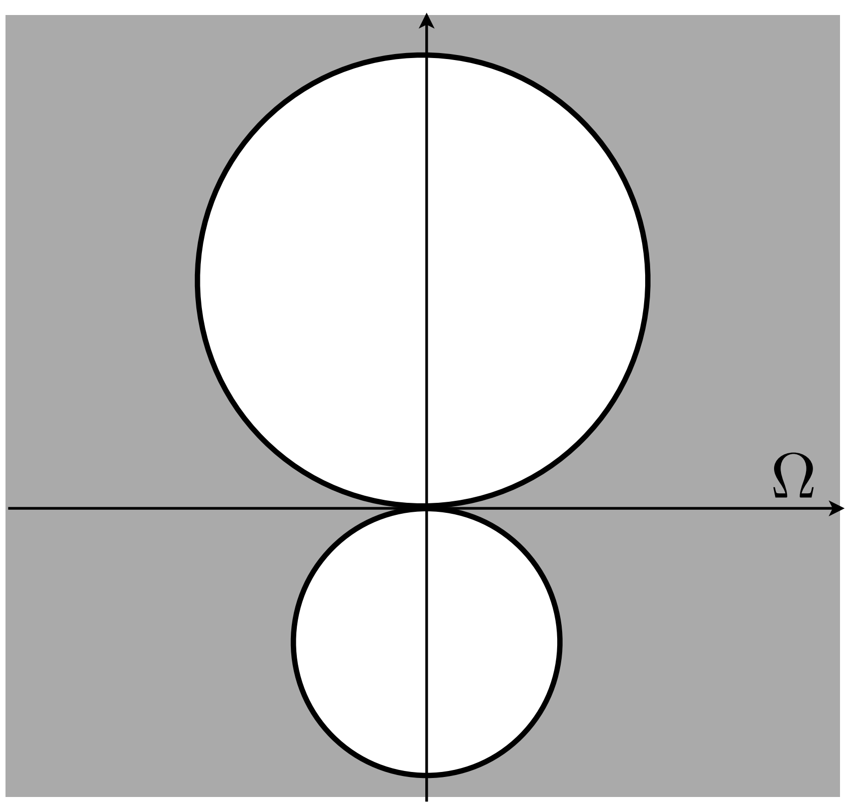
	Without Change of Variable	With Change of Variable
Regular Coordinates	Linear	Linear
Curvilinear Coordinates	Linear	Quadratic

Asymptotic Laplace-Young Equation

(in a Circular Cusp domain)

$$\nabla \cdot \frac{\nabla v}{|\nabla v|^2} = v \qquad \text{in } \Omega$$

$$\nu \cdot \frac{\nabla v}{|\nabla v|^2} = \cos\gamma \qquad \text{on } \partial\Omega$$

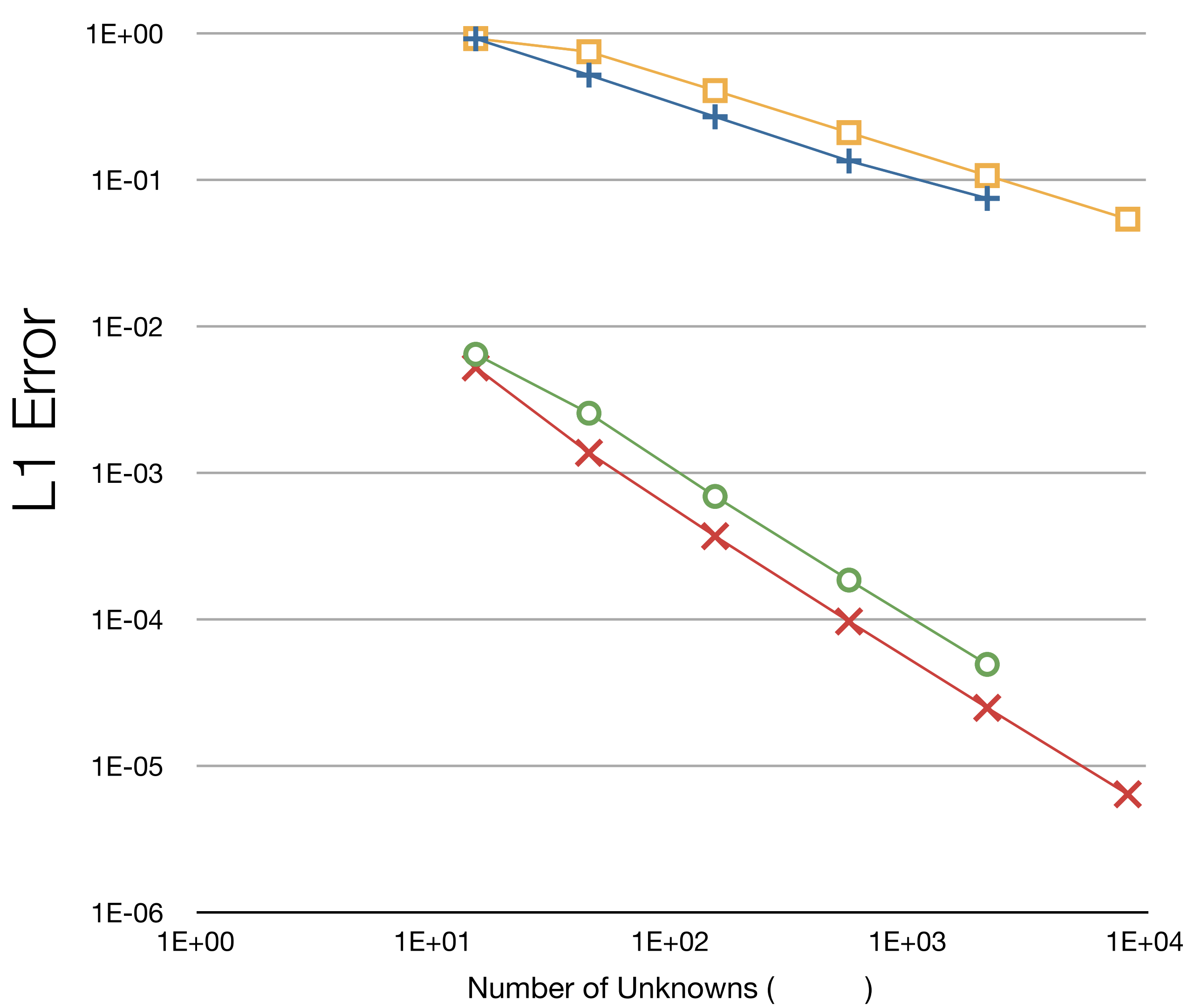


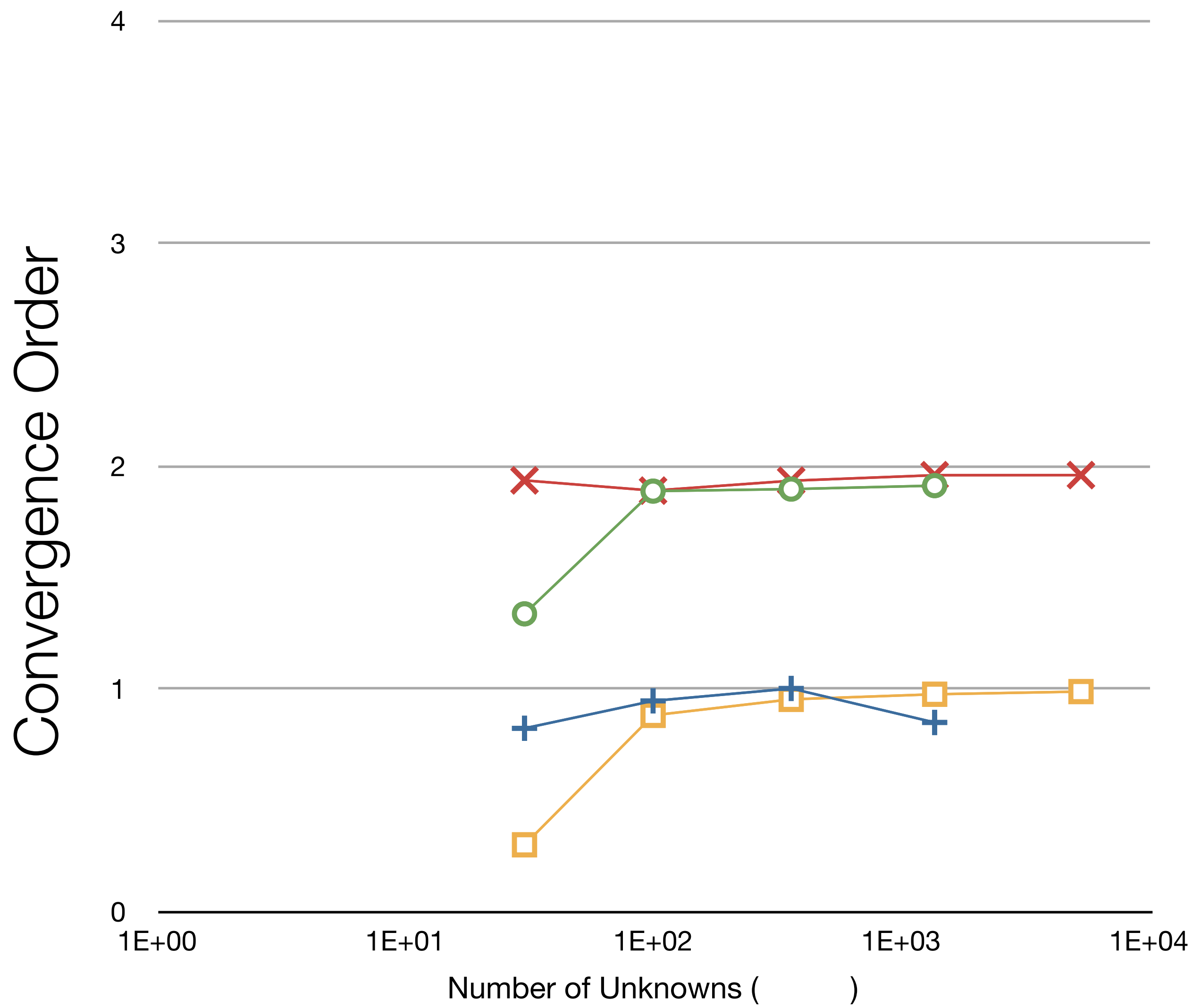
$$v(p,q) = Ap^2 - 2\sqrt{1 - A^2(q-q_0)^2}\; p - A(q-q_0)^2 + Aq_0^2$$

$$u(p,q) = v(p,q) + O(p^{-5}) \qquad \text{as } p \to \infty$$

Convergence Study

(Asymptotic Laplace-Young Equation in a Circular Cusp domain)



