Smoothed Aggregation and Algebraic Multigrid Methods for Markov Chains

Hans De Sterck, Killian Miller

Department of Applied Mathematics, University of Waterloo

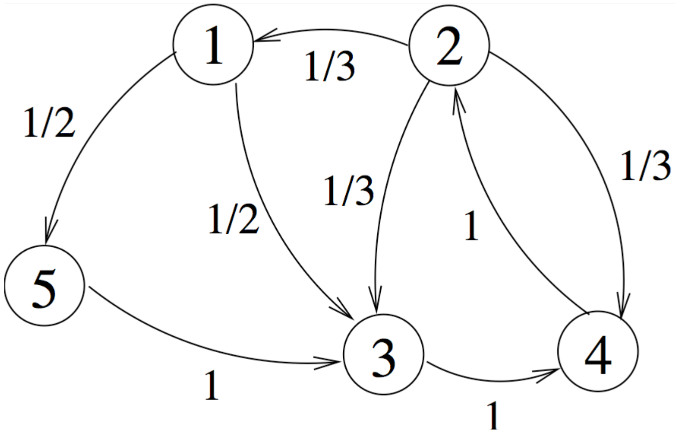
Steve McCormick, John Ruge, Tom Manteuffel, Geoff Sanders

Department of Applied Mathematics, University of Colorado at Boulder

University of Waterloo

SIAM CSE 2009

1. Simple Markov Chain Example



$$B = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/2 & 1/3 & 0 & 0 & 1 \\ 0 & 1/3 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- start in one state with probability 1: what is the stationary probability vector after ∞ number of steps?

$$\mathbf{x}_{i+1} = B\,\mathbf{x}_i$$

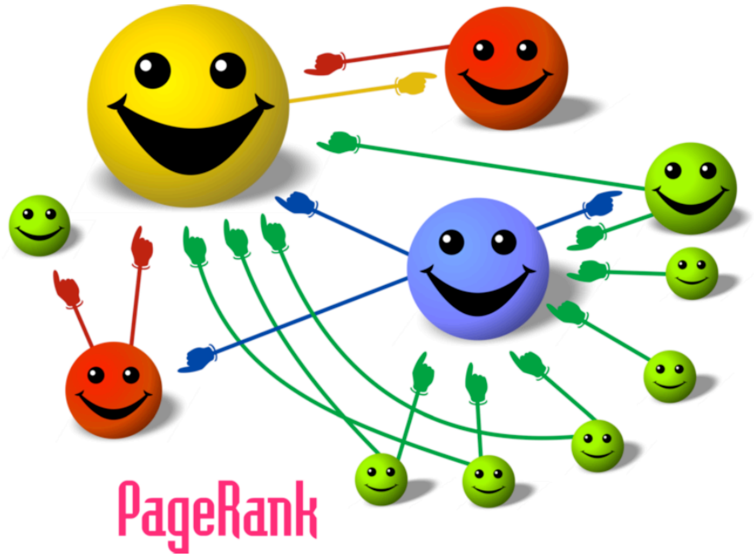
- stationary probability:

$$B\,\mathbf{x} = \mathbf{x} \qquad \|\mathbf{x}\|_1 = 1$$

$$\mathbf{x}^T = [2/19\ 6/19\ 4/19\ 6/19\ 1/19]$$

Applications of Markov Chains

- information retrieval and web ranking
- performance modelling of computer systems
- analysis of biological systems
- dependability and security analysis
- ...



2. Problem Statement

$$B\,\mathbf{x} = \mathbf{x} \qquad \|\mathbf{x}\|_1 = 1 \qquad x_i \geq 0\;\forall i$$

- *B* is column-stochastic

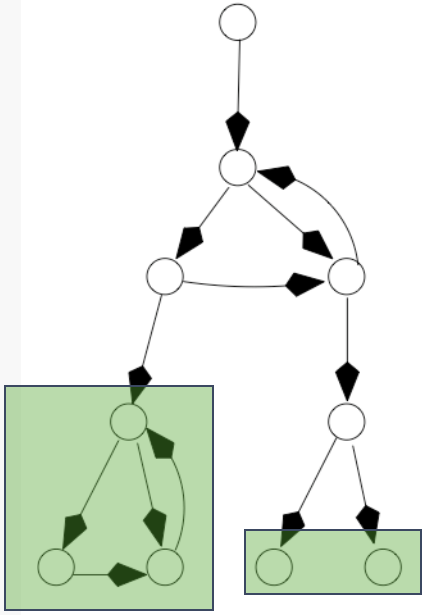
$$0 \leq b_{ij} \leq 1\;\forall i, j \qquad \mathbf{1}^T B = \mathbf{1}^T$$

- *B* is irreducible (every state can be reached from every other state in the directed graph)

$$\Longrightarrow$$

$$\exists!\;\mathbf{x} : \quad B\,\mathbf{x} = \mathbf{x} \qquad \|\mathbf{x}\|_1 = 1 \qquad x_i > 0\;\forall i$$

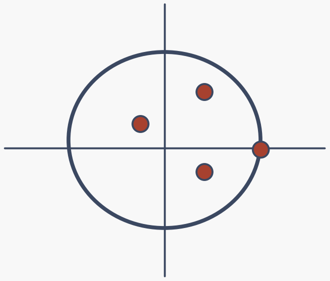
(no probability sinks!)



probability sinks not irreducible

3. Power Method

$B\,\mathbf{x} = \mathbf{x}$ or $(I - B)\,\mathbf{x} = 0$ or $A\,\mathbf{x} = 0$

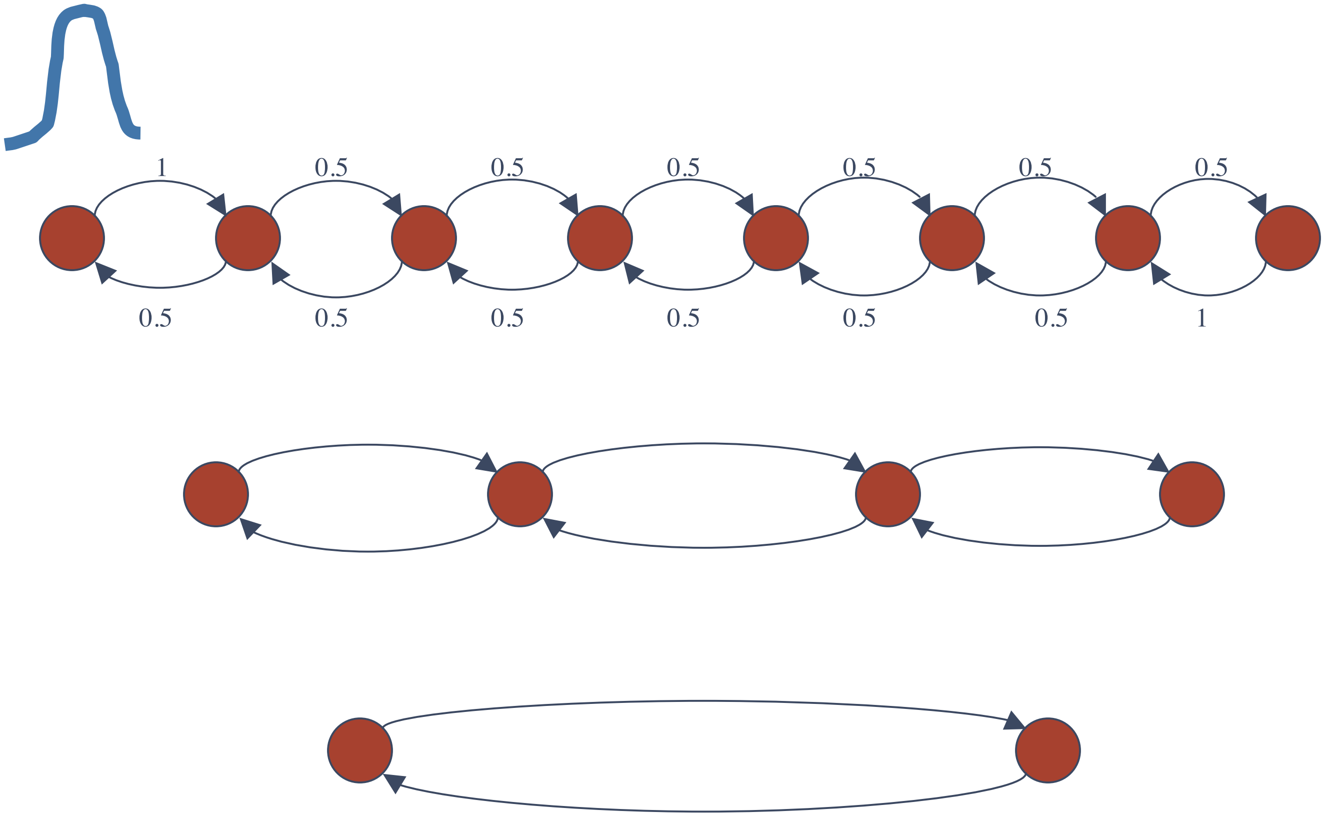


- largest eigenvalue of *B*: $\lambda_1 = 1$
- power method: $\mathbf{x}_{i+1} = B\mathbf{x}_i$
  - convergence factor: $|\lambda_2|$
  - convergence is very slow when

