Smoothed Aggregation Multigrid for Slowly Mixing Markov Chains

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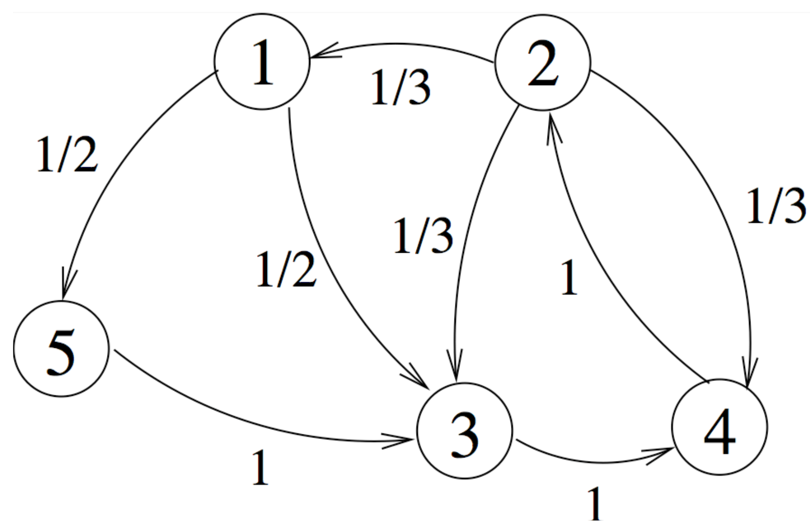
Tuesday seminar, 4 March 2008

1. Simple Markov Chain Example

- 5 states
- each outgoing edge same probability (random walk on directed graph)



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Simple Markov Chain Example

- start in one state with probability 1: what is the stationary probability vector after ∞ number of steps?

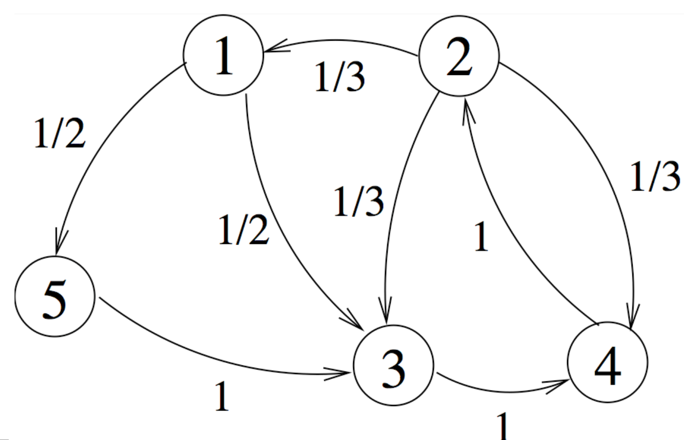
$$\mathbf{x}_{i+1} = B\,\mathbf{x}_i$$

- stationary probability:

$$B\,\mathbf{x} = \mathbf{x} \qquad \|\mathbf{x}\|_1 = 1$$

$$\mathbf{x}^T = [2/19\ 6/19\ 4/19\ 6/19\ 1/19]$$

$$B = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/2 & 1/3 & 0 & 0 & 1 \\ 0 & 1/3 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \end{bmatrix}$$



2. Problem Statement

$$B\,\mathbf{x} = \mathbf{x} \qquad \|\mathbf{x}\|_1 = 1 \qquad x_i \geq 0\,\forall i$$

- *B* is column-stochastic

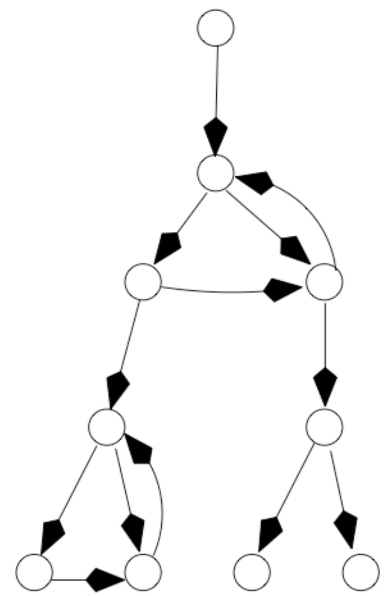
$$b_{i,j} \geq 0, \qquad \sum_i b_{i,j} = 1 \;\; \forall j$$

- *B* is irreducible (every state can be reached from every other state in the directed graph)

$$\Longrightarrow$$

$$\exists!\; \mathbf{x} : \quad B\,\mathbf{x} = \mathbf{x} \qquad \|\mathbf{x}\|_1 = 1 \qquad x_i > 0 \;\forall i$$

(no probability sinks!)