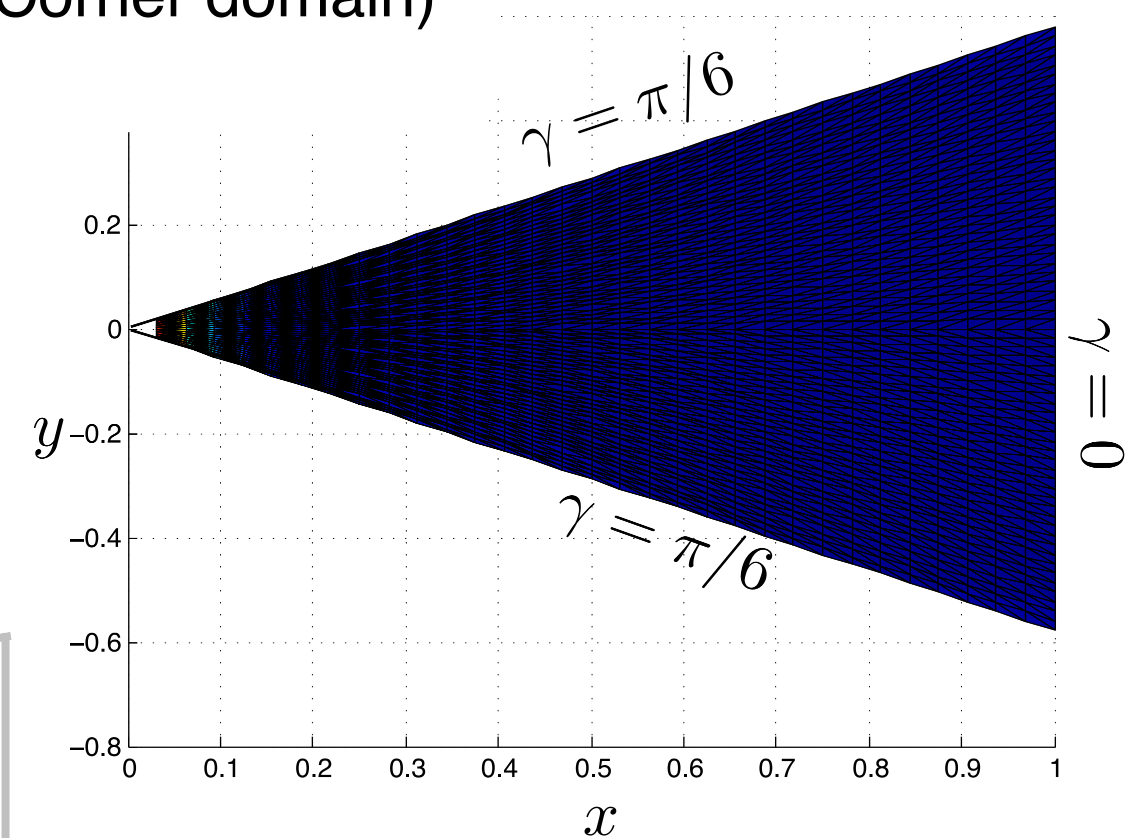


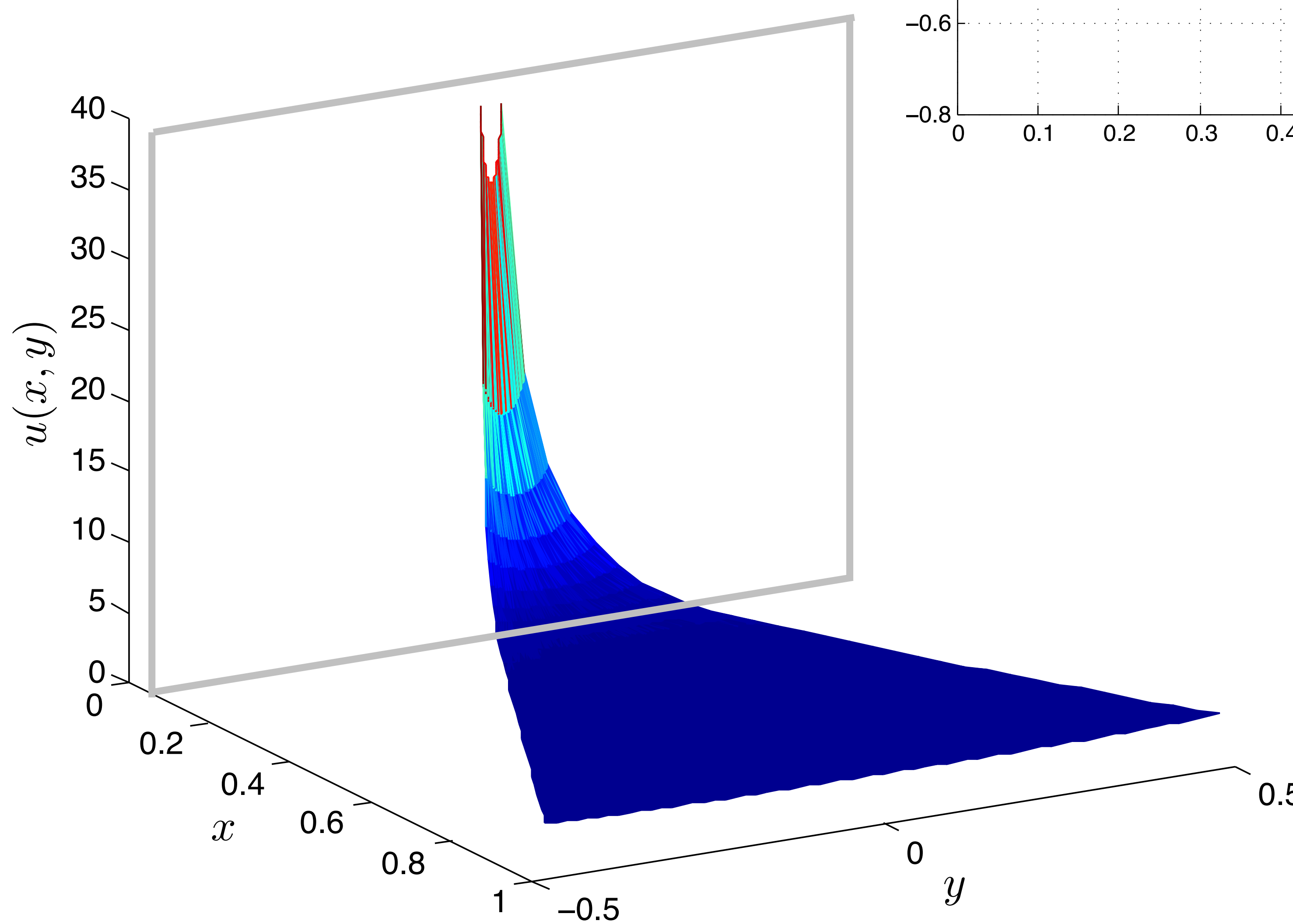
Numerical Experiment

(Laplace Young Equation in a Corner domain)

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

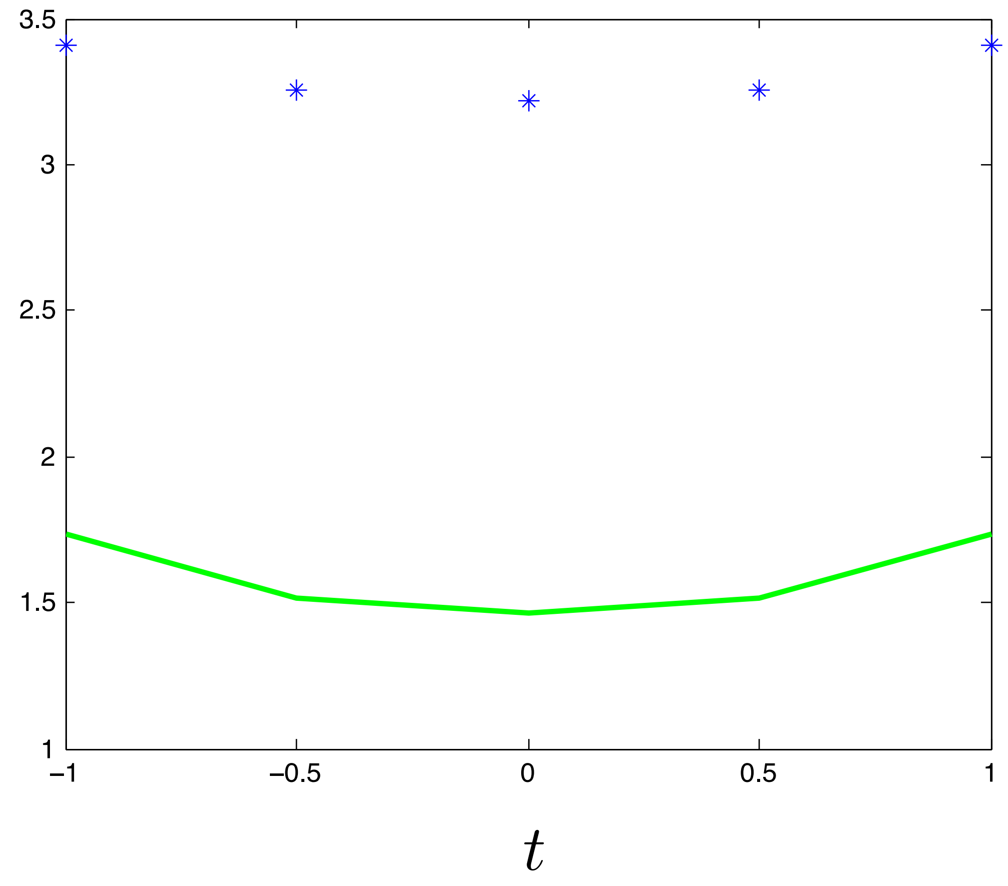
$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$





Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)