    $|\lambda_2| \approx 1$

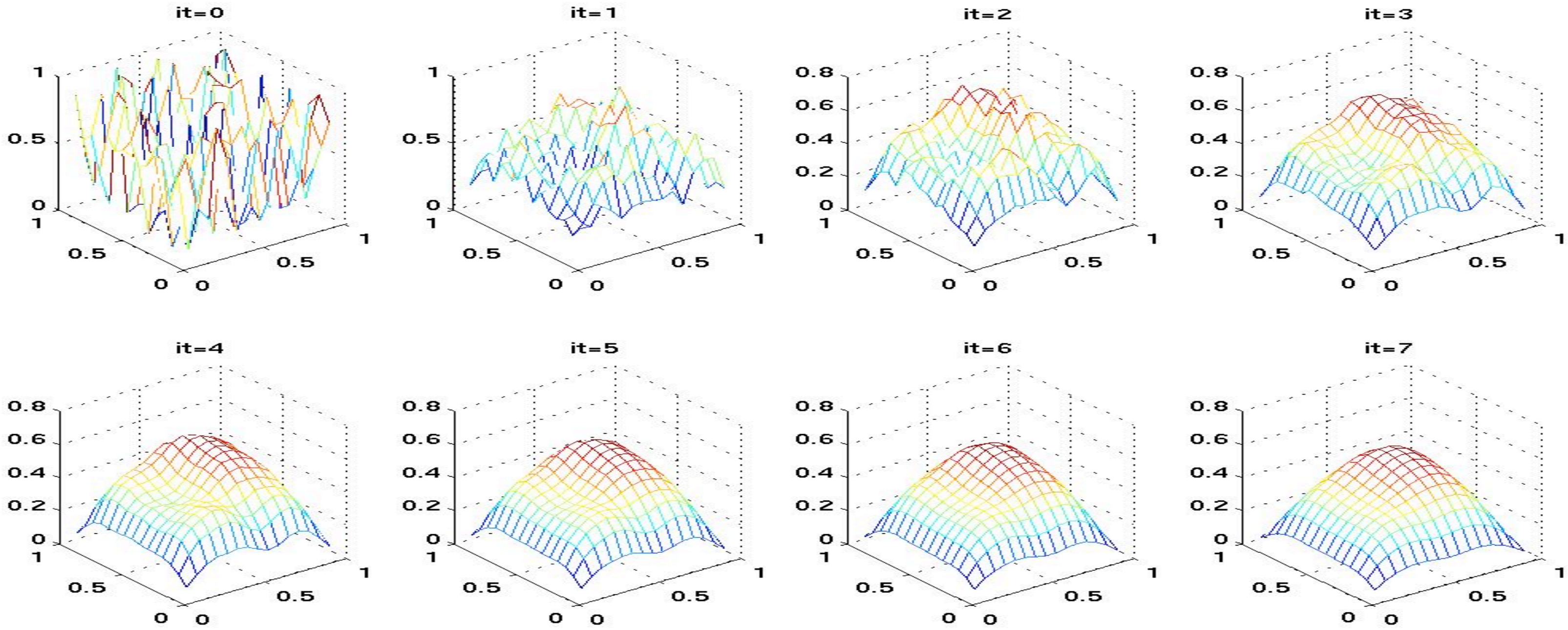
    (slowly mixing Markov chain) (JAC, GS also slow)

Why/When is Power Method Slow?
Why Multilevel Methods?



Principle of Multigrid (for PDEs)

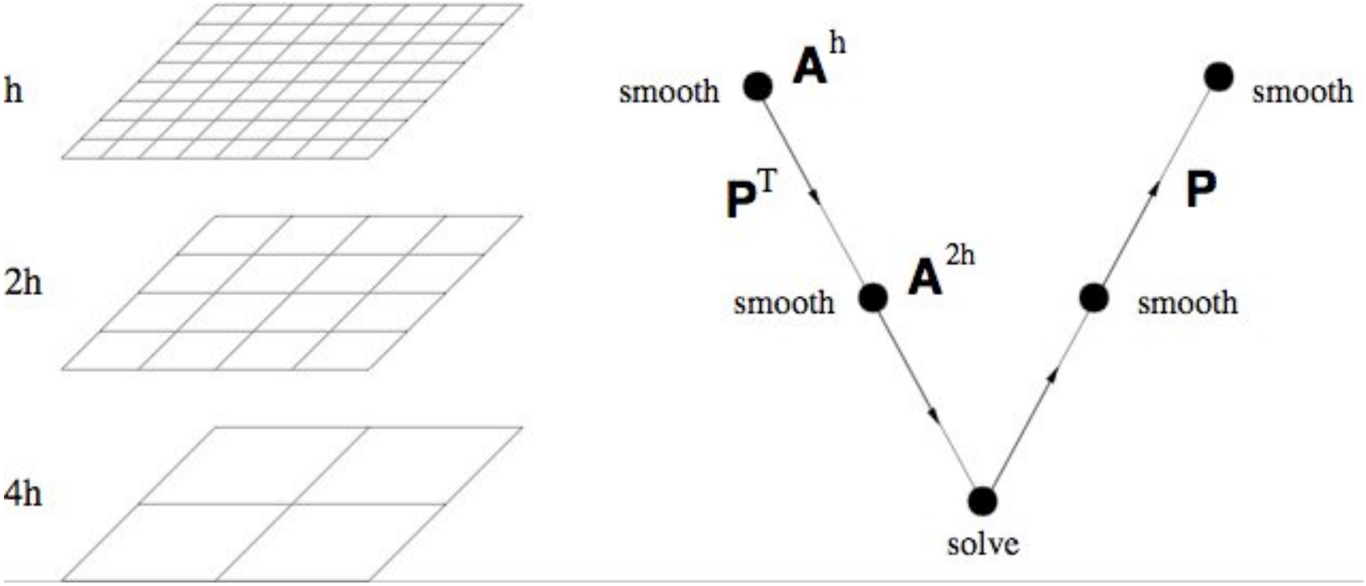
$$-u_{xx} - u_{yy} = f(x,y) \qquad Ax = b$$



- high-frequency error is removed by relaxation (weighted Jacobi, Gauss-Seidel, ... power method)
- low-frequency-error needs to be removed by coarse-grid correction

University of Waterloo

Multigrid Hierarchy: V-cycle



- multigrid V-cycle:
  - relax (=smooth) on successively coarser grids
  - transfer error using restriction ($R=P^T$) and interpolation (P)
- W=O(n) : (optimally) scalable method

SIAM CSE 2009
hdesterck@uwaterloo.ca

4. Aggregation for Markov Chains

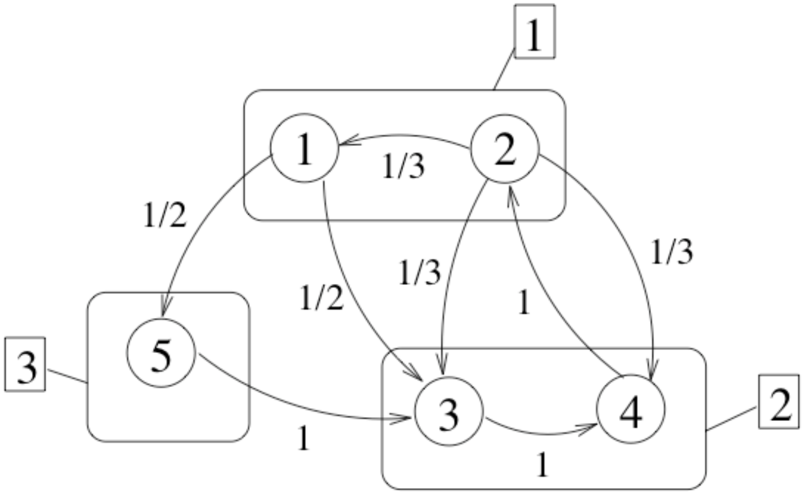
- form three coarse, aggregated states

$$x_{c,I} = \sum_{i \in I} x_i$$

$$\mathbf{x}_c^T = [8/19 \;\; 10/19 \;\; 1/19]$$

$$B_c \, \mathbf{x}_c = \mathbf{x}_c$$

$$b_{c,IJ} = \frac{\displaystyle\sum_{j \in J} x_j \left(\sum_{i \in I} b_{ij} \right)}{\displaystyle\sum_{j \in J} x_j}$$



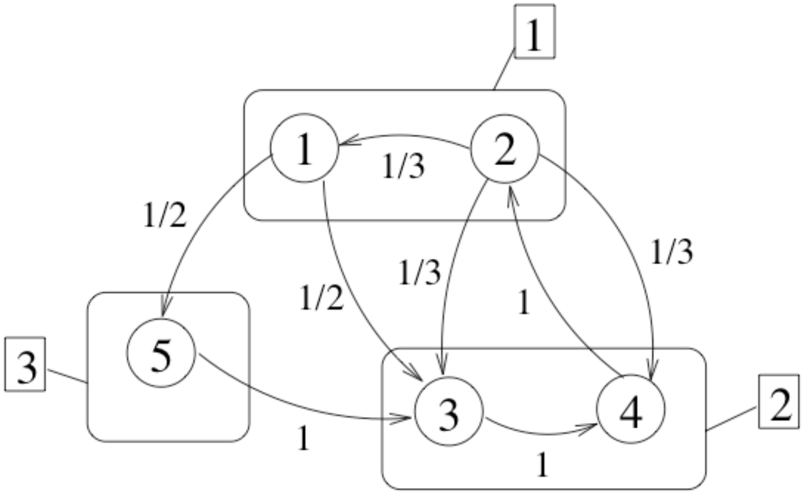
$$B_c = \begin{bmatrix} 1/4 & 3/5 & 0 \\ 5/8 & 2/5 & 1 \\ 1/8 & 0 & 0 \end{bmatrix}$$

(Simon and Ando, 1961)

Aggregation for Markov Chains

$$B_c \, \mathbf{x}_c = \mathbf{x}_c$$

$$b_{c,IJ} = \frac{\sum\limits_{j \in J} x_j \left(\sum\limits_{i \in I} b_{ij} \right)}{\sum\limits_{j \in J} x_j}$$



$$B_c = Q^T \, B \, \mathrm{diag}(\mathbf{x}) \, Q \, \mathrm{diag}(Q^T \, \mathbf{x})^{-1}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_{c,I} = \textstyle\sum_{i \in I} x_i$$

$$\mathbf{x}_c = Q^T \, \mathbf{x}$$

(Krieger, Horton, ... 1990s)

Error Equation

- multiplicative error: $\mathbf{x} = \text{diag}(\mathbf{x}_i)\, \mathbf{e}_i$
- error equation: $A\, \text{diag}(\mathbf{x}_i)\, \mathbf{e}_i = 0$

- coarse grid equation:

$$Q^T A\, \text{diag}(\mathbf{x}_i)\, Q\, \mathbf{e}_c = 0$$

$$A_c\, \mathbf{e}_c = 0$$

- restriction and interpolation:

$$R = Q^T \qquad P = \text{diag}(\mathbf{x}_i)\, Q$$

$$A_c = R\, A\, P$$

- coarse grid correction: $\mathbf{x}_{i+1} = P\, \mathbf{e}_c$

Error Equation

- important properties of A_c:

