Efficient solution of stationary Euler flows with critical points and shocks

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1. Introduction

- consider stationary solutions of hyperbolic conservation law

\[
\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F}(U) = 0
\]

- in particular, compressible Euler equations

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \mathbf{v} \\ \frac{p}{\gamma-1} + \frac{\rho v^2}{2} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \mathbf{v} \\ \rho \mathbf{v}\mathbf{v} + I \rho \\ \frac{\gamma p}{\gamma-1} + \frac{\rho v^2}{2} \end{bmatrix} = 0
\]
Transonic steady Euler flows

\[ \frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0 \]

\[ \nabla \cdot \vec{F}(U) = 0 \]
Standard approach for steady flow simulation

- time marching (often implicit)
  \[
  \frac{U^{n+1} - U^n}{\Delta t} + \nabla \cdot \vec{F}(U^{n+1}) = 0
  \]

- **Newton**: linearize \( \vec{F} \)
- **Krylov**: iterative solution of linear system in every Newton step
- **Schwarz**: parallel (domain decompositioning), or multigrid

⇒ **NKS** methodology for steady flows
Main advantages of NKS

- use the hyperbolic BCs for steady problem
- ‘physical’ way to find suitable initial conditions for the Newton method in every timestep
- it works! (in the sense that it allows one to converge to a solution, in many cases, with some trial-and-error)
Disadvantages of NKS

- number of Newton iterations required for convergence can grow as a function of resolution
- number of Krylov iterations required for convergence of the linear system in each Newton step grows as a function of resolution
- grid sequencing/nested iteration: often does not work as well as it could (need many Newton iterations on each level)
- robustness, hard to find general strategy to increase timestep

→ NKS methodology not very scalable, and expensive
Why not solve the steady equations directly?

\[ \frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0 \quad \text{ and } \quad \nabla \cdot \vec{F}(U) = 0 \]

- too hard! (BCs, elliptic-hyperbolic, ...)
- let’s try anyway:
  - maybe we can understand why it is difficult
  - maybe we can find a method that is more efficient than implicit time marching
- start in 1D
2. 1D model problems

- radial outflow from extrasolar planet

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho r^2 \\ \rho u r^2 \\ (\frac{p}{\gamma - 1} + \frac{\rho u^2}{2}) r^2 \end{bmatrix} + \frac{\partial}{\partial r} \begin{bmatrix} \rho u r^2 \\ \rho u^2 r^2 + p r^2 \\ (\frac{\gamma p}{\gamma - 1} + \frac{\rho u^2}{2}) u r^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho GM + 2 pr \\ -\rho GM u + q_{heat} r^2 \end{bmatrix}
\]
Radial outflow from exoplanet

- http://exoplanet.eu
- 346 extrasolar planets known, as of April 2009
- 37 multiple planet systems
- many exoplanets are gas giants ("hot Jupiters")
- many orbit very close to star (~0.05 AU)
- hypothesis: strong irradiation leads to supersonic hydrogen escape
Transonic radial outflow solution of Euler equations of gas dynamics

subsonic ⇛|⇠ supersonic
Use time marching method (explicit)

\[ v - c = 0 \]
Simplified 1D problem: radial isothermal Euler

- 2 equations (ODEs), 2 unknowns ($u$, $\rho$)

\[
\frac{d}{dr}(\rho ur^2) = 0
\]

\[
\frac{du}{dr} = \frac{2 u c^2 (r-r_c)}{r^2 (u^2-c^2)}
\]
Solving the steady ODE system is hard...

- **critical point:**
  2 equations, 2 unknowns, but only 1 BC needed: $\rho_0$ ! (along with transonic solution requirement)
  (no $u_0$ required!)

- solving ODE from the left does not work...

- but... integrating outward from the critical point does work!!!

\[
\frac{d}{dr}(\rho ur^2) = 0 \\
\frac{du}{dr} = \frac{2 u c^2 (r-r_c)}{r^2 (u^2-c^2)}
\]
3. Newton Critical Point (NCP) method for steady transonic Euler flows

- First component of NCP: integrate outward from critical point, using dynamical systems formulation

\[ \frac{du}{dr} = \frac{2 u c^2 (r-r_c)}{r^2 (u^2-c^2)} \]
First component of NCP

\[ \frac{du}{dr} = \frac{2u c^2 (r-r_c)}{r^2 (u^2-c^2)} \]

1. Write as dynamical system...

\[ \begin{align*}
\frac{du(s)}{ds} &= -2u c^2 \left( r - \frac{GM}{2c^2} \right) \\
\frac{dr(s)}{ds} &= -r^2 (u^2 - c^2) \\
\frac{dV}{ds} &= G(V)
\end{align*} \]