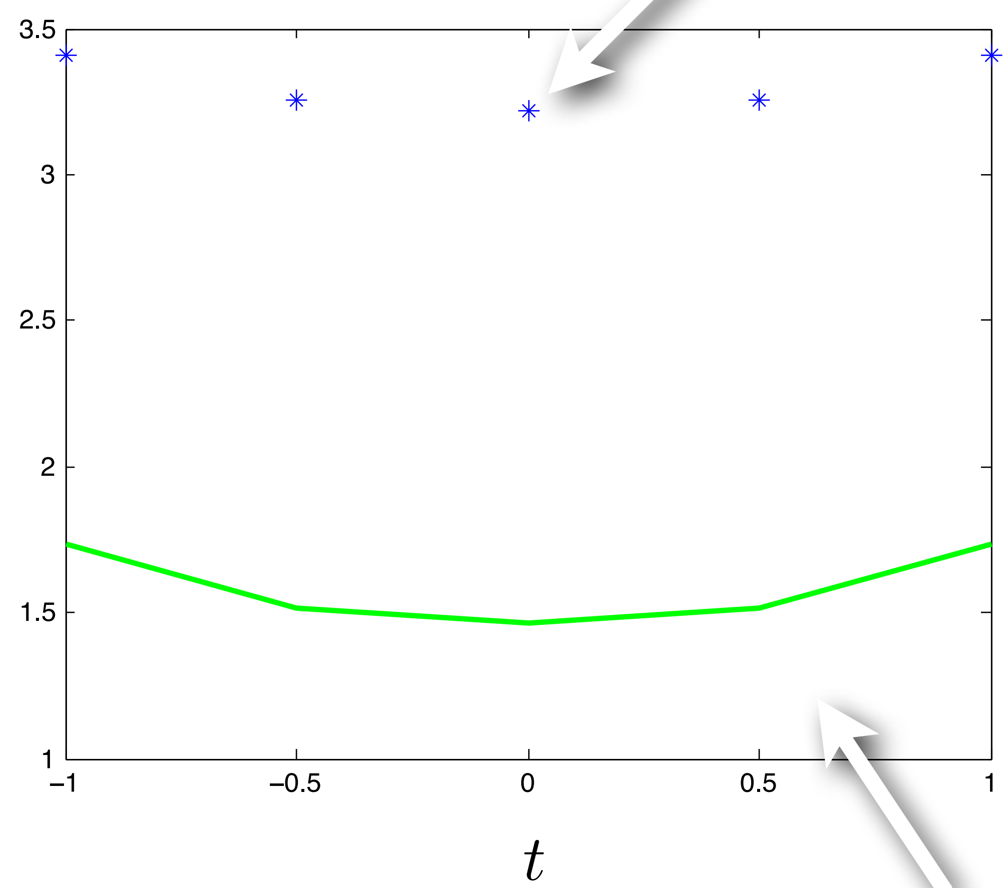


Numeri

(Finite Volume Element approximation wi s)

Numerical Solution

$$u^h(s,t)|_{s=h}$$

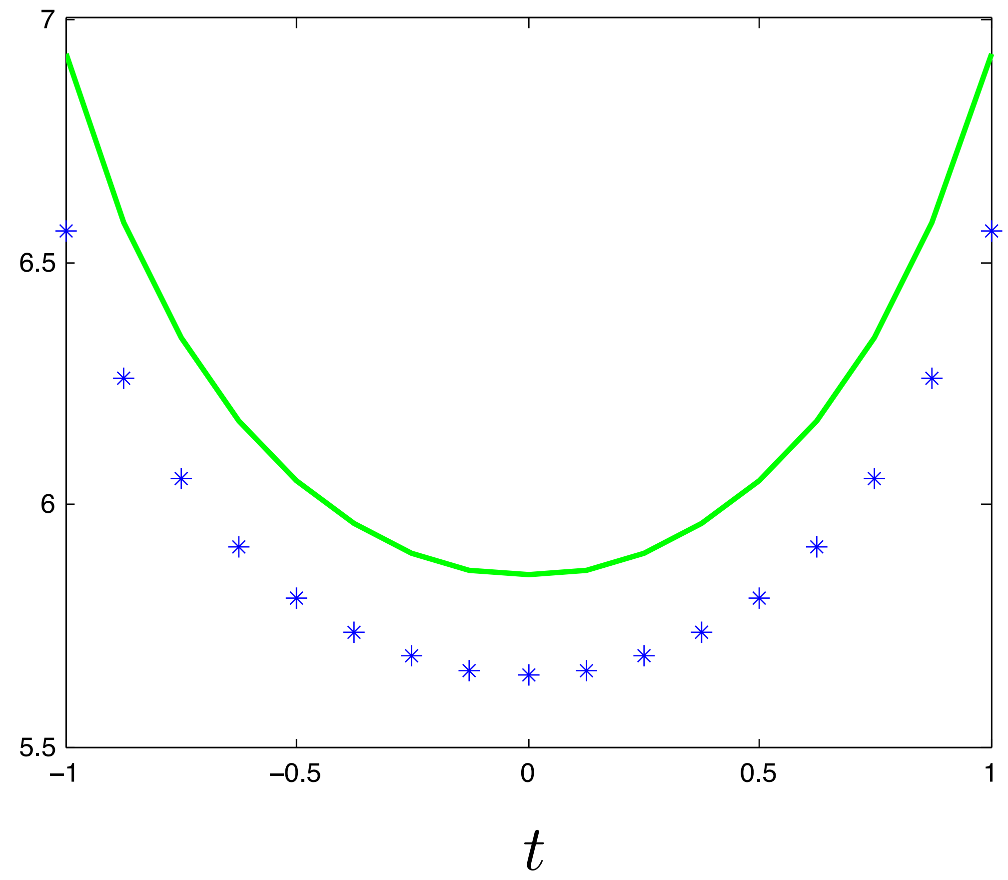


Analytically Proven Asymptotic Behaviour

$$\left.\frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{kr}\right|_{s=h}$$

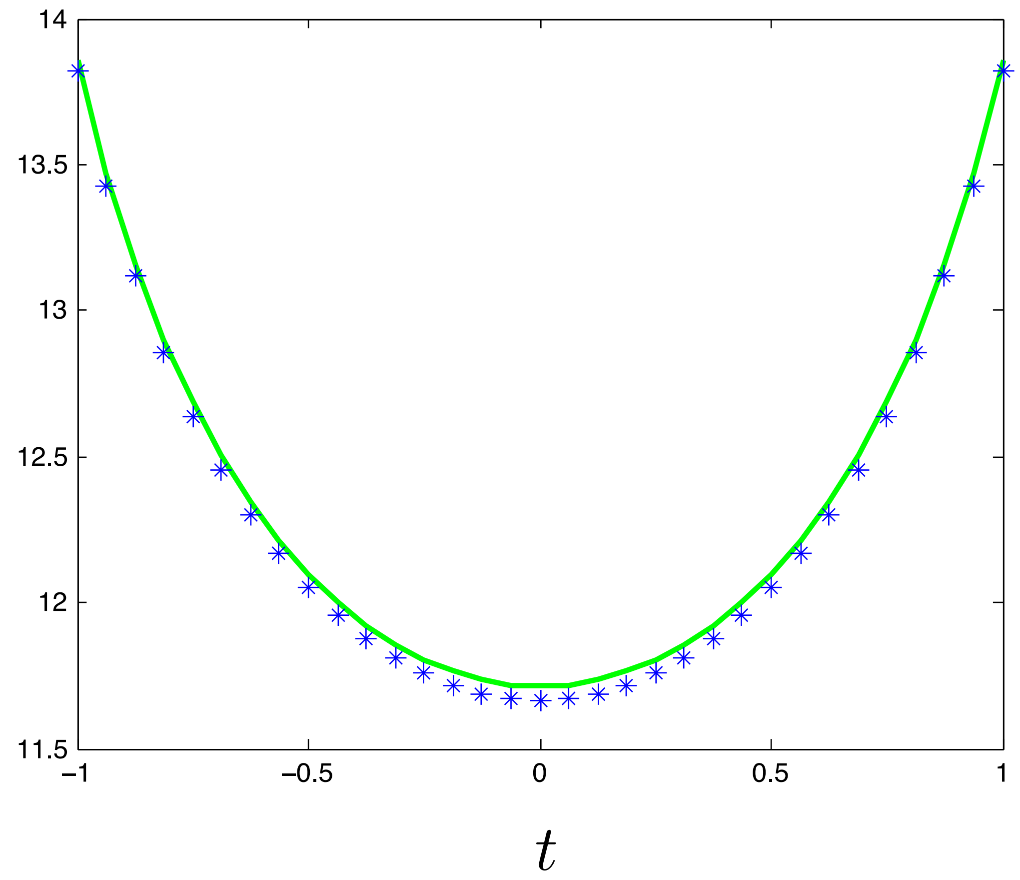
Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)



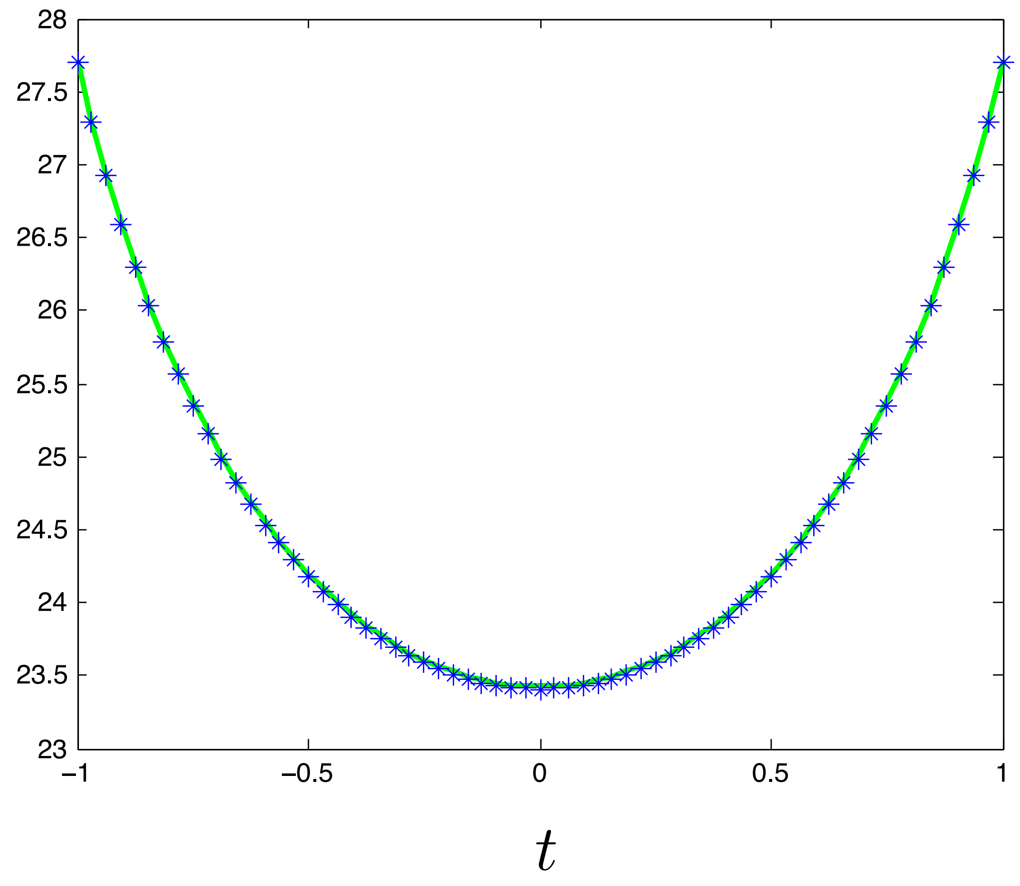
Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)