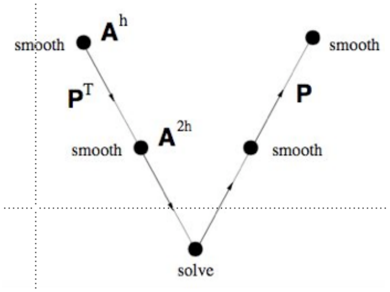
$$(1)\ \mathbf{1}_c^T A_c = 0 \quad \forall\, \mathbf{x}_i$$

$$(\text{since } \mathbf{1}_c^T R = \mathbf{1}^T \text{ and } \mathbf{1}^T A = 0)$$

$$(2)\ A_c \mathbf{1}_c = 0 \quad \text{for } \mathbf{x}_\text{i} = \mathbf{x}$$

Multilevel Aggregation Algorithm

Algorithm: Multilevel Adaptive Aggregation method (V-cycle)



```
$\mathbf{x} = \text{AM\_V}(A, \mathbf{x}, \nu_1, \nu_2)$
    begin
        $\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x})$    $\nu_1$ times
        build $Q$ based on $\mathbf{x}$ and $A$    ($Q$ is rebuilt every level and cycle)
        $R = Q^T$ and $P = \text{diag}(\mathbf{x})\, Q$
        $A_c = R\, A\, P$
        $\mathbf{x}_c = \text{AM\_V}(A_c\, \text{diag}(P^T\, \mathbf{1})^{-1}, P^T\, \mathbf{1}, \nu_1, \nu_2)$    (coarse-level solve)
        $\mathbf{x} = P\, (\text{diag}(P^T\, \mathbf{1}))^{-1} \mathbf{x}_c$    (coarse-level correction)
        $\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x})$    $\nu_2$ times
    end
```

(Krieger, Horton 1994, but no good way to build Q, convergence not good)

Algebraic Aggregation Mechanism

$\bar{A} = A\,\mathrm{diag}(\mathbf{x}_i)$ (scaled problem matrix)

$$S_{jk} = \begin{cases} 1 & \text{if } j \neq k \text{ and } -\bar{a}_{jk} \geq \theta \max\limits_{l \neq j}(-\bar{a}_{jl}) \\ 0 & \text{otherwise,} \end{cases}$$ (strength matrix)

```
ALGORITHM. AGGREGATION BASED ON STRENGTH MATRIX $S$
repeat
    • among the unassigned states, choose state $j$ which has the largest value
      in current iterate $\mathbf{x}_i$ as the seed point of a new aggregate
    • add all unassigned states $k$ that are strongly influenced by seed point $j$
      (i.e., $S_{kj} = 1$) to the new aggregate
until all states are assigned to aggregates
```

Well-posedness: Singular M-matrices

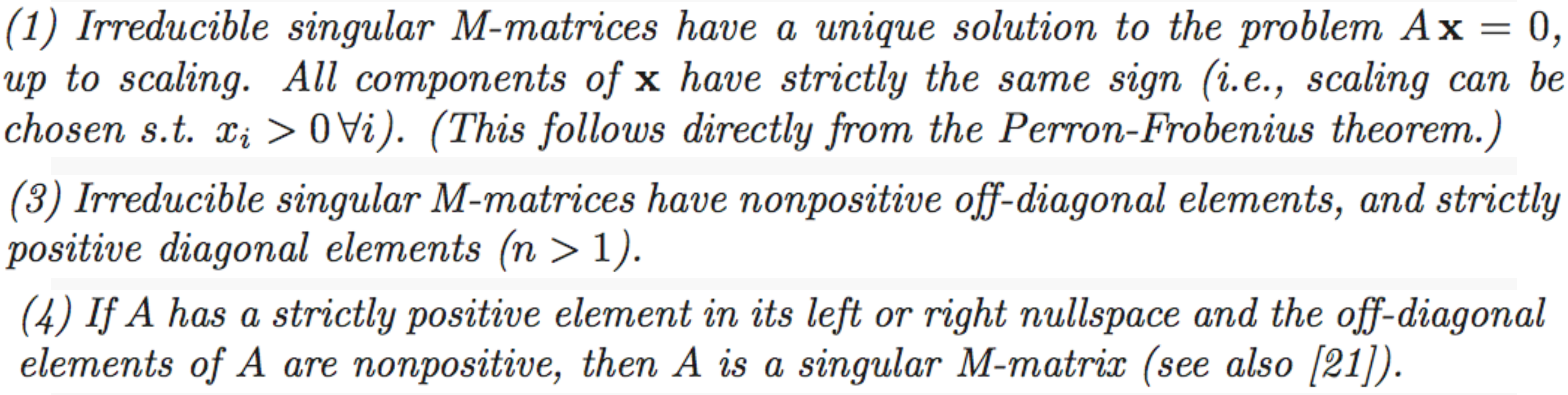
- singular M-matrix:

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

$A \in \mathbb{R}^{n\times n}$ *is a singular M-matrix* $\Leftrightarrow$

$$\exists\, B \in \mathbb{R}^{n\times n},\ b_{ij} \geq 0\,\forall i,j\ :\ A = \rho(B)\, I - B$$

- our *A*=*I*-*B* is a singular M-matrix on all levels



(1) Irreducible singular M-matrices have a unique solution to the problem $A\mathbf{x} = 0$, up to scaling. All components of $\mathbf{x}$ have strictly the same sign (i.e., scaling can be chosen s.t. $x_i > 0\,\forall i$). (This follows directly from the Perron-Frobenius theorem.)

(3) Irreducible singular M-matrices have nonpositive off-diagonal elements, and strictly positive diagonal elements ($n > 1$).

(4) If A has a strictly positive element in its left or right nullspace and the off-diagonal elements of A are nonpositive, then A is a singular M-matrix (see also [21]).

University of Waterloo

Well-posedness: Unsmoothed Method

THEOREM 3.1 (Singular M-matrix property of AM coarse-level operators). A_c *is an irreducible singular M-matrix on all coarse levels, and thus has a unique right kernel vector* $\mathbf{e}_c$ *with strictly positive components (up to scaling) on all levels.*

(1) $\mathbf{1}_c^T A_c = 0 \quad \forall\, \mathbf{x}_i$

(since $\mathbf{1}_c^T R = \mathbf{1}^T$ and $\mathbf{1}^T A = 0$)

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

THEOREM 3.2 (Fixed-point property of AM V-cycle). *Exact solution* $\mathbf{x}$ *is a fixed point of the AM V-cycle.*

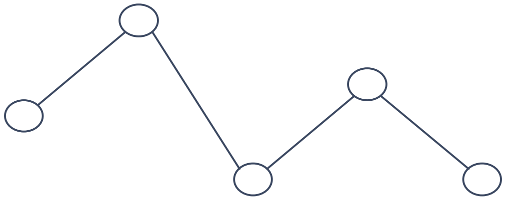
(2) $A_c \mathbf{1}_c = 0 \quad$ for $\mathbf{x}_\mathrm{i} = \mathbf{x}$

$$A_c \mathbf{e}_c = 0$$

$$\mathbf{x}_{i+1} = P\,\mathbf{e}_c$$

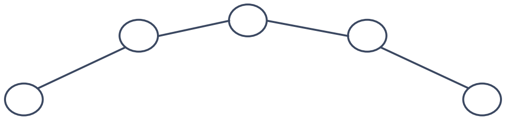
We Need ‘Smoothed Aggregation’...

(Vanek, Mandel, and Brezina, Computing, 1996)

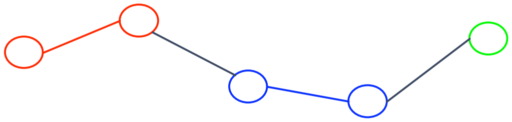


$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

after smoothing:

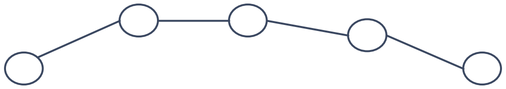


coarse grid correction with Q:



$$Q_s = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$$

coarse grid correction with Q_s:



Smoothed Aggregation

$$A = D - (L + U)$$

- smooth the columns of *P* with weighted Jacobi:

$$P_s = (1 - w)\,\mathrm{diag}(\mathbf{x}_i)\,Q + w\,D^{-1}\,(L + U)\,\mathrm{diag}(\mathbf{x}_i)\,Q$$

- smooth the rows of *R* with weighted Jacobi:

$$R_s = R\,(1 - w) + R\,w\,(L + U)\,D^{-1}$$

Smoothed Aggregation

- smoothed coarse level operator:

$$\begin{aligned} A_{cs} &= R_s \left(D - (L+U) \right) P_s \\ &= R_s \, D \, P_s - R_s \, (L+U) \, P_s \end{aligned}$$

$$\begin{aligned} \mathbf{1}_c^T \, A_{cs} &= 0 \quad \forall \, \mathbf{x}_i, \\ A_{cs} \, \mathbf{1}_c &= 0 \quad \text{for } \mathbf{x}_\mathrm{i} = \mathbf{x} \end{aligned}$$

- problem: A_{cs} is not a singular M-matrix (signs wrong)

- solution: lumping approach on S in

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

$$A_{cs} = S - G \qquad \hat{A}_{cs} = \hat{S} - G$$

SIAM CSE 2009
hdesterck@uwaterloo.ca

Smoothed Aggregation

$$A_{cs} = S - G \qquad \hat{A}_{cs} = \hat{S} - G$$

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

- we want as little lumping as possible
- only lump ‘offending’ elements *(i,j):*

$$\mathbf{1}_c^T\, \hat{A}_{cs} = 0 \quad \forall\, \mathbf{x}_i,$$
$$\hat{A}_{cs}\, \mathbf{1}_c = 0 \quad \text{for } \mathbf{x}_\text{i} = \mathbf{x}$$

$$s_{ij} \neq 0,\ i \neq j \text{ and } s_{ij} - g_{ij} \geq 0$$

(we consider both off-diagonal signs and reducibility here!)