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3. Power Method

$$B\,\mathbf{x} = \mathbf{x} \quad \text{or} \quad (I - B)\,\mathbf{x} = 0 \quad \text{or} \quad A\,\mathbf{x} = 0$$

- largest eigenvalue of *B*: $\lambda_1 = 1$
- power method: $\mathbf{x}_{i+1} = B\mathbf{x}_i$
  - convergence factor: $|\lambda_2|$
  - convergence is very slow when

    $$|\lambda_2| \approx 1$$

    (slowly mixing Markov chain) (JAC, GS also slow)

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4. Aggregation for Markov Chains

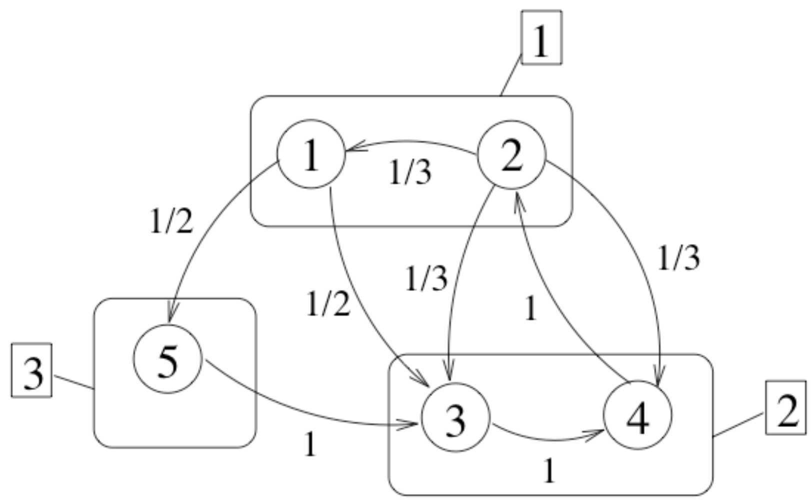
- form three coarse, aggregated states

$$x_{c,I} = \sum_{i \in I} x_i$$

$$\mathbf{x}_c^T = [8/19 \;\; 10/19 \;\; 1/19]$$

$$B_c \mathbf{x}_c = \mathbf{x}_c$$

$$b_{c,IJ} = \frac{\sum\limits_{j \in J} x_j \left(\sum\limits_{i \in I} b_{ij} \right)}{\sum\limits_{j \in J} x_j}$$



$$B_c = \begin{bmatrix} 1/4 & 3/5 & 0 \\ 5/8 & 2/5 & 1 \\ 1/8 & 0 & 0 \end{bmatrix}$$

(Simon and Ando, 1961)

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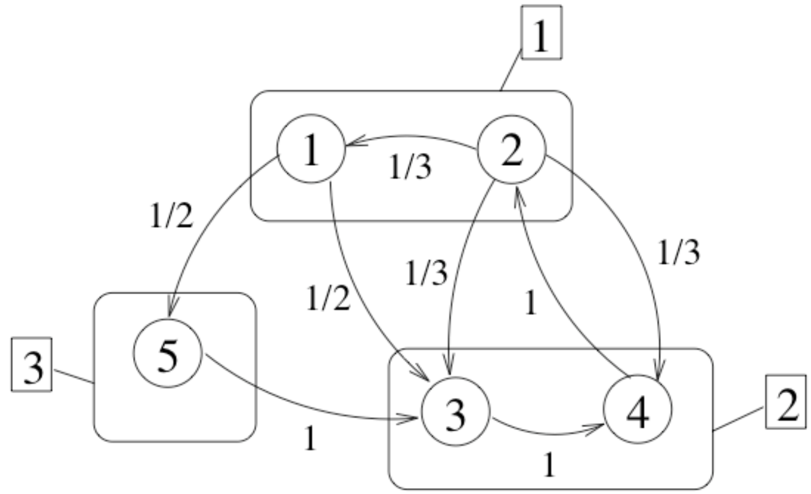
Aggregation for Markov Chains

$$B_c \mathbf{x}_c = \mathbf{x}_c$$

$$b_{c,IJ} = \frac{\sum_{j \in J} x_j \left(\sum_{i \in I} b_{ij} \right)}{\sum_{j \in J} x_j}$$



$$B_c = Q^T \, B \, \mathrm{diag}(\mathbf{x}) \, Q \, \mathrm{diag}(Q^T \mathbf{x})^{-1}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_{c,I} = \textstyle\sum_{i \in I} x_i$$

$$\mathbf{x}_c = Q^T \mathbf{x}$$

(Krieger, Horton, ... 1990s)