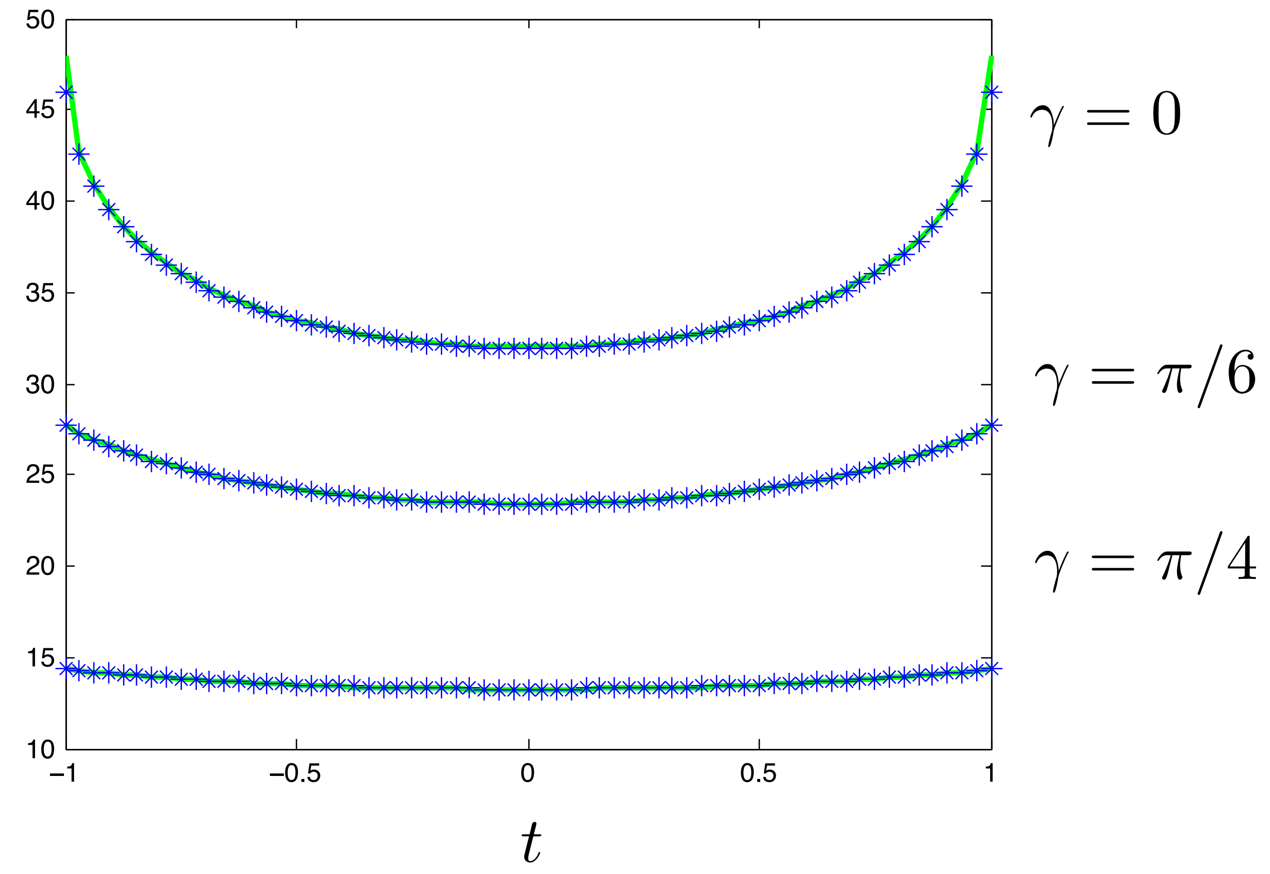
Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)



Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)

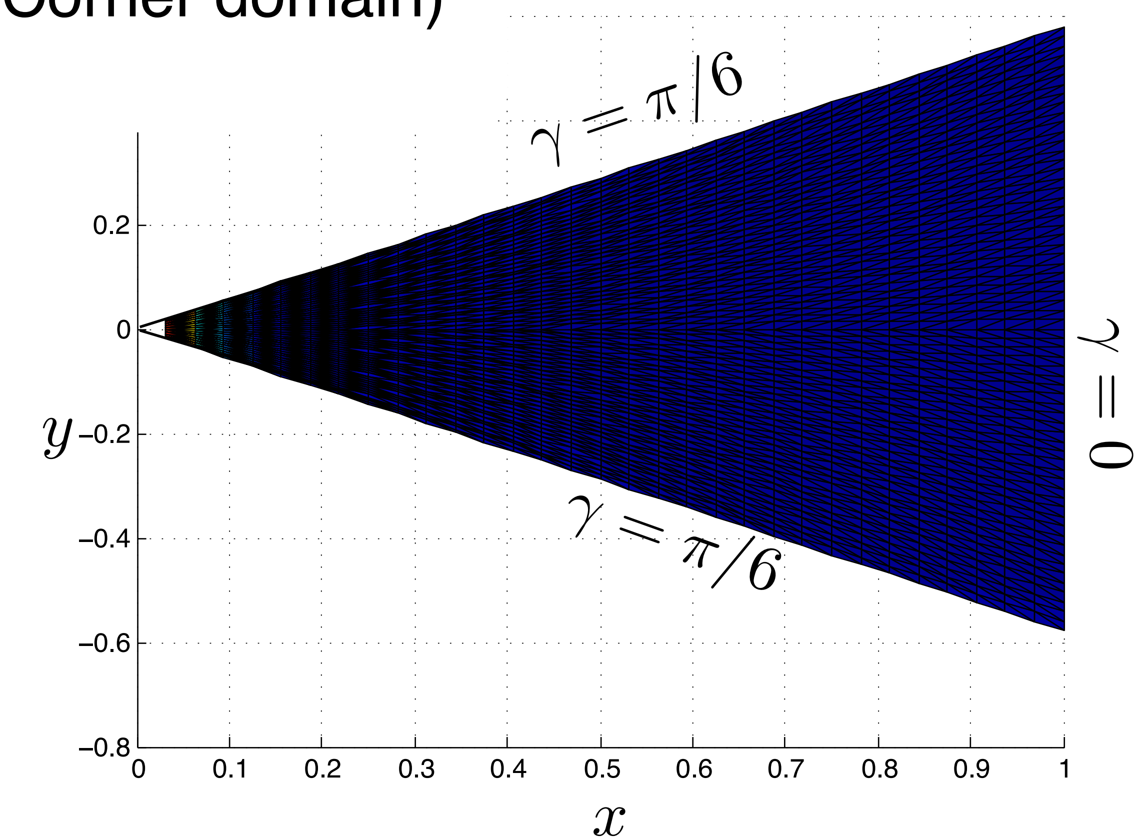


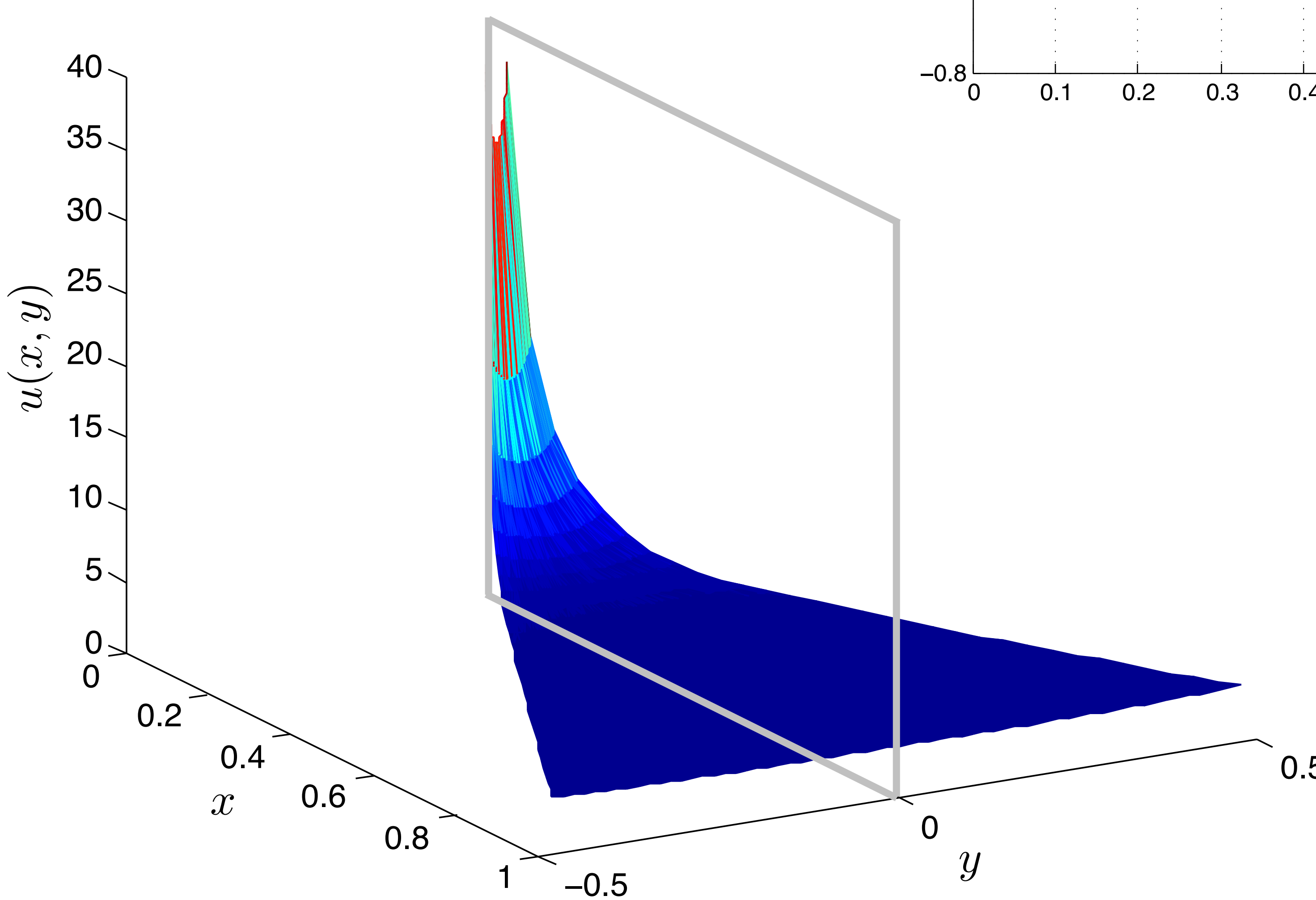
Numerical Experiment

(Laplace Young Equation in a Corner domain)