- for ‘offending’ elements *(i,j)*, add $S_{\{i,j\}}$ to *S*:

$$S_{\{i,j\}} = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{array}{ccccc} & i & & j & \\ \end{array}$$

$$S_{\{i,j\}} = \begin{array}{c|ccccc|} & & i & & j & \\ & \ddots & \vdots & & \vdots & \\ i & \cdots & \beta_{\{i,j\}} & \cdots & -\beta_{\{i,j\}} & \cdots \\ & & \vdots & & \vdots & \\ j & \cdots & -\beta_{\{i,j\}} & \cdots & \beta_{\{i,j\}} & \cdots \\ & & \vdots & & \vdots & \end{array}$$

$$s_{ij} - g_{ij} - \beta_{\{i,j\}} < 0$$
$$s_{ji} - g_{ji} - \beta_{\{i,j\}} < 0$$

conserves both row and column sums

Lumped Smoothed Method is Well-posed

(A-SAM: Algebraic Smoothed Aggregation for Markov Chains)

THEOREM 4.1 (Singular M-matrix property of lumped SAM coarse-level operators). *$\hat{A}_{cs}$ is an irreducible singular M-matrix on all coarse levels, and thus has a unique right kernel vector $\mathbf{e}_c$ with strictly positive components (up to scaling) on all levels.*

THEOREM 4.2 (Fixed-point property of lumped SAM V-cycle). *Exact solution $\mathbf{x}$ is a fixed point of the SAM V-cycle (with lumping).*

$$\mathbf{1}_c^T \, \hat{A}_{cs} = 0 \quad \forall \mathbf{x}_i,$$
$$\hat{A}_{cs} \, \mathbf{1}_c = 0 \quad \text{for } \mathbf{x}_\mathrm{i} = \mathbf{x}$$

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

(De Sterck et al., SISC (accepted, 2009), ‘Smoothed aggregation multigrid for Markov chains’)

5. Algebraic Multigrid for Markov Chains

- scaled problem matrix: $\bar{A} := A\,\mathrm{diag}(\mathbf{x}_i)$

- multiplicative error eqaution: $\bar{A}\,\mathbf{e}_i = \mathbf{0}$

  At convergence, $\mathbf{1}$ lies in the nullspace of $\bar{A}$

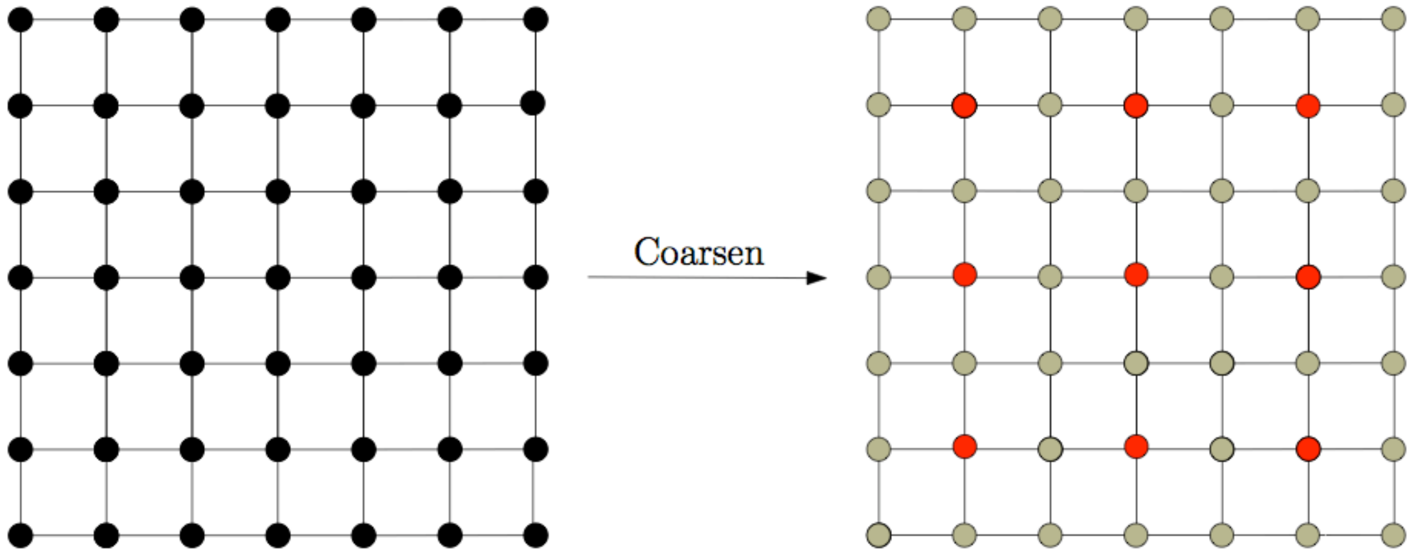
- we can use 'standard' AMG on $\bar{A}\,\mathbf{e}_i = \mathbf{0}$

- define AMG coarsening and interpolation

  $$R\,\bar{A}\,P\,\mathbf{e}_c = \mathbf{0} \quad \text{or} \quad \bar{A}_c\,\mathbf{e}_c = \mathbf{0}$$

  $$R = P^T$$

AMG (two-pass) Coarsening and Interpolation



Current fine level

Fine points F in grey
Coarse points C in red

$$(P\,\mathbf{e}_c)_i = \begin{cases} (\mathbf{e}_c)_i & \text{if } i \in C, \\ \sum_{j\in C_i} w_{ij}(\mathbf{e}_c)_j & \text{if } i \in F. \end{cases}$$

$$\bar{w}_{ij} = \frac{\bar{a}_{ij} + \sum_{m\in D_i^s}\left(\frac{\bar{a}_{im}\bar{a}_{mj}}{\sum_{k\in C_i}\bar{a}_{mk}}\right)}{\sum_{j\in C_i}\bar{a}_{ij} + \sum_{r\in D_i^s}\bar{a}_{ir}}$$

AMG Properties

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

- we can show: all elements of *P >= 0*
- lumping can be done as in the Smoothed Aggregation case:

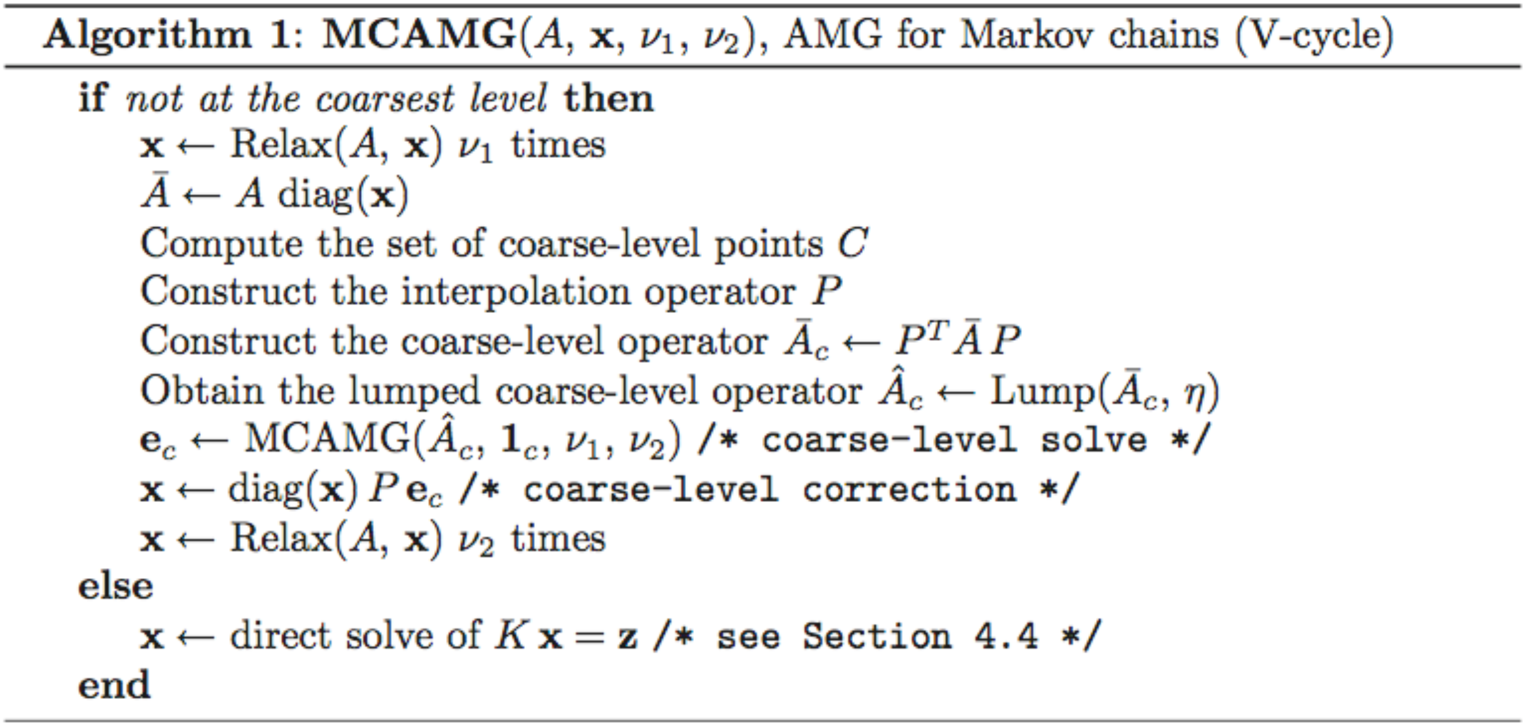
$$\bar{A}_c = P^T \bar{A}\, P = P^T D\, P - P^T (L + U)\, P = S - G$$

$$\hat{A}_c = \hat{S} - G.$$

- lumping conserves row and column sums:

$$\mathbf{1}_c^T \hat{A}_c = \mathbf{1}_c^T \bar{A}_c + \mathbf{1}_c^T (\hat{S} - S) = \mathbf{1}_c^T \bar{A}_c = \mathbf{0} \quad \forall\, \mathbf{x}_i,$$
$$\hat{A}_c\, \mathbf{1}_c = \bar{A}_c\, \mathbf{1}_c + (\hat{S} - S)\, \mathbf{1}_c = \bar{A}_c\, \mathbf{1}_c = \mathbf{0} \quad \text{for } \mathbf{x}_i = \mathbf{x}$$

Algebraic Multigrid for Markov Chains (MCAMG)



Algorithm 1: MCAMG(A, $\mathbf{x}$, ν_1, ν_2), AMG for Markov chains (V-cycle)

```
if not at the coarsest level then
    $\mathbf{x} \leftarrow$ Relax($A$, $\mathbf{x}$) $\nu_1$ times
    $\bar{A} \leftarrow A\,\mathrm{diag}(\mathbf{x})$
    Compute the set of coarse-level points $C$
    Construct the interpolation operator $P$
    Construct the coarse-level operator $\bar{A}_c \leftarrow P^T \bar{A}\, P$
    Obtain the lumped coarse-level operator $\hat{A}_c \leftarrow \mathrm{Lump}(\bar{A}_c, \eta)$
    $\mathbf{e}_c \leftarrow$ MCAMG($\hat{A}_c$, $\mathbf{1}_c$, $\nu_1$, $\nu_2$) /* coarse-level solve */
    $\mathbf{x} \leftarrow \mathrm{diag}(\mathbf{x})\, P\, \mathbf{e}_c$ /* coarse-level correction */
    $\mathbf{x} \leftarrow$ Relax($A$, $\mathbf{x}$) $\nu_2$ times
else
    $\mathbf{x} \leftarrow$ direct solve of $K\,\mathbf{x} = \mathbf{z}$ /* see Section 4.4 */
end
```