5. Error Equation

- multiplicative correction: error equation, coarse level error equation, and coarse grid correction

$$\mathbf{x} = \mathrm{diag}(\mathbf{x}_i)\,\mathbf{e}_i$$

$$A\,\mathrm{diag}(\mathbf{x}_i)\,\mathbf{e}_i = 0$$

$$Q^T A\,\mathrm{diag}(\mathbf{x}_i)\,Q\,\mathbf{e}_c = 0$$

$$A_c\,\mathbf{e}_c = 0$$

$$R = Q^T \qquad P = \mathrm{diag}(\mathbf{x}_i)\,Q$$

$$A_c = R\,A\,P$$

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Error Equation

- coarse grid correction

$$\mathbf{x} = \text{diag}(\mathbf{x}_i)\, \mathbf{e}_i$$
$$A\, \text{diag}(\mathbf{x}_i)\, \mathbf{e}_i = 0$$

$$Q^T\, A\, \text{diag}(\mathbf{x}_i)\, Q\, \mathbf{e}_c = 0$$
$$A_c\, \mathbf{e}_c = 0$$

$$R = Q^T \qquad P = \text{diag}(\mathbf{x}_i)\, Q$$
$$A_c = R\, A\, P$$

$$\mathbf{x}_c = \text{diag}(P^T\, \mathbf{1})\, \mathbf{e}_c$$
$$A_c\, (\text{diag}(P^T\, \mathbf{1}))^{-1}\, \mathbf{x}_c = 0$$

$$\mathbf{x}_{i+1} = P\, (\text{diag}(P^T\, \mathbf{1}))^{-1} \mathbf{x}_c$$
$$\mathbf{x}_{i+1} = P\, \mathbf{e}_c$$

Error Equation

- important properties of A_c:

$$\mathbf{x} = \text{diag}(\mathbf{x}_i)\,\mathbf{e}_i$$
$$A\,\text{diag}(\mathbf{x}_i)\,\mathbf{e}_i = 0$$

$$Q^T A\,\text{diag}(\mathbf{x}_i)\,Q\,\mathbf{e}_c = 0$$
$$A_c\,\mathbf{e}_c = 0$$

$$R = Q^T \qquad P = \text{diag}(\mathbf{x}_i)\,Q$$
$$A_c = R\,A\,P$$

(1) $\mathbf{1}_c^T A_c = 0 \quad \forall\, \mathbf{x}_i$
(since $\mathbf{1}_c^T R = \mathbf{1}^T$ and $\mathbf{1}^T A = 0$)

(2) $A_c\,\mathbf{1}_c = 0$ for $\mathbf{x}_\mathrm{i} = \mathbf{x}$

6. Multilevel Aggregation Algorithm

Algorithm: Multilevel Adaptive Aggregation method (V-cycle)

$\mathbf{x} = \text{AM\_V}(A, \mathbf{x}, \nu_1, \nu_2)$

begin

$\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x})$ $\quad \nu_1$ times

build Q based on $\mathbf{x}$ and A $\quad$ (Q is rebuilt every level and cycle)

$R = Q^T$ and $P = \text{diag}(\mathbf{x})\, Q$

$A_c = R\, A\, P$

$\mathbf{x}_c = \text{AM\_V}(A_c \,\text{diag}(P^T \mathbf{1})^{-1}, P^T \mathbf{1}, \nu_1, \nu_2)$ $\quad$ (coarse-level solve)

$\mathbf{x} = P\, (\text{diag}(P^T \mathbf{1}))^{-1} \mathbf{x}_c$ $\quad$ (coarse-level correction)

$\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x})$ $\quad \nu_2$ times

end

(Krieger, Horton 1994, but no good way to build Q)

7. Well-posedness: Singular M-matrices

- singular M-matrix:

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

$A \in \mathbb{R}^{n \times n}$ *is a singular M-matrix* $\Leftrightarrow$

$$\exists B \in \mathbb{R}^{n \times n},\, b_{ij} \geq 0\, \forall i,j \,:\, A = \rho(B)\, I - B$$