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \quad u \qquad \text{in } \Omega$$

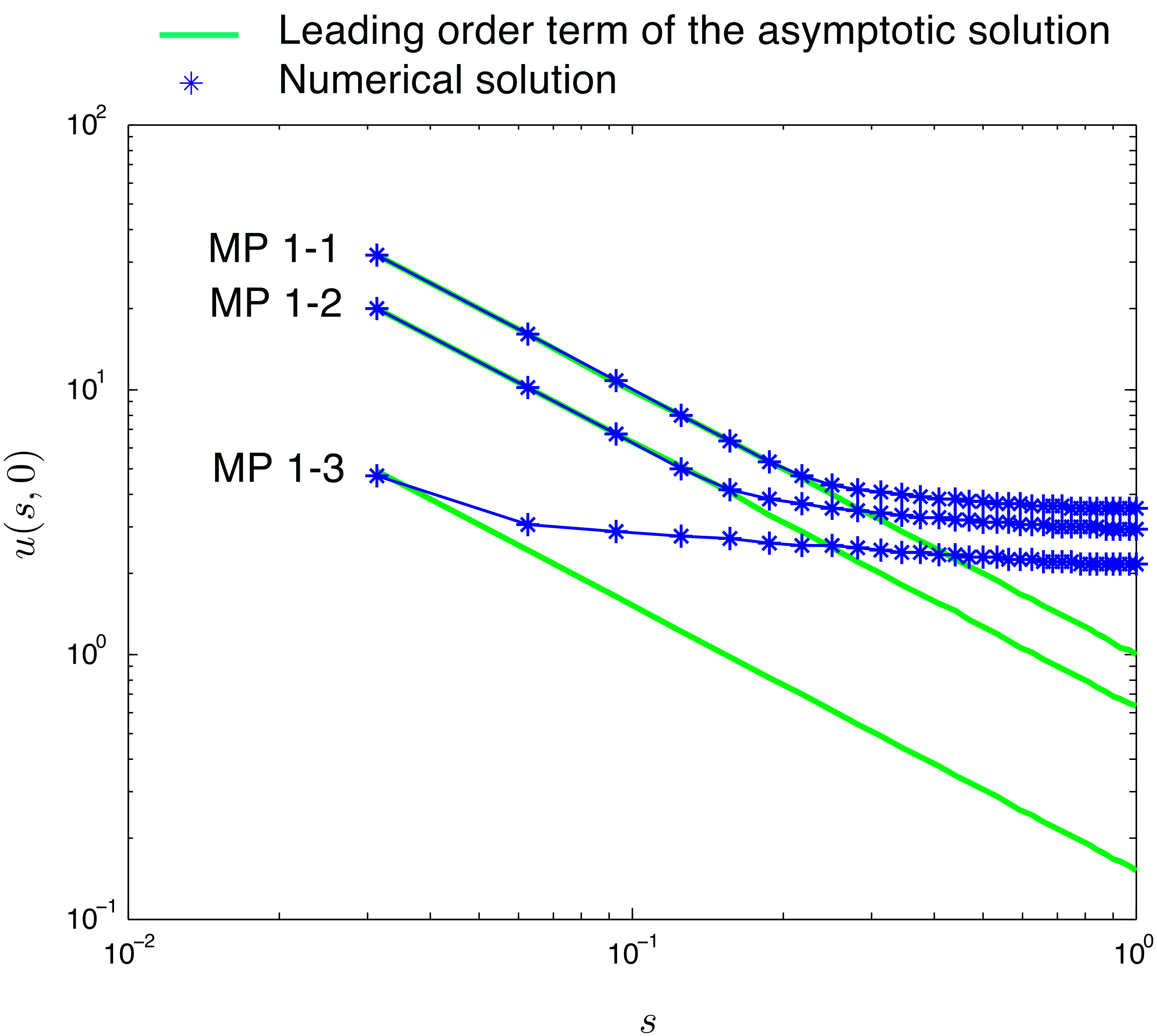
$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$





Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)

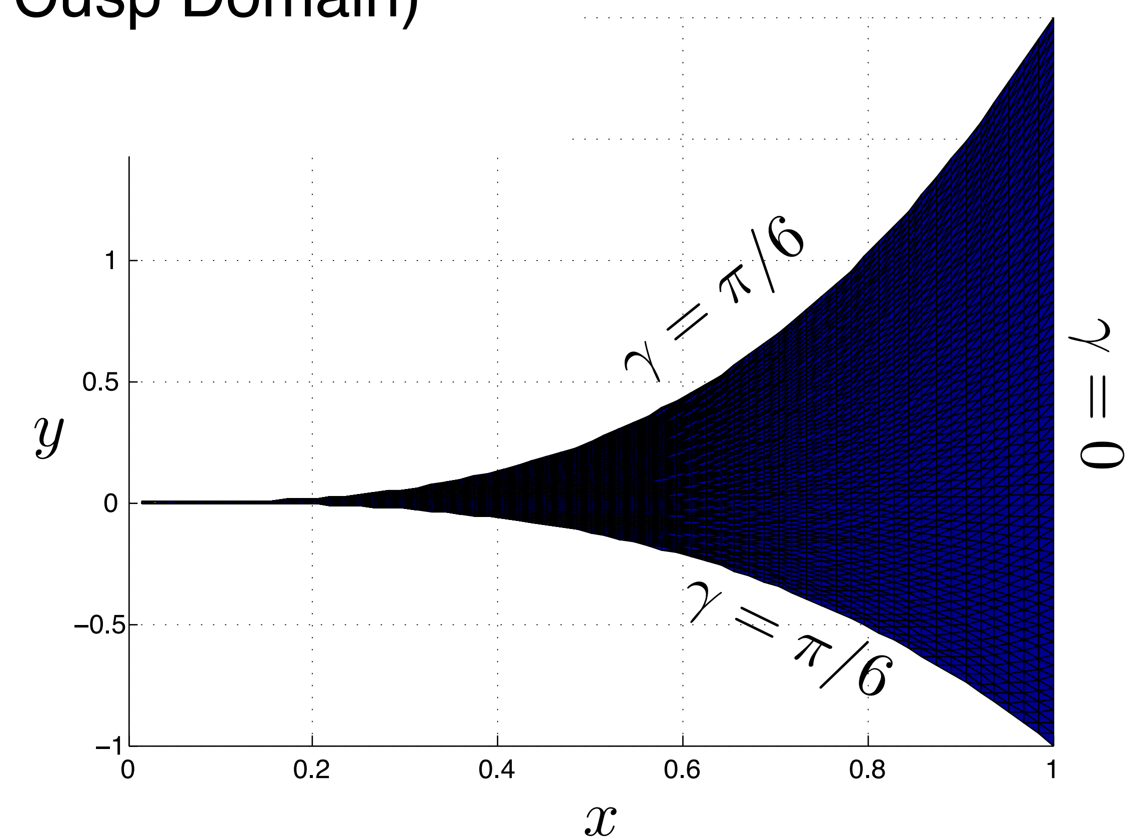


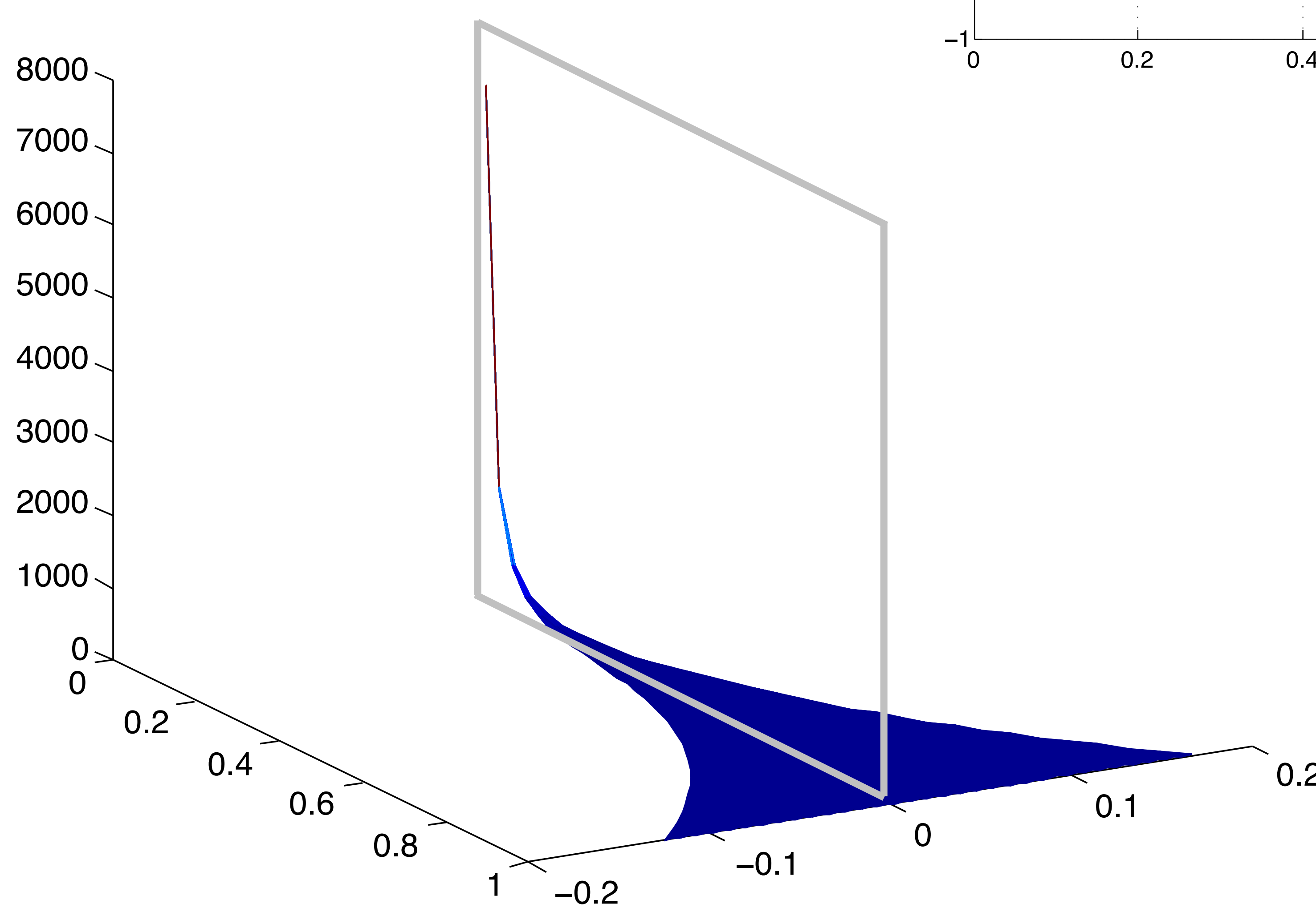
Numerical Experiment

(Laplace-Young Equation in a Cusp Domain)