MCAMG Properties

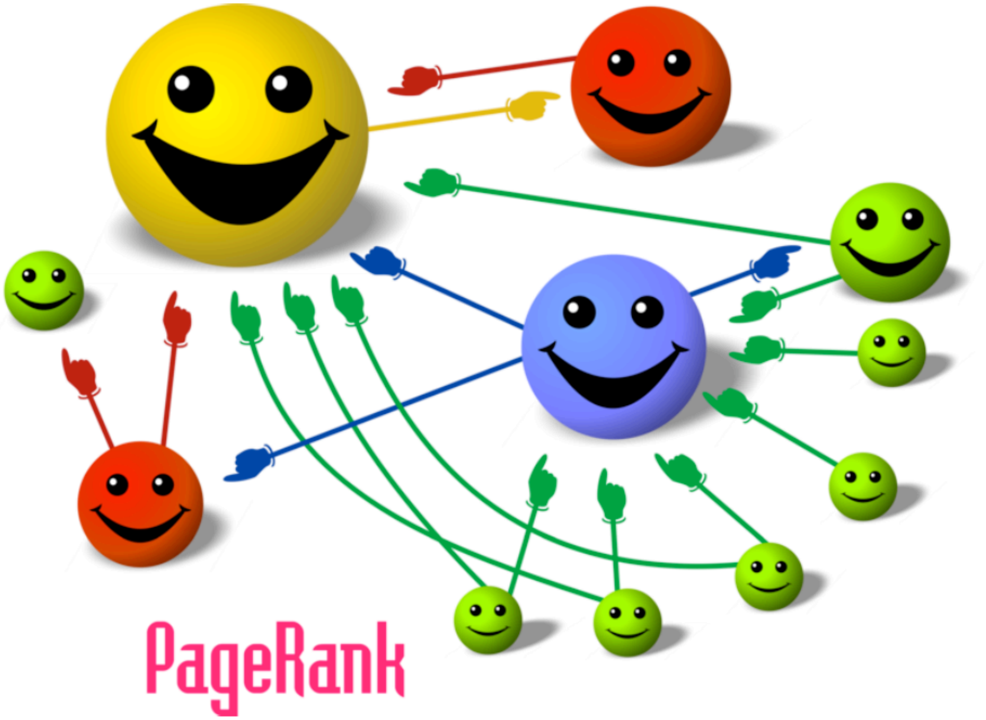
THEOREM 4.2 (Singular M-matrix property of lumped coarse-level operator). *$\hat{A}_c$ is an irreducible singular M-matrix on all coarse levels and, thus, has a unique right-kernel vector with positive components (up to scaling) on all levels.*

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

THEOREM 4.3 (Fixed-point property of MCAMG V-cycle). *The exact solution, $\mathbf{x}$, is a fixed point of the MCAMG V-cycle.*

(De Sterck et al., 'Algebraic Multigrid for Markov Chains', preprint)

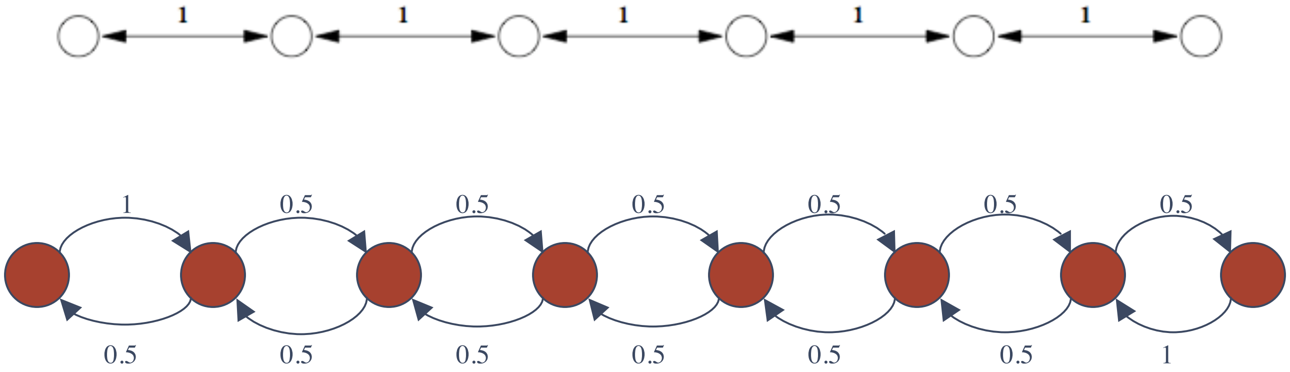
6. Test Problems



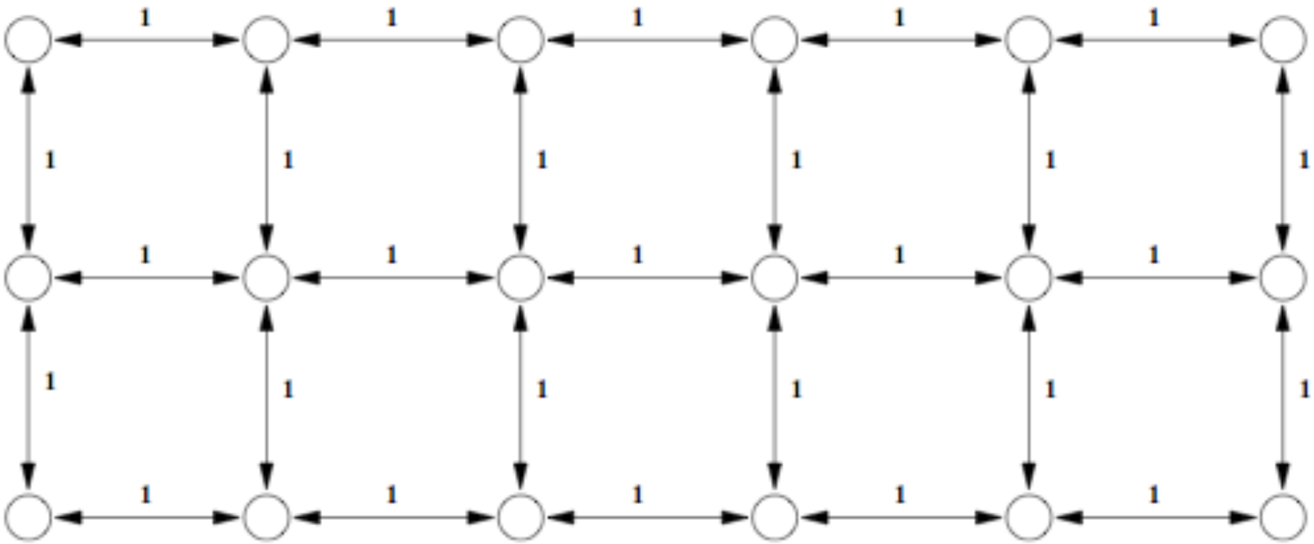
(De Sterck et al., SISC, 2008, ‘Multilevel adaptive aggregation for Markov chains, with application to web ranking’)

6.1 Uniform 1D Chain

- random walk on (undirected) graph
- all edges have the same weight
- transition probability for directed edge =
  weight of edge / sum of weights of outgoing edges
- solution trivial - test problem
- random walk on undirected graph gives real-spectrum *B*