- our *A=I-B* is a singular M-matrix on all levels

(1) Irreducible singular M-matrices have a unique solution to the problem $A\,\mathbf{x} = 0$*, up to scaling. All components of* $\mathbf{x}$ *have strictly the same sign (i.e., scaling can be chosen s.t.* $x_i > 0\, \forall i$*). (This follows directly from the Perron-Frobenius theorem.)*

(3) Irreducible singular M-matrices have nonpositive off-diagonal elements, and strictly positive diagonal elements $(n > 1)$*.*

(4) If A has a strictly positive element in its left or right nullspace and the off-diagonal elements of A are nonpositive, then A is a singular M-matrix (see also [21]).

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Well-posedness: Singular M-matrices

THEOREM 3.1 (Singular M-matrix property of AM coarse-level operators). A_c *is an irreducible singular M-matrix on all coarse levels, and thus has a unique right kernel vector* $\mathbf{e}_c$ *with strictly positive components (up to scaling) on all levels.*

(1) $\mathbf{1}_c^T A_c = 0 \quad \forall \mathbf{x}_i$

(since $\mathbf{1}_c^T R = \mathbf{1}^T$ and $\mathbf{1}^T A = 0$)

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

THEOREM 3.2 (Fixed-point property of AM V-cycle). *Exact solution* $\mathbf{x}$ *is a fixed point of the AM V-cycle.*

(2) $A_c \mathbf{1}_c = 0$ for $\mathbf{x}_i = \mathbf{x}$

$$A_c \mathbf{e}_c = 0$$
$$\mathbf{x}_{i+1} = P \mathbf{e}_c$$

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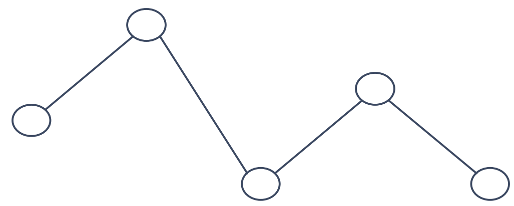
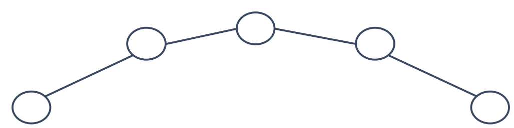
8. We Need 'Smoothed Aggregation'...

(Vanek, Mandel, and Brezina, Computing, 1996)

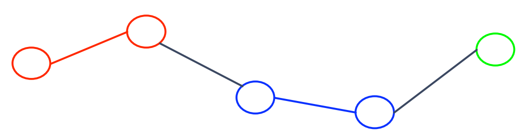


$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

after relaxation:

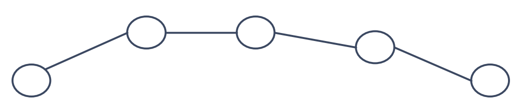


coarse grid correction with Q:



$$Q_s = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$$

coarse grid correction with Q_s:



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Smoothed Aggregation

$$A = D - (L + U)$$

- smooth the columns of *P* with weighted Jacobi:

$$P_s = (1-w)\,\mathrm{diag}(\mathbf{x}_i)\,Q + w\,D^{-1}\,(L+U)\,\mathrm{diag}(\mathbf{x}_i)\,Q$$

- take $R=P^T$, and normalize its column sums to 1

$$R_s = P_s^T\,(\mathrm{diag}(\mathbf{1}_c^T\,P_s^T))^{-1}$$

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Smoothed Aggregation

- smoothed coarse level operator:

$$A_{cs} = R_s\,(D - (L+U))\,P_s$$
$$= R_s\,D\,P_s - R_s\,(L+U)\,P_s$$

$$\mathbf{1}_c^T\,A_{cs} = 0 \quad \forall\,\mathbf{x}_i,$$
$$A_{cs}\,\mathbf{1}_c = 0 \quad \text{for } \mathbf{x}_\mathrm{i} = \mathbf{x}$$

- problem: A_{cs} is not a singular M-matrix (signs wrong)