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = u \qquad \text{in } \Omega$$

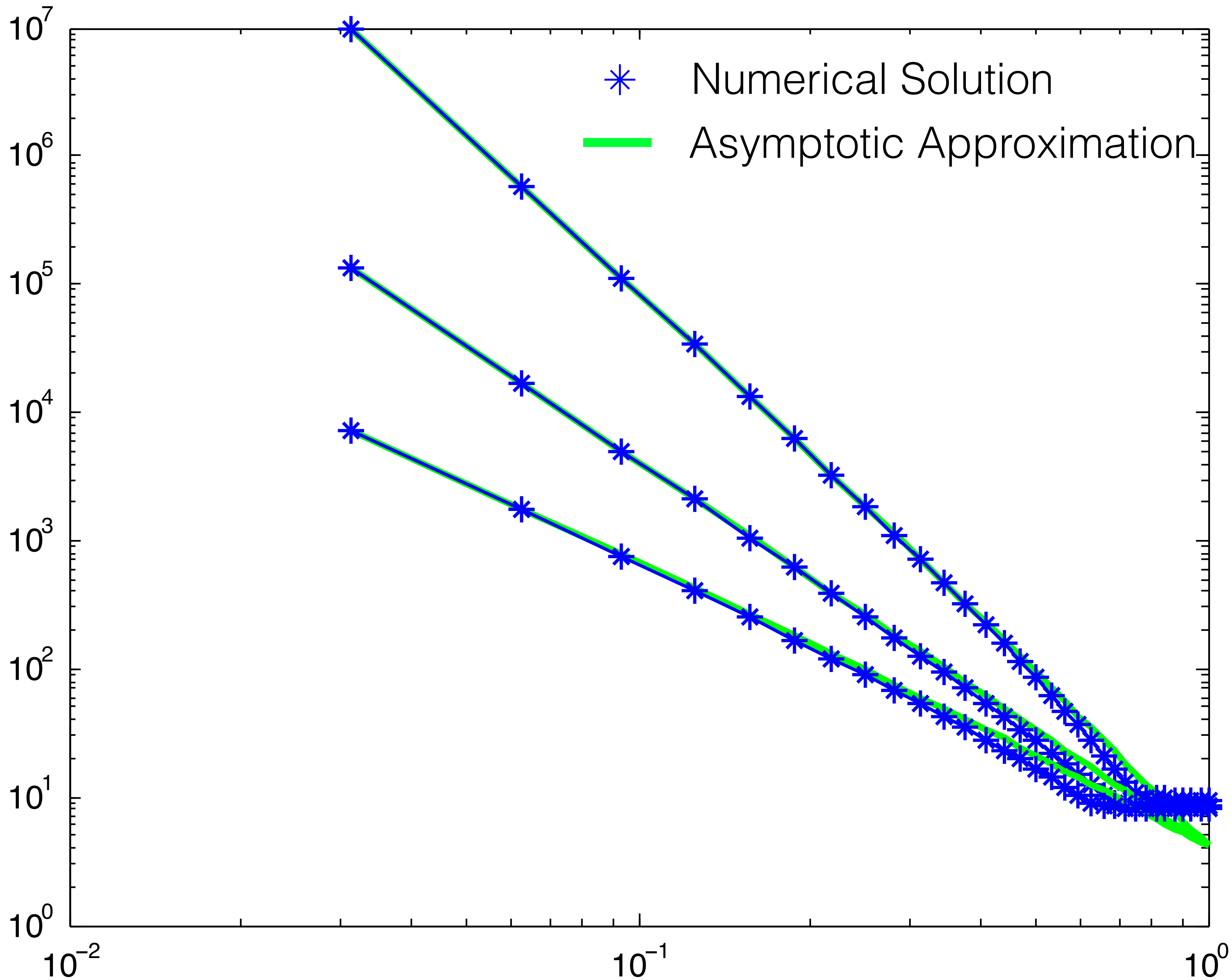
$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{on } \partial\Omega$$





Numerical Experiment

(Finite Volume Element approximation with change of variable and with change of coordinates)



Open Problems

Theorem 2: u is bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

and $f_{1,2,}(x) \in C^2([0,\infty))$.

Y.A. and David Siegel, Bounded and Unbounded Capillary Surfaces in a Cusp Domain, to appear in Pacific Journal of Mathematics, accepted on December 31st 2011.

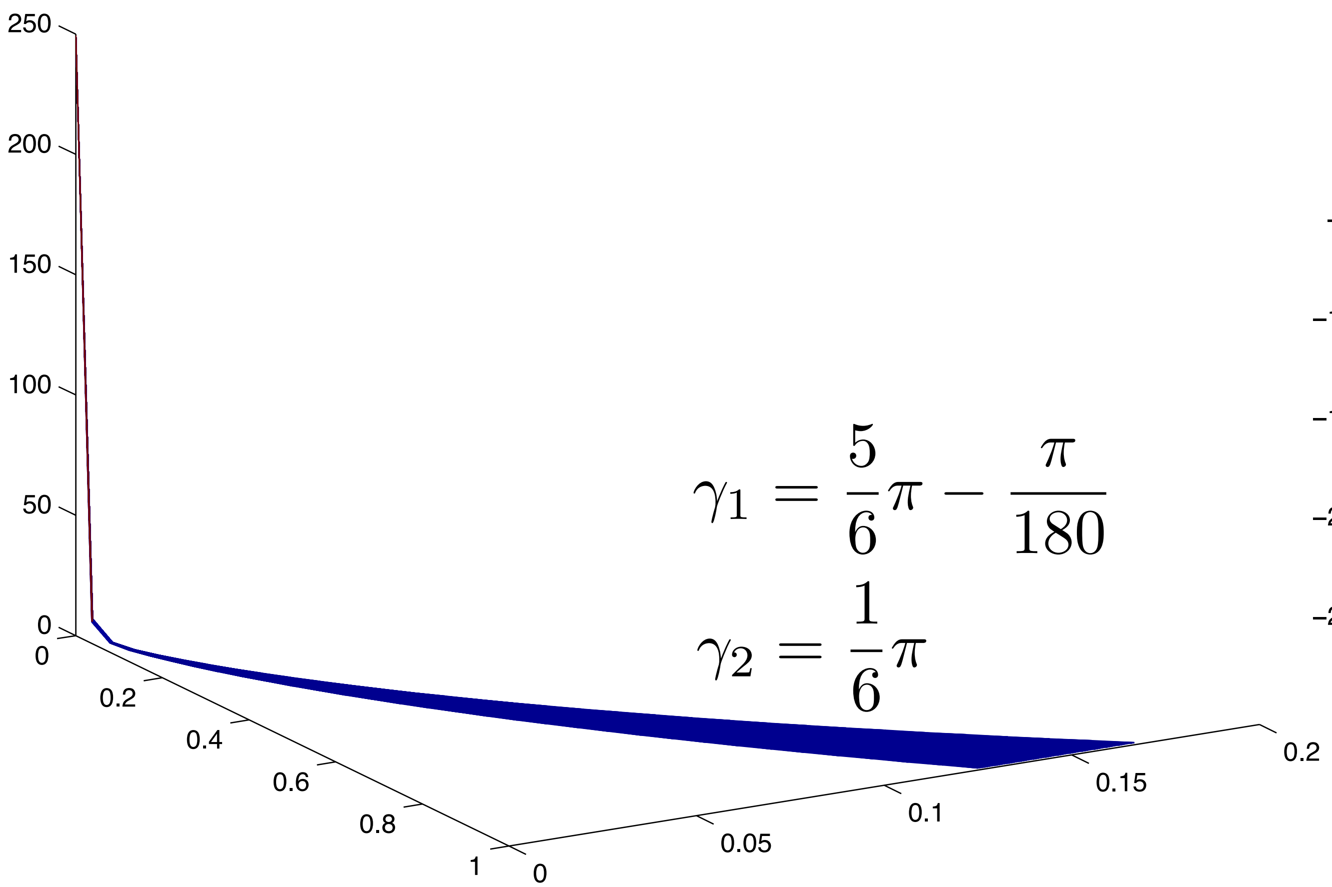
Open Problem 1: Is u bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

and $f_{1,2}(x) \notin C^2([0,\infty))$.