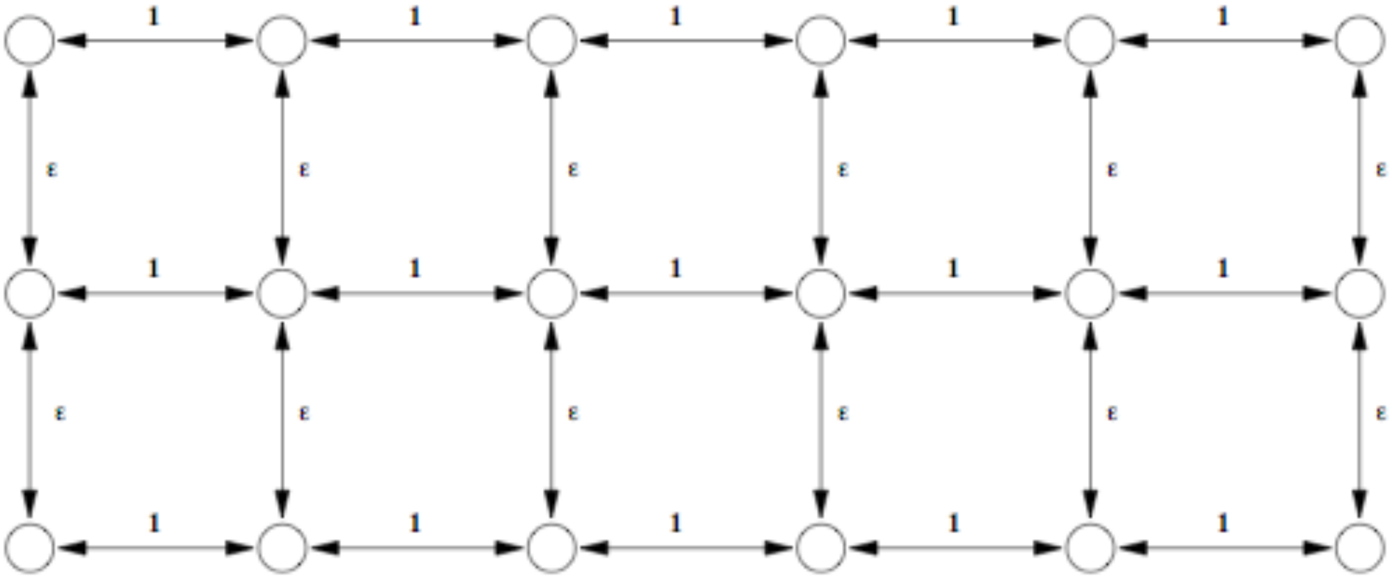


6.2 Uniform 2D Lattice

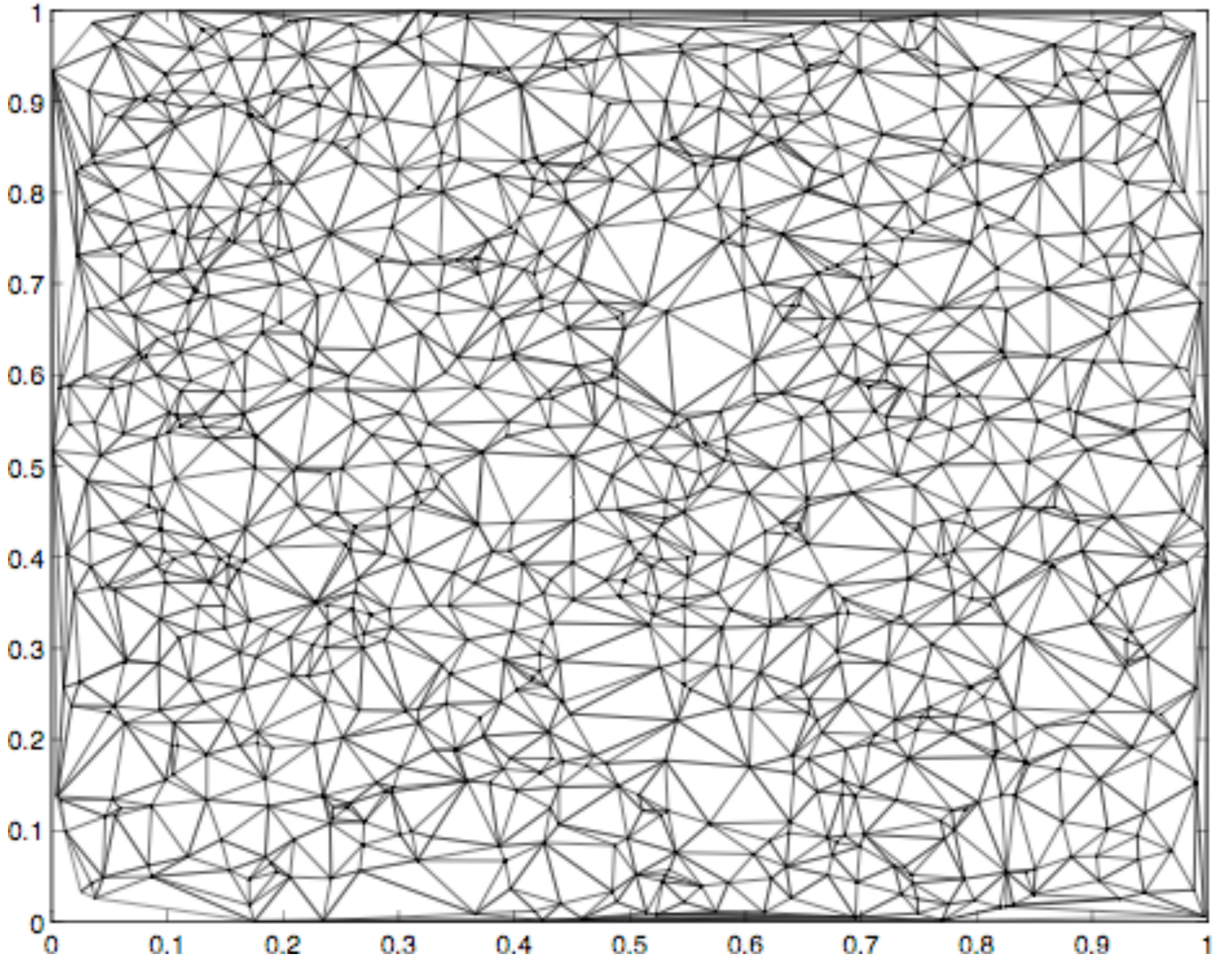


SIAM CSE 2009
hdesterck@uwaterloo.ca

6.3 Anisotropic 2D Lattice



6.4 Unstructured Planar Graph



Size of Subdominant Eigenvalue

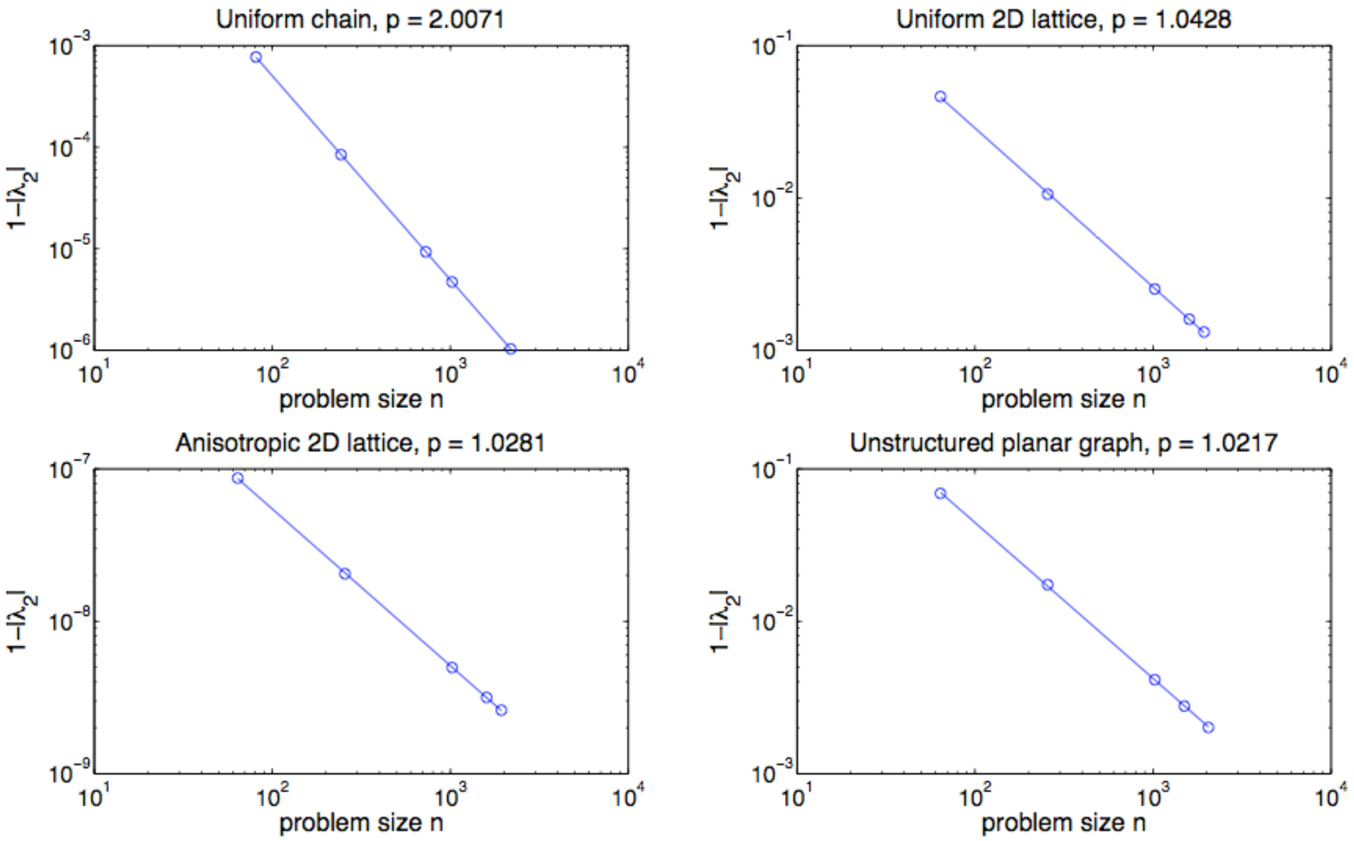


FIG. 5.1. *Magnitude of subdominant eigenvalue as a function of problem size.*

University of Waterloo

6.5 Tandem Queueing Network

FIG. 5.6. *Tandem queueing network.*

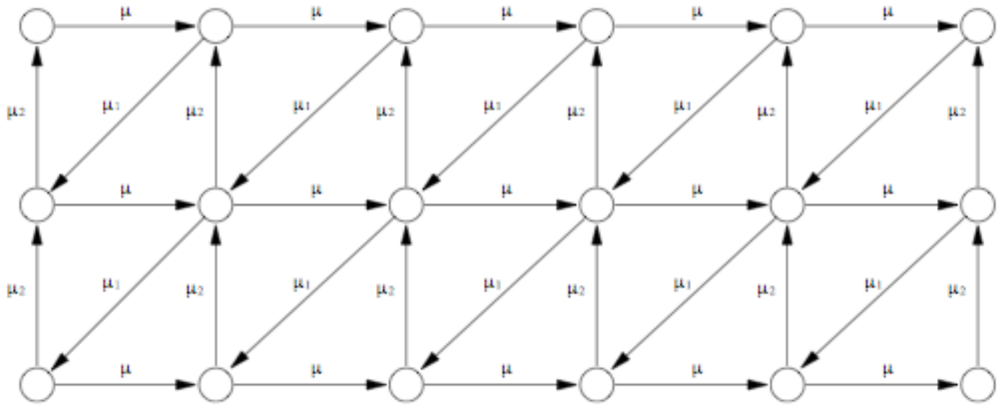
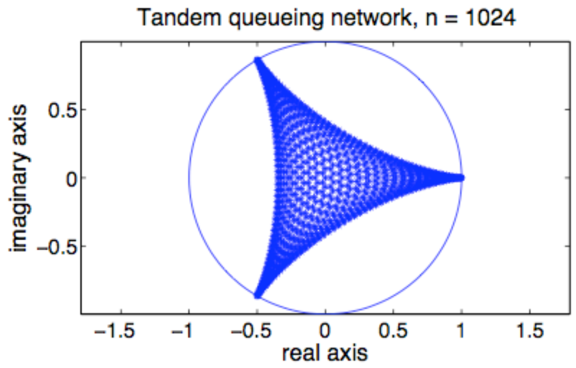
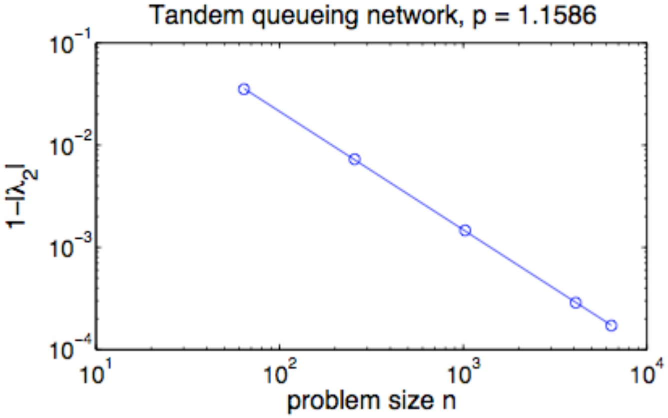