2. find critical point: \( G(V) = 0 \)

3. linearize about critical point, eigenvectors

\[ \frac{\partial G}{\partial V}\bigg|_{V_{\text{crit}}} = \begin{bmatrix}
0 & 2c^3 \\
\frac{(GM)^2}{2c^3} & 0
\end{bmatrix} \]

4. integrate outward from critical point
For the Full Euler Equations

\[
\frac{d}{dr} \left[ \begin{array}{c}
\rho u r^2 \\
\rho u^2 r^2 + p r^2 \\
\left( \frac{\gamma p}{\gamma - 1} + \frac{\rho u^2}{2} \right) u r^2 
\end{array} \right] = \left[ \begin{array}{c}
0 \\
-\rho G M + 2 p r \\
-\rho G M u + q_{\text{heat}} r^2 
\end{array} \right]
\]

- 3 equations, 3 unknowns, but only 2 inflow BC \((\rho_0, p_0)\) 
  \((u_0 \text{ results from simulation})\)
- problem: there are many possible critical points! (two-parameter family)
Full Euler dynamical system

\[ \frac{dF}{ds} = 0, \]
\[ \frac{du}{ds} = 2u c^2 \left( r - \frac{GM}{2c^2} \right) - (\gamma - 1) q_{\text{heat}} \frac{r^4 u}{F}, \]
\[ \frac{dr}{ds} = r^2 \left( u^2 - c^2 \right), \]
\[ \frac{dT}{ds} = (\gamma - 1) T \left( GM - 2u^2 r \right) - \]
\[ (\gamma - 1) q_{\text{heat}} \frac{r^4}{F} (T - u^2). \]

\[ \Rightarrow \quad T_{\text{crit}} = \frac{GM}{2\gamma r_{\text{crit}}} + (\gamma - 1) \frac{q_{\text{heat}} r_{\text{crit}}^3}{2\gamma F_{\text{crit}}}, \]
\[ u_{\text{crit}} = \sqrt{\gamma T_{\text{crit}}}. \]
Second component of NCP: use Newton method to match critical point with BCs

guess initial critical point
1. use adaptive ODE integrator to find trajectory (RK45)
2. modify guess for critical point depending on deviation from desired inflow boundary conditions (2x2 Newton method)
   \[
   \text{find } C \text{ s.t. } B^* = F(C) \\
   C^{(k+1)} = C^{(k)} + (J_{C^{(k)}})^{-1} (B^* - B^{(k)})
   \]
3. repeat
Quadratic Newton convergence

<table>
<thead>
<tr>
<th>Newton step $k$</th>
<th>error $|B^{(k)} - B^*|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.41106268600662</td>
</tr>
<tr>
<td>2</td>
<td>2.28831581534917</td>
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<tr>
<td>3</td>
<td>1.43924405447424</td>
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<tr>
<td>4</td>
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<td>5</td>
<td>0.00125578478131</td>
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<tr>
<td>6</td>
<td>0.000000037420499</td>
</tr>
</tbody>
</table>
NCP method for 1D steady flows

- it is possible to solve steady equations directly, if one uses critical point and dynamical systems knowledge
- (Newton) iteration is still needed
- NCP Newton method solves a 2x2 nonlinear system (adaptive integration of trajectories is explicit)
- much more efficient than solving a 1500x1500 nonlinear system, and more well-posed


A fast and accurate algorithm for computing radial transonic flows
Hans De Sterck a, Scott Rostrup a, Feng Tian b
4. Extension to problems with shocks

- consider quasi-1D nozzle flow

\[
\frac{\partial}{\partial t} \begin{bmatrix}
    \rho A \\
    \rho u A \\
    \left( \frac{p}{\gamma-1} + \frac{\rho u^2}{2} \right) A
\end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix}
    \rho u A \\
    \rho u^2 A + p A \\
    \left( \frac{\gamma p}{\gamma-1} + \frac{\rho u^2}{2} \right) u A
\end{bmatrix} = \begin{bmatrix}
    0 \\
    p \frac{dA}{dx} \\
    0
\end{bmatrix}.
\]

\[
\Rightarrow \quad u_{crit} = \sqrt{\gamma T_{crit}} = c_{crit},
\]

\[
\frac{dA}{dx}(x_{crit}) = 0.
\]
NCP method for nozzle flow with shock (Scott Rostrup)

- subsonic in: 2 BC
- subsonic out: 1 BC

- NCP from critical point to match 2 inflow BC

- Newton method to match shock location to outflow BC (using Rankine-Hugoniot relations, 1 free parameter)
Other application: black hole accretion
Some thoughts