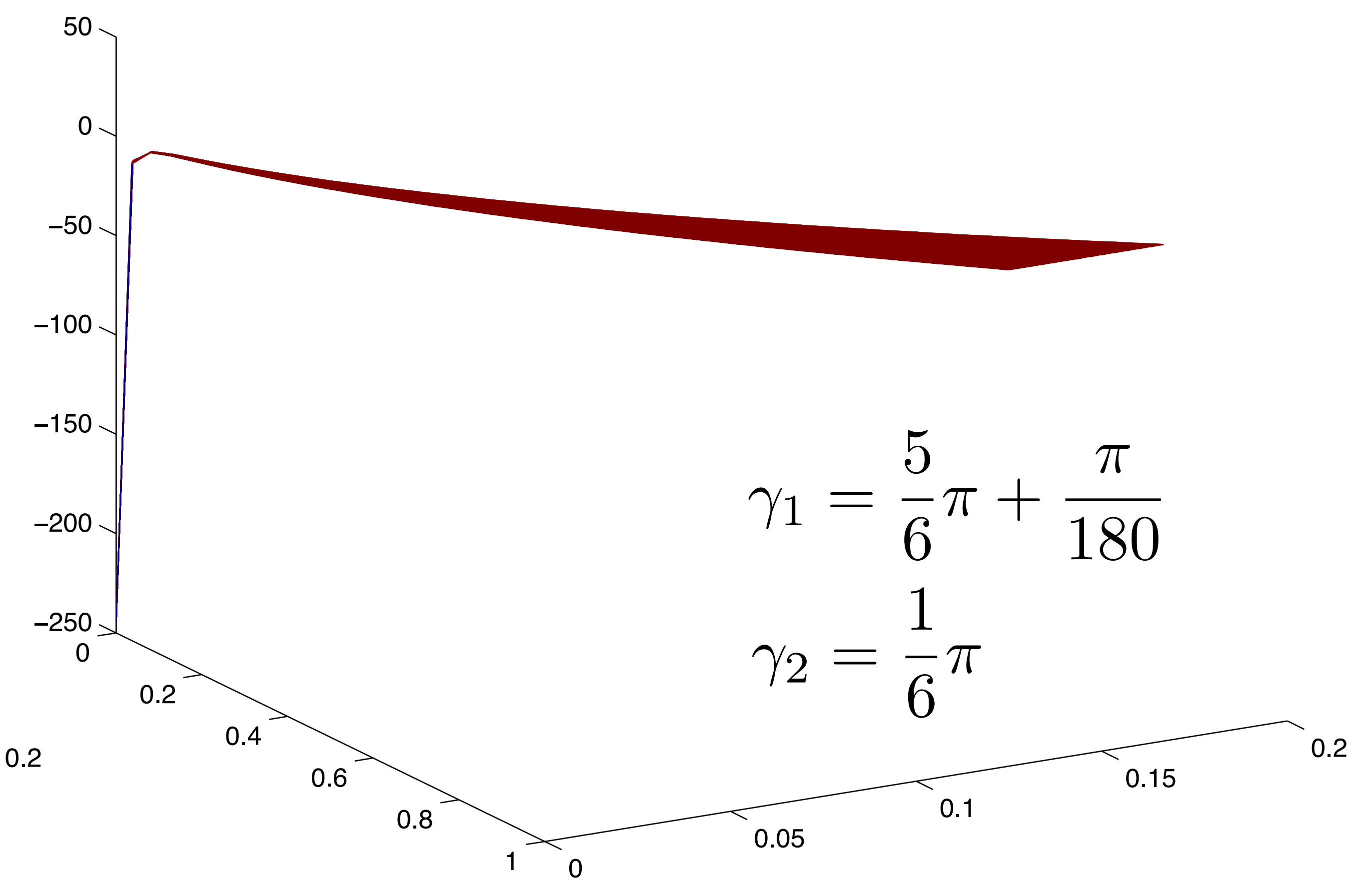
For example $f_1(x) = \frac{1}{6}x^{\frac{3}{2}}$ $f_2(x) = \frac{1}{8}x^{\frac{3}{2}}$

Open Problem 1: Is u bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$, and $f_{1,2}(x) \notin C^2([0,\infty))$.

$$f_1(x) = \frac{1}{6}x^{\frac{3}{2}} \qquad\qquad f_2(x) = \frac{1}{8}x^{\frac{3}{2}}$$



$$\gamma_1 = \frac{5}{6}\pi - \frac{\pi}{180}$$
$$\gamma_2 = \frac{1}{6}\pi$$

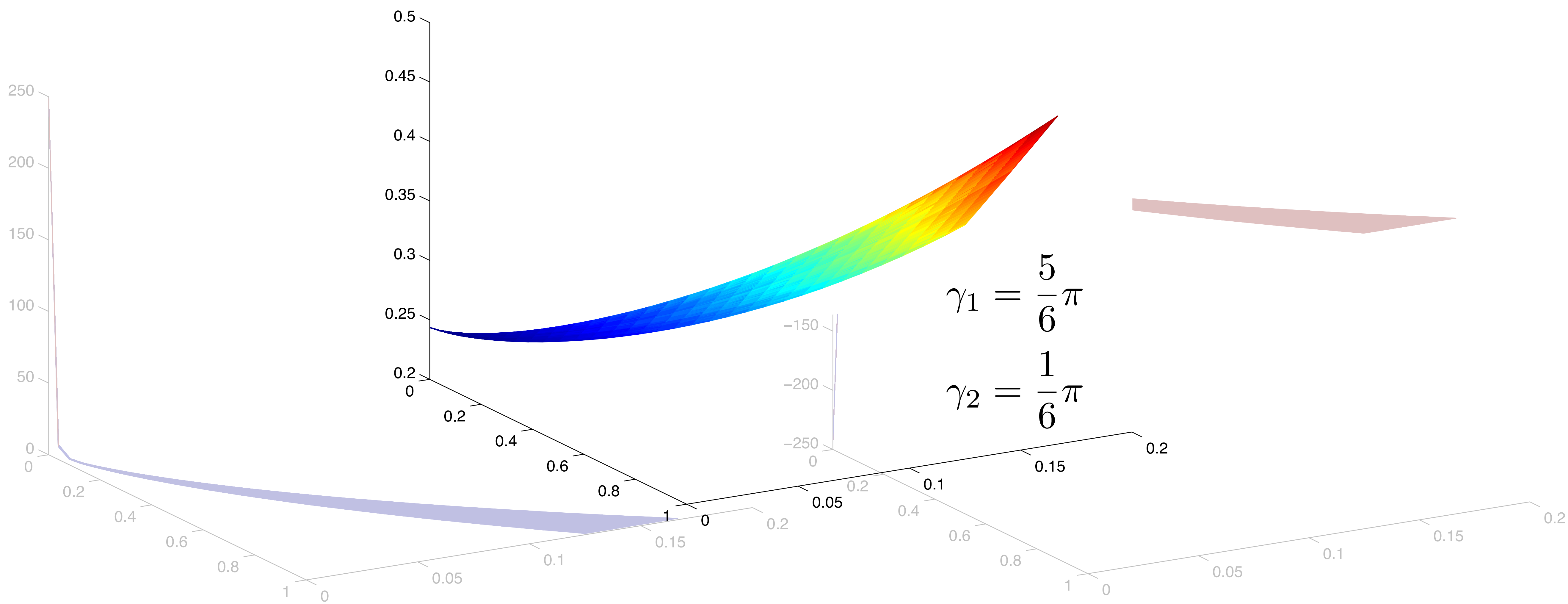


$$\gamma_1 = \frac{5}{6}\pi + \frac{\pi}{180}$$
$$\gamma_2 = \frac{1}{6}\pi$$

Open Problem 1: Is u bounded if $\cos\gamma_1 + \cos\gamma_2 = 0$,

and $f_{1,2}(x) \notin C^2([0,\infty))$.

$$f_1(x) = \frac{1}{6}x^{\frac{3}{2}} \qquad f_2(x) = \frac{1}{8}x^{\frac{3}{2}}$$



$$\gamma_1 = \frac{5}{6}\pi$$

$$\gamma_2 = \frac{1}{6}\pi$$

Open Problem 2:

What is the formal asymptotic series
for the double cusp domain?

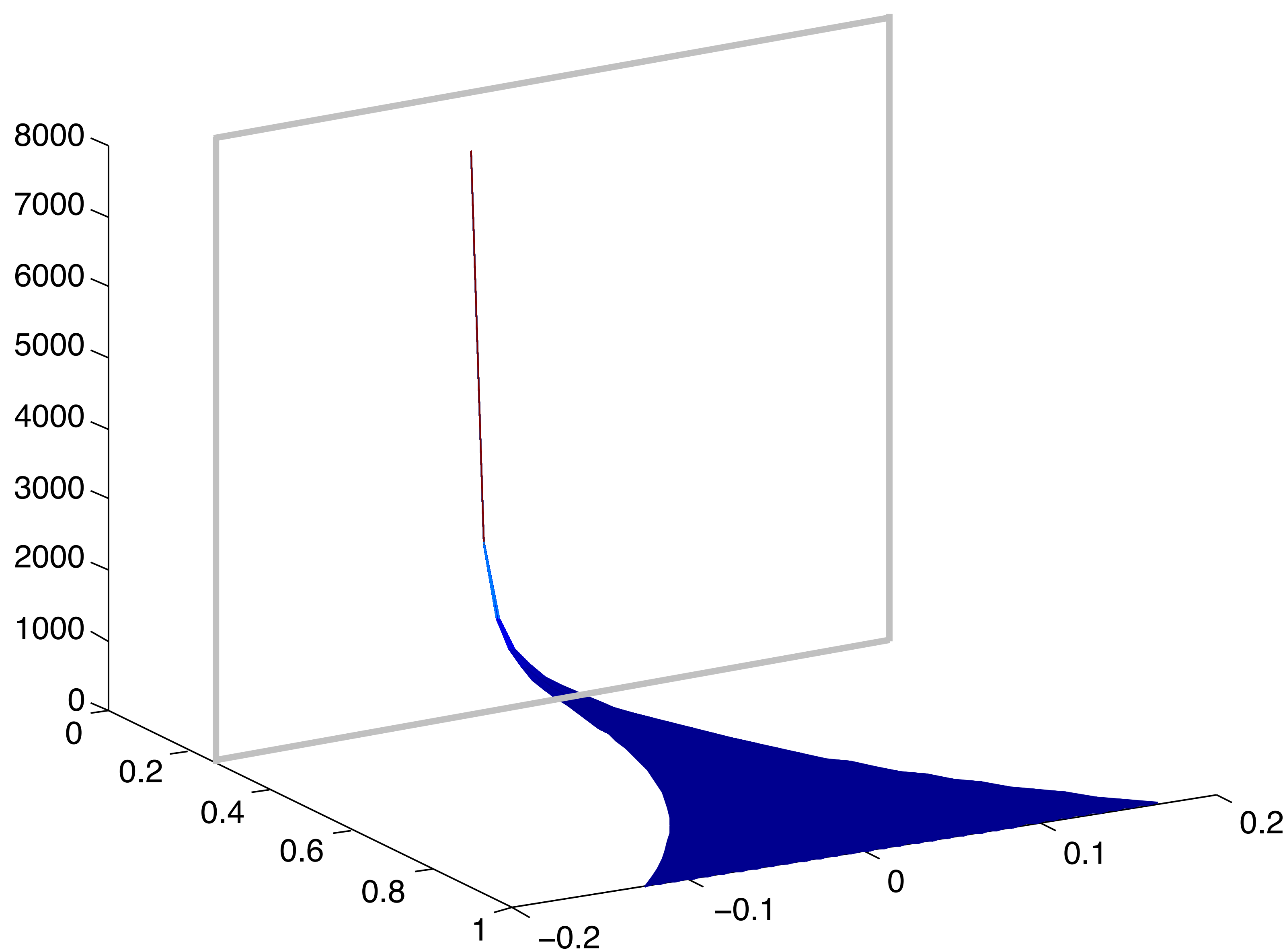
The formal asymptotic series solution for the **regular** cusp domain:

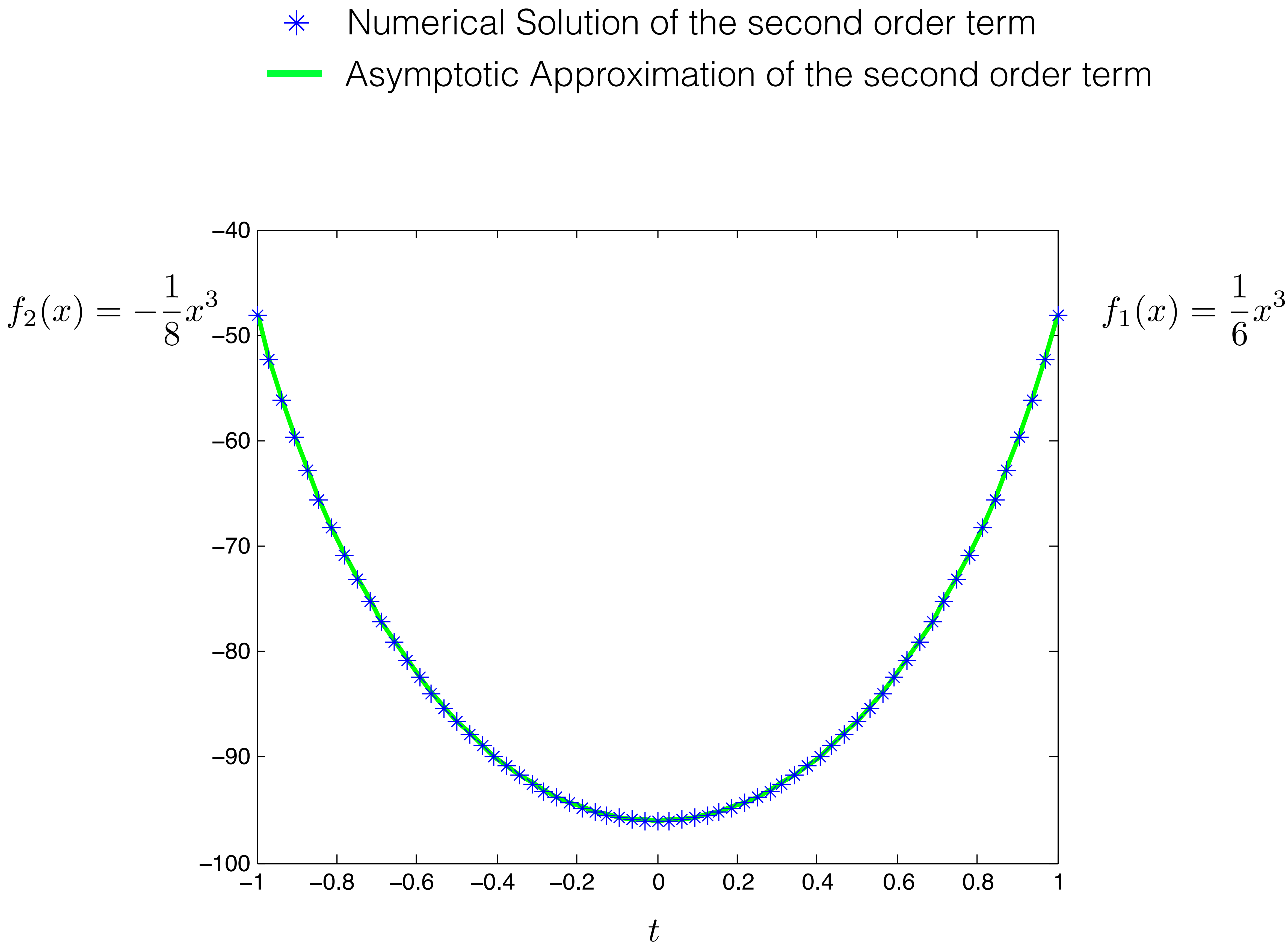
$$v = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

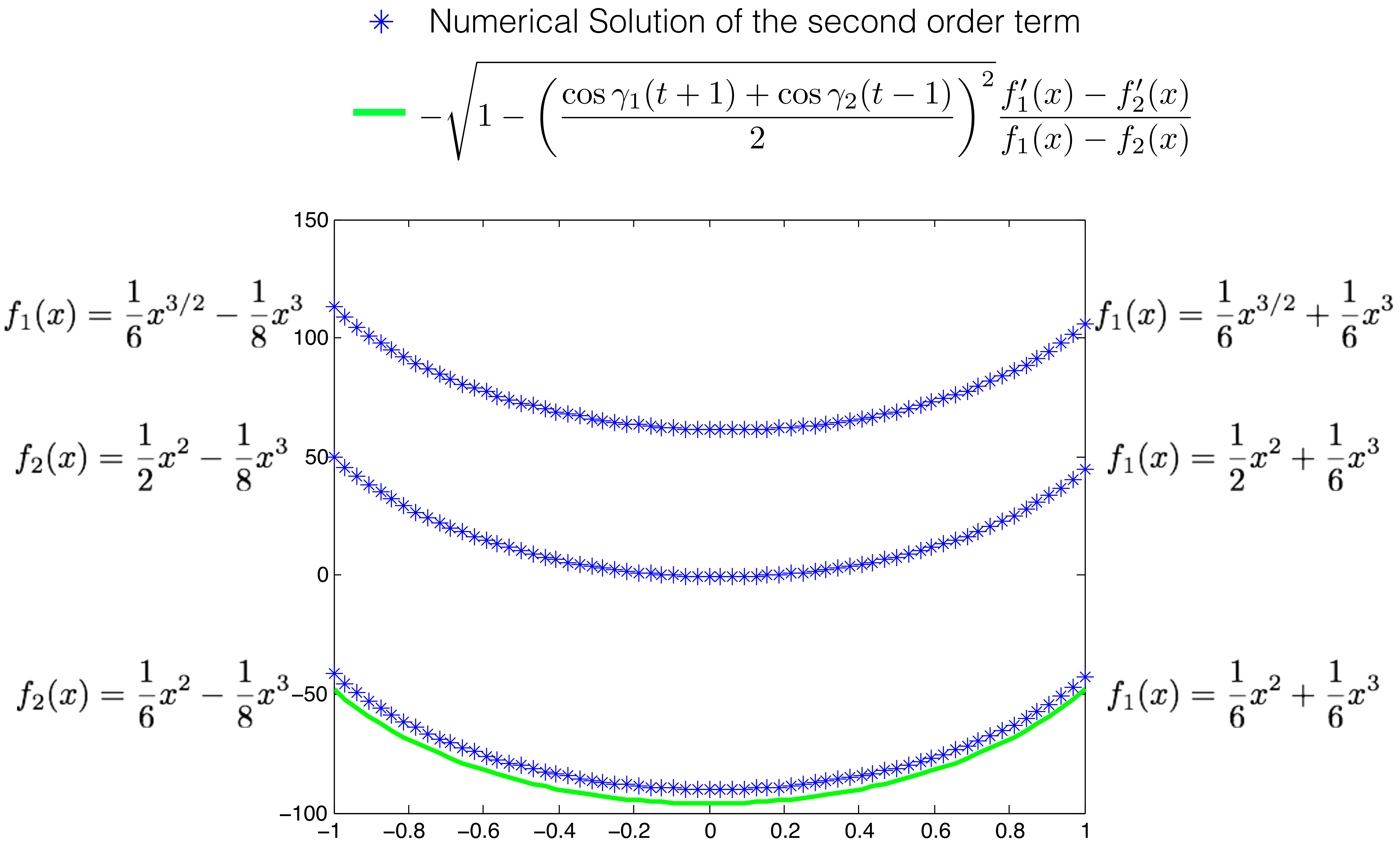
$$g(x,y) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2}$$

$$h(x,y) = -\frac{\cos\gamma_1 + \cos\gamma_2}{4}\left(\delta t + \frac{t^2}{2}\right) + \frac{1-\alpha}{2(\cos\gamma_1 + \cos\gamma_2}g(x,y)^2$$

$$t = \frac{2y - (f_1(x) + f_2(x))}{f_1(x) - f_2(x)}$$


8000
7000
6000
5000
4000
3000
2000
1000
0
0
0.2
0.4
0.6
0.8
1
−0.2
−0.1
0
0.1
0.2


Numerical Solution of the second order term
Asymptotic Approximation of the second order term
$f_2(x) = -\frac{1}{8}x^3$
$f_1(x) = \frac{1}{6}x^3$
−40
−50
−60
−70
−80
−90
−100
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
t



Numerical Solution of the second order term
$-\sqrt{1-\left(\frac{\cos\gamma_1(t+1)+\cos\gamma_2(t-1)}{2}\right)^2}\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)}$
$f_1(x)=\frac{1}{6}x^{3/2}-\frac{1}{8}x^3$
$f_1(x)=\frac{1}{6}x^{3/2}+\frac{1}{6}x^3$
$f_2(x)=\frac{1}{2}x^2-\frac{1}{8}x^3$
$f_1(x)=\frac{1}{2}x^2+\frac{1}{6}x^3$
$f_2(x)=\frac{1}{6}x^2-\frac{1}{8}x^3$
$f_1(x)=\frac{1}{6}x^2+\frac{1}{6}x^3$
150
100
50
0
−50
−100
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

Open Problem 2:

What is the formal asymptotic series
for the double cusp domain?

The formal asymptotic series solution for the **double cusp** domain:

$$v = \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} + g(x,y)\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} + h(x,y)\frac{(f_1'(x) - f_2'(x))^2}{f_1(x) - f_2(x)}$$

$$g(x,y) = -\sqrt{1 - \left(\frac{\cos\gamma_1(t+1) + \cos\gamma_2(t-1)}{2}\right)^2} + C$$

$$h(x,y) = ?$$

$$t = \frac{2y - (f_1(x) + f_2(x))}{f_1(x) - f_2(x)}$$

Conclusion

Asymptotic Analysis

Asymptotic Analysis

Change of Variable

Asymptotic Analysis

Change of Variable + Curvilinear Coordinate System

Asymptotic Analysis

Change of Variable + Curvilinear Coordinate System

Finite Volume Element method
or
Finite Element method

Asymptotic Analysis

Change of Variable + Curvilinear Coordinate System

Finite Volume Element method
or
Finite Element method

Numerical Approximation valid for the **entire** domain