FIG. 5.7. *Graph for tandem queueing network.*

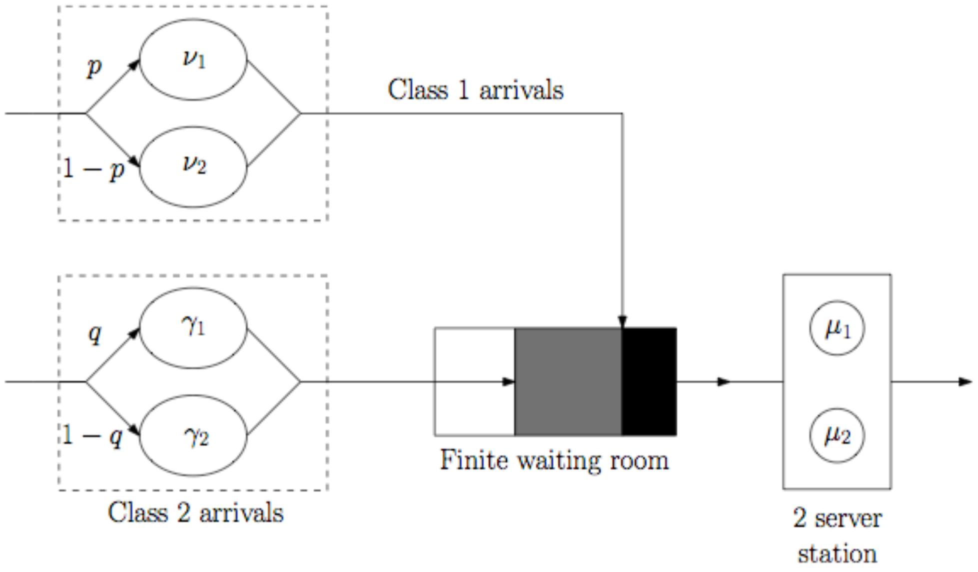


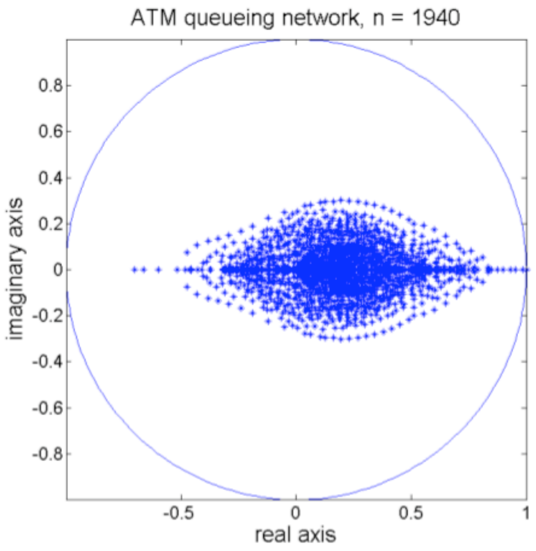
Tandem queueing network, n = 1024

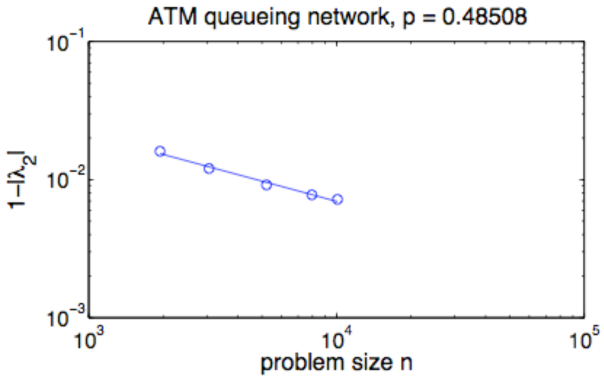


Tandem queueing network, p = 1.1586

6.6 ATM Queueing Network

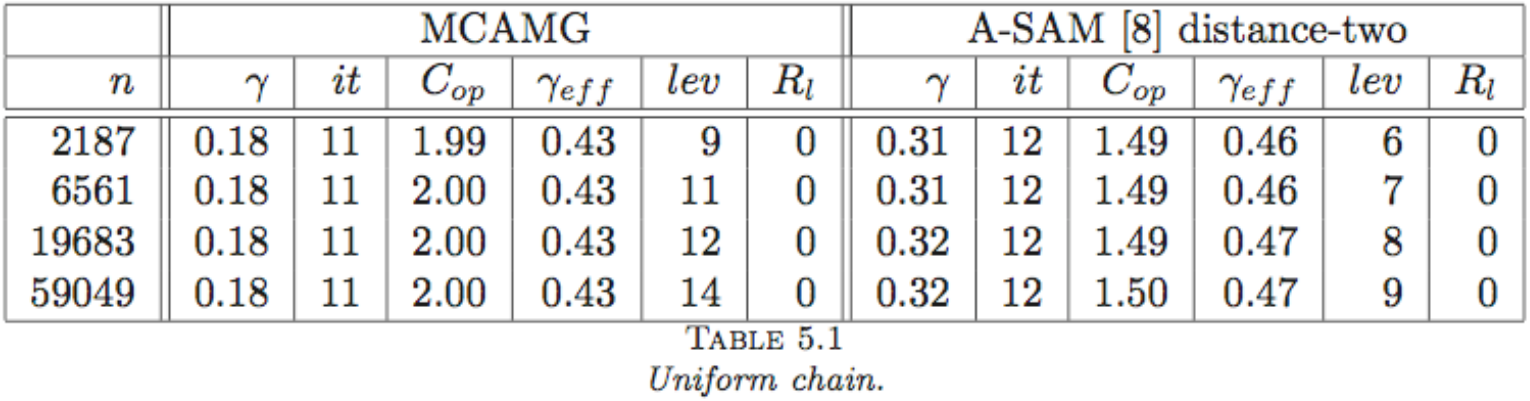






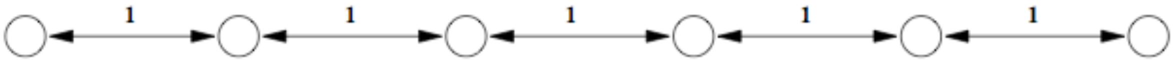
7. Numerical Results

7.1 Uniform 1D Chain

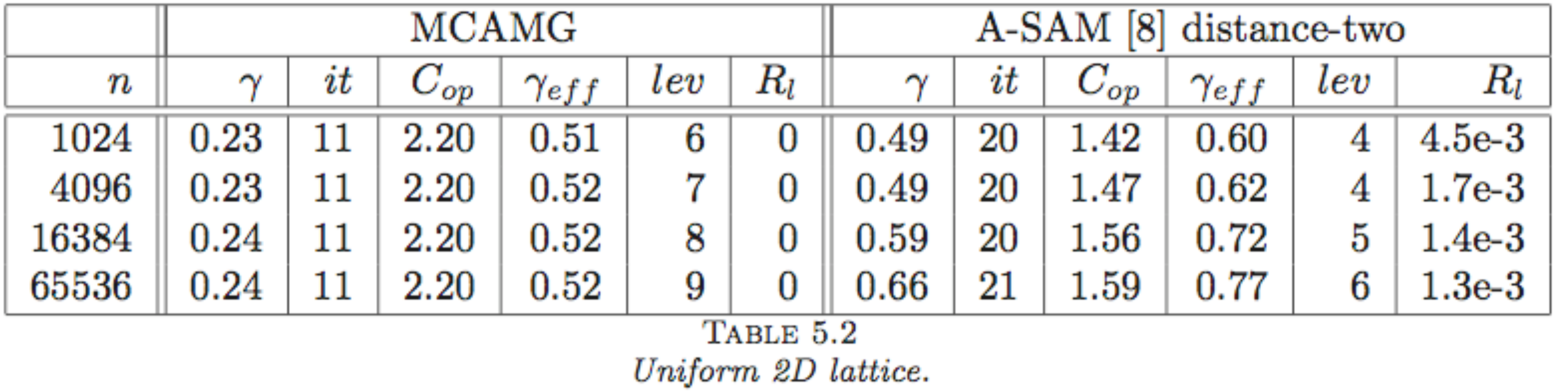


	MCAMG						A-SAM [8] distance-two					
n	γ	it	C_{op}	γ_{eff}	lev	R_l	γ	it	C_{op}	γ_{eff}	lev	R_l
2187	0.18	11	1.99	0.43	9	0	0.31	12	1.49	0.46	6	0
6561	0.18	11	2.00	0.43	11	0	0.31	12	1.49	0.46	7	0
19683	0.18	11	2.00	0.43	12	0	0.32	12	1.49	0.47	8	0
59049	0.18	11	2.00	0.43	14	0	0.32	12	1.50	0.47	9	0

TABLE 5.1
Uniform chain.

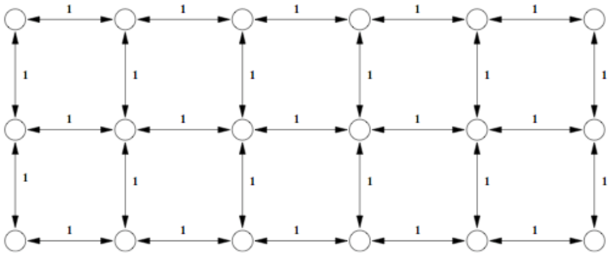


7.2 Uniform 2D Lattice

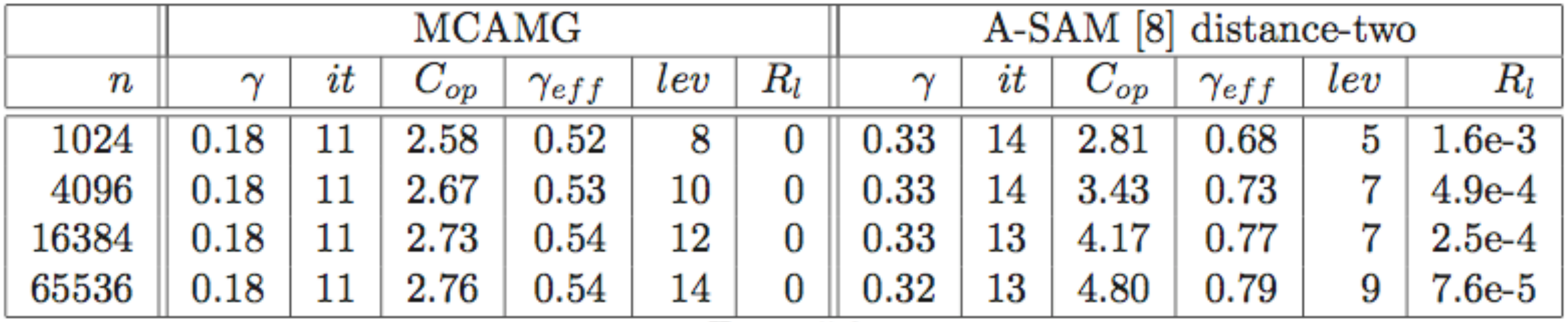


	MCAMG						A-SAM [8] distance-two					
n	γ	it	C_{op}	γ_{eff}	lev	R_l	γ	it	C_{op}	γ_{eff}	lev	R_l
1024	0.23	11	2.20	0.51	6	0	0.49	20	1.42	0.60	4	4.5e-3
4096	0.23	11	2.20	0.52	7	0	0.49	20	1.47	0.62	4	1.7e-3
16384	0.24	11	2.20	0.52	8	0	0.59	20	1.56	0.72	5	1.4e-3
65536	0.24	11	2.20	0.52	9	0	0.66	21	1.59	0.77	6	1.3e-3

TABLE 5.2
Uniform 2D lattice.