- positive at shock... (monotone, no oscillations)
- no limiter was required... (=no headache)
- as accurate as you want, with error control (adaptive RK45 in smooth parts, Newton with small tolerance at singularities)
- small Newton systems at singularities (one dimension smaller than problem)
- if only we could do something like this in 2D, 3D, time-dependent!
- ‘dream on...’ ;-)}
5. Extension to problems with heat conduction

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho r^2 \\ \rho u r^2 \\ \left(\frac{\rho}{\gamma-1} + \frac{\rho u^2}{2}\right) r^2 \end{bmatrix} + \frac{\partial}{\partial r} \begin{bmatrix} \rho u r^2 \\ \rho u^2 r^2 + \rho r^2 \\ \left(\frac{\gamma \rho}{\gamma-1} + \frac{\rho u^2}{2}\right) u r^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho GM + 2\rho r \\ -\rho GM u + q_{\text{heat}} r^2 + \frac{\partial}{\partial r} \left(\kappa r^2 \frac{\partial T}{\partial r}\right) \end{bmatrix}
\]
Dynamical system for Euler with heat conduction

$$\phi = \kappa r^2 \frac{dT}{dr}$$

\[
\begin{align*}
\frac{dr}{ds} &= -r^2(u^2 - c^2)(u^2 - T), \\
\frac{dF}{ds} &= 0, \\
\frac{du}{ds} &= -2uc^2 \left( r - \frac{GM}{2c^2} \right) (u^2 - T) + \frac{\phi u (u^2 - c^2)}{\kappa} \\
&\quad - (\gamma - 1)uT(GM - 2u^2r), \\
\frac{dT}{ds} &= -\phi (u^2 - c^2)(u^2 - T) \\
\frac{d\phi}{ds} &= \frac{\kappa}{(\gamma - 1)\kappa} \frac{\phi F (u^2 - c^2)^2}{\gamma} \\
&\quad + FT(GM - 2u^2r)(u^2 - c^2) + q_{\text{heat}}r^4(u^2 - c^2)(u^2 - T).
\end{align*}
\]
Two types of critical points!

- sonic critical point (node):
  \[ u_{\text{crit}} = \sqrt{\gamma} T_{\text{crit}} = c_{\text{crit}} \]

- thermal critical point (saddle point):
  \[ u_{\text{crit}} = \sqrt{T_{\text{crit}}} = c_{\text{crit}}/\sqrt{\gamma}, \]
  \[ \frac{\phi_{\text{crit}}}{\kappa} + GM - 2u_{\text{crit}}^2 r_{\text{crit}} = 0. \]
Transonic flow with heat conduction

- subsonic inflow: 3 BC ($\rho$, $p$, $\phi$)
- supersonic outflow: 0 BC
- 3-parameter family of thermal critical points
- NCP matches thermal critical point with 3 inflow BC
6. Some extensions being considered

- **viscosity:**
  - some preliminary investigation indicates that no new critical points are introduced
  - needs further investigation
- **robustness:**
  - Newton method can ‘shoot’ to negative density or pressure when approaching inner boundary
  - often, desired solutions lie very close to ‘border’ of feasible/physical parameter domain
  - need a more robust nonlinear system solver (line search, trust region, ...)
- if **topology** is not known in advance: level sets?
7. Extension to 2D, 3D: bow shock flows

- assume isothermal flow: \( \rho, u, v \)
- parametrize shock curve: \( r(\theta) \)
- discretize: \( r_i = r(\theta_i) \)
- given \( \rho_\infty, u_\infty, v_\infty \) and \( r(\theta) \), use RH relations to get \( \rho_r, u_r, v_r \)
- solve PDE using (nonlinear) FD method in smooth region on right of shock, with BC \( \rho_r, u_r, v_r \)
- adjust \( r_i \) until \( v_n = 0 \) at wall (1D Newton procedure on \( F(r_i) = 0 \), dense matrix)
- does not work since marching FD is unstable in elliptic region!
bow shock flows

• solution: solve PDE using (nonlinear) FD method in smooth region on right of shock, with BC $\rho_r, v_{n,r}, v_{\text{par},r} = 0$, this gives $v_{\text{par},r}^*$

• adjust $r_i$ until $v_{\text{par},r}^* = v_{\text{par},r}$ at shock (1D Newton procedure on $F(r_i)=0$, dense matrix)
bow shock flows