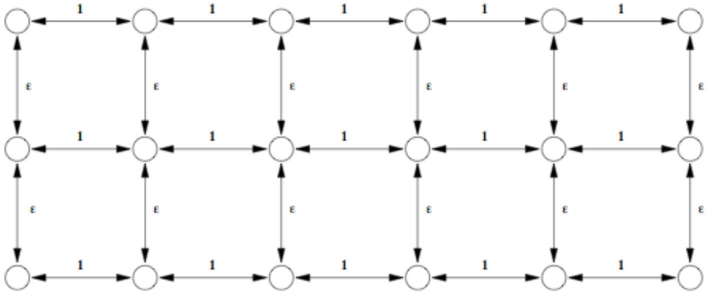


7.3 Anisotropic 2D Lattice



	MCAMG						A-SAM [8] distance-two					
n	γ	it	C_{op}	γ_{eff}	lev	R_l	γ	it	C_{op}	γ_{eff}	lev	R_l
1024	0.18	11	2.58	0.52	8	0	0.33	14	2.81	0.68	5	1.6e-3
4096	0.18	11	2.67	0.53	10	0	0.33	14	3.43	0.73	7	4.9e-4
16384	0.18	11	2.73	0.54	12	0	0.33	13	4.17	0.77	7	2.5e-4
65536	0.18	11	2.76	0.54	14	0	0.32	13	4.80	0.79	9	7.6e-5

TABLE 5.3
Anisotropic 2D lattice ($\varepsilon = 1e - 6$).



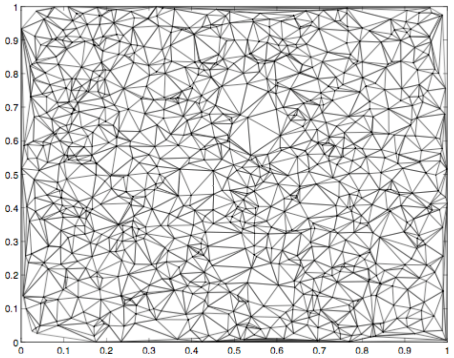
University of
Waterloo

7.4 Unstructured Planar Graph

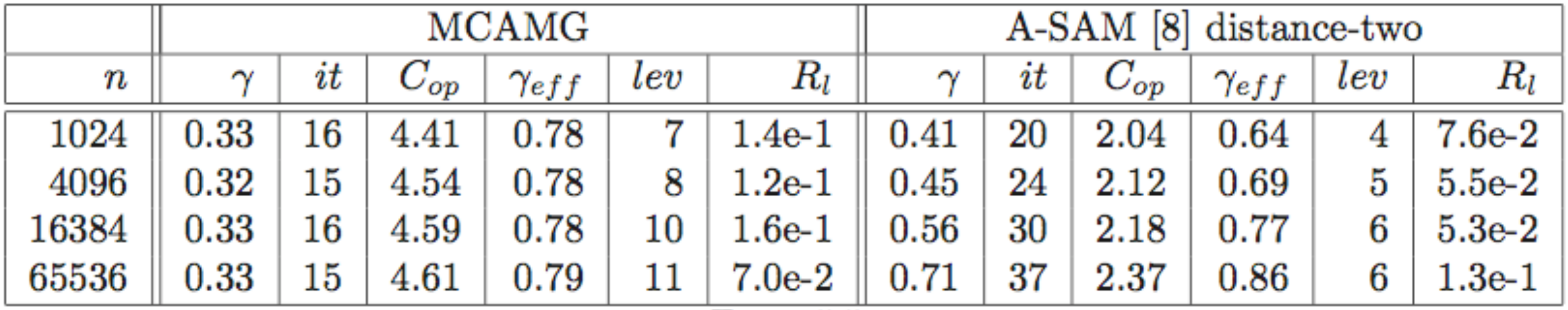
	MCAMG						A-SAM [8] distance-one					
n	γ	it	C_{op}	γ_{eff}	lev	R_l	γ	it	C_{op}	γ_{eff}	lev	R_l
1024	0.40	15	2.13	0.65	6	0	0.53	20	1.69	0.68	5	2.6e-2
2048	0.33	14	2.22	0.61	7	6.3e-5	0.52	19	1.68	0.68	5	2.1e-2
4096	0.40	15	2.19	0.66	7	6.3e-5	0.61	21	1.80	0.76	5	2.4e-2
8192	0.40	15	2.25	0.66	8	9.3e-5	0.64	22	1.92	0.79	7	2.5e-2
16384	0.37	14	2.26	0.65	9	7.0e-5	0.76	30	2.03	0.87	7	2.4e-2
32768	0.37	14	2.28	0.65	9	1.3e-4	0.74	28	2.08	0.86	7	2.4e-2

TABLE 5.4

Unstructured planar graph.



7.5 Tandem Queueing Network



	MCAMG						A-SAM [8] distance-two					
n	γ	it	C_{op}	γ_{eff}	lev	R_l	γ	it	C_{op}	γ_{eff}	lev	R_l
1024	0.33	16	4.41	0.78	7	1.4e-1	0.41	20	2.04	0.64	4	7.6e-2
4096	0.32	15	4.54	0.78	8	1.2e-1	0.45	24	2.12	0.69	5	5.5e-2
16384	0.33	16	4.59	0.78	10	1.6e-1	0.56	30	2.18	0.77	6	5.3e-2
65536	0.33	15	4.61	0.79	11	7.0e-2	0.71	37	2.37	0.86	6	1.3e-1

TABLE 5.5
Tandem queueing network.

FIG. 5.6. *Tandem queueing network.*

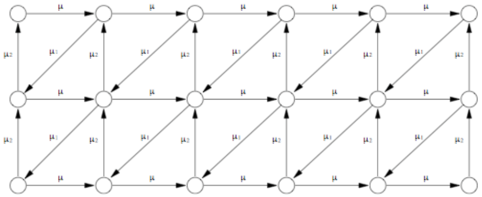
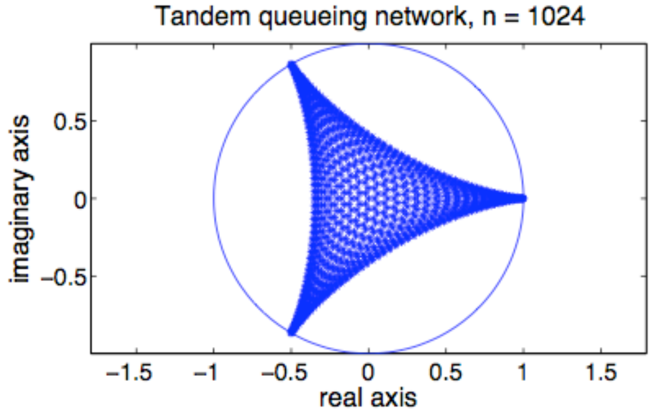


FIG. 5.7. *Graph for tandem queueing network.*



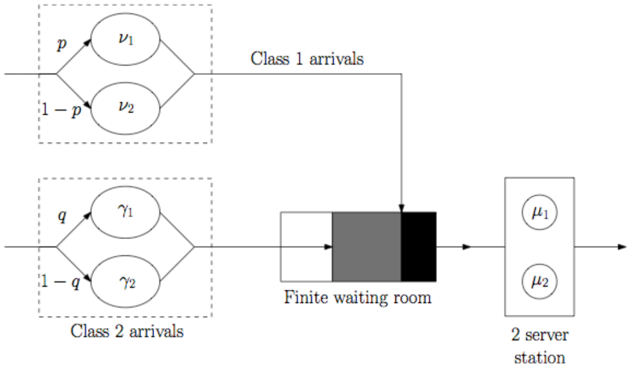
SIAM CSE 2009
hdesterck@uwaterloo.ca

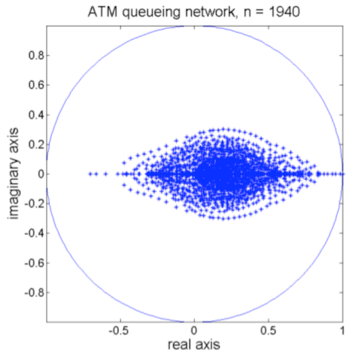
7.6 ATM Queueing Network (MCAMG)

n	γ	it	C_{op}	γ_{eff}	lev	R_l
1940	0.37	19	7.06	0.87	9	3.65e-2
3060	0.43	19	7.46	0.89	12	3.29e-2
5220	0.44	21	7.62	0.90	15	3.11e-2
10100	0.46	20	7.64	0.90	18	2.87e-2
13796	0.47	21	8.08	0.91	22	2.68e-2
19620	0.48	21	8.12	0.91	27	2.58e-2
32276	0.45	21	8.58	0.91	29	2.35e-2

TABLE 5.7

ATM queueing network.





8. Conclusions

- A-SAM (Smoothed Aggregation for Markov Chains) and MCAMG (Algebraic Multigrid for Markov Chains) are scalable: they are algorithms for calculating the stationary vector of slowly mixing Markov chains with near-optimal complexity
- smoothing is essential for aggregation for many problems
- appropriate theoretical framework (well-posedness)
- no theory yet on (optimal) convergence (non-symmetric matrices)
- this can be done in parallel
- other presentations in this mini-symposium: other multilevel methods for the stationary Markov problem

- Questions?