- we keep from 1D:
  - smaller-size Newton problem (1D instead of 2D)
  - we can use simple high-order FD method for smooth flow region
- worse than in 1D:
  - dense Jacobian
  - need to iterate to solve nonlinear PDE in smooth region
- this may work
- note similarity with shock capturing
- efficiency?; robustness?
Extension to 2D, 3D: critical curves

\[ \sin \Psi = 1 / M \]

\[ v_n - c = 0 \]

\[ r(\theta) \]

\[ \rho_0 \]

no \( u_0 \)!
Extension to 2D, 3D: critical curves

- assume isothermal flow: \( \rho, u, v \)
- simple case: \( v_\theta = 0 \)
- critical curve
  - transition from subsonic to supersonic
  - transition from elliptic to hyperbolic
  - limiting line for the characteristics
    (envelope of characteristics, \( v_n - c = 0 \))
- guess critical curve: \( r(\theta) \)
- discretize: \( r_i = r(\theta_i) \)
- solve PDE using (nonlinear) FD method in smooth region inside critical curve
- adjust \( r_i \) until boundary conditions are satisfied (1D Newton procedure on \( F(r_i) = 0 \), dense matrix)

\[
\sin \Psi = \frac{1}{M}
\]
Extension to 2D, 3D: critical curves

- assume isothermal flow: \( \rho, u, v \)
- general case: \( v_\theta \neq 0 \)
- critical curve
  - limiting line for the characteristics (envelope of characteristics, \( v_n - c = 0 \))
- critical curve
  - transition from subsonic to supersonic,
  - transition from elliptic to hyperbolic (\( v_{\text{tot}} - c = 0 \))
- guess critical curve: \( r(\theta) \), gives \( v_n, v_{\text{par}} \), guess \( v_{n0} \)
- solve PDE using (nonlinear) FD method in smooth region inside critical curve (can integrate through ell-hyp boundary), with BC \( v_n \rho_0, v_{n0} \), this gives \( v_{\text{par},0}^*, v_{\text{par}}^* \)
- adjust \( r_i \) and \( v_{n0} \) until \( v_{\text{par},0} = v_{\text{par},0}^*, v_{\text{par}} = v_{\text{par}}^* \)
  - (1D Newton procedure on \( F(r_i, v_{n0})=0 \), dense matrix)
Extension to 2D, 3D: critical curves

- guess critical curve: $r(\theta)$, gives $v_n, v_{\text{par}}$
- guess $v_{n0}$
- discretize: $r_i = r(\theta_i)$
- solve PDE using (nonlinear) FD method in smooth region inside critical curve (can integrate through ell-hyp boundary), with BC $v_n \rho_0, v_{n0}$, this gives $v_{\text{par},0}^*, v_{\text{par}}^*$
- adjust $r_i$ and $v_{n0}$ until $v_{\text{par},0} = v_{\text{par},0}^*, v_{\text{par}} = v_{\text{par}}^*$ (1D Newton procedure on $F(r_i, v_{n0})=0$, dense matrix)

- open problems:
  - how to derive $v_n, v_{\text{par}}$ from limiting line condition
  - how to continueate solution from limiting line (dynamical system?)
8. Conclusions

- solving steady Euler equations directly is superior to time-marching methods for 1D transonic flows

- NCP uses
  - adaptive integration outward from critical point
  - dynamical system formulation
  - 2x2 Newton method to match critical point with BC

- 1D: so what?
  - can use inefficient methods (?)
  - there are real 1D applications!
1D applications: exoplanet and early earth

THE ASTROPHYSICAL JOURNAL, 621:1049–1060, 2005 March 10
TRANSONIC HYDRODYNAMIC ESCAPE OF HYDROGEN FROM EXTRASOLAR PLANETARY ATMOSPHERES
Feng Tian,1,2 Owen B. Toon,2,3 Alexander A. Pavlov,2 and H. De Sterck4

A Hydrogen-Rich Early Earth Atmosphere
Feng Tian,1,2* Owen B. Toon,2,3 Alexander A. Pavlov,2 H. De Sterck3
Conclusions

- 2D, 3D, time-dependent: future work (‘dream on’ ;-) )
  - integrate separately in different domains of the flow, ‘outward’ from critical curves
  - match conditions at critical curves with BCs using Newton method
- issues:
  - change of topology (level sets?)
  - solve nonlinear PDEs in different regions (cost?)
  - smaller but dense Newton system
  - conditions at limiting lines and continuation?
  - time-dependent (do the same in space-time?)
Conclusions

• 2D, 3D, time-dependent : future work (‘dream on’ ;-) )
  – issues:
    – change of topology (level sets?)
    – solve nonlinear PDEs in different regions (cost?)
    – smaller but dense Newton system
    – conditions at limiting lines and continuation?
    – time-dependent (do the same in space-time?)
  – potential advantages are significant: problem more well-posed
    – fixed number of Newton steps, linear iterations (scalable)
    – better grid sequencing (nested iteration)
    – can use simple high-order methods in smooth flow, no limiters (at least not that headache)
Transiting exoplanet