- solution: lumping approach on S in

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

$$A_{cs} = S - G$$

$$\hat{A}_{cs} = \hat{S} - G$$

Smoothed Aggregation

$$A_{cs} = R_s\,(D - (L + U))\,P_s$$
$$= R_s\,D\,P_s - R_s\,(L + U)\,P_s$$

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

$$A_{cs} = S - G$$

$$\hat{A}_{cs} = \hat{S} - G$$

- we want to retain crucial properties

$$\mathbf{1}_c^T\,\hat{A}_{cs} = 0 \quad \forall\,\mathbf{x}_i,$$
$$\hat{A}_{cs}\,\mathbf{1}_c = 0 \quad \text{for } \mathbf{x}_\text{i} = \mathbf{x}$$

- note: *S* is symmetric!
- we can lump to diagonal in symmetric way, conserving both row and column sums

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$$\hat{s}_{ij} - g_{ij} \leq 0 \quad \forall i \neq j$$

Smoothed Aggregation

$$A_{cs} = S - G \qquad \hat{A}_{cs} = \hat{S} - G$$

$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

- we want as little lumping as possible
- only lump 'offending' elements *(i,j)*:

$$s_{ij} \neq 0,\ i \neq j \text{ and } s_{ij} - g_{ij} \geq 0$$

(we consider both off-diagonal signs and reducibility here!)

- for 'offending' elements *(i,j)*, choose $\eta \in (0,1]$ s.t.

$$\hat{s}_{ij} - g_{ij}^m = -\eta\, g_{ij}^m$$
$$\hat{s}_{ji} - g_{ji}^m = -\eta\, g_{ji}^m$$

with $g_{ij}^m = g_{ji}^m = \max(\mathrm{g_{ij}}, \mathrm{g_{ji}})$

- η=1 means lump full value of offending elements of *S* ($\hat{s}_{ij}$ = 0)

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9. Lumped Smoothed Method is Well-posed

THEOREM 4.1 (Singular M-matrix property of lumped SAM coarse-level operators). *$\hat{A}_{cs}$ is an irreducible singular M-matrix on all coarse levels, and thus has a unique right kernel vector $\mathbf{e}_c$ with strictly positive components (up to scaling) on all levels.*

THEOREM 4.2 (Fixed-point property of lumped SAM V-cycle). *Exact solution $\mathbf{x}$ is a fixed point of the SAM V-cycle (with lumping).*

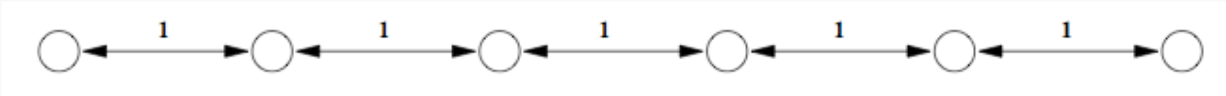
$$\mathbf{1}_c^T \hat{A}_{cs} = 0 \quad \forall \mathbf{x}_i,$$

$$\hat{A}_{cs} \mathbf{1}_c = 0 \quad \text{for } \mathbf{x}_\text{i} = \mathbf{x}$$

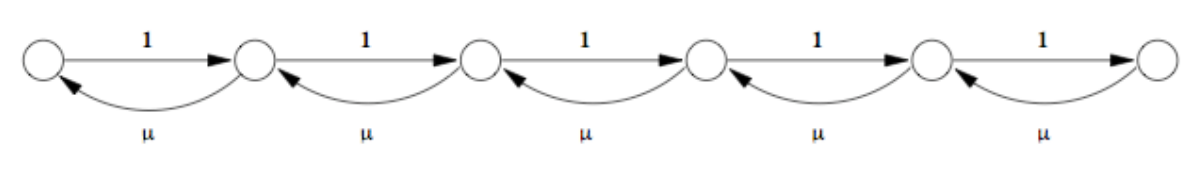
$$A = \begin{bmatrix} + & - & - & - & - \\ - & + & - & - & - \\ - & - & + & - & - \\ - & - & - & + & - \\ - & - & - & - & + \end{bmatrix}$$

10. Numerical Results: Test Problems

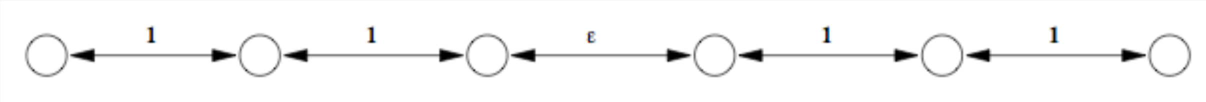
- uniform chain



- birth-death chain



- uniform chain with two weak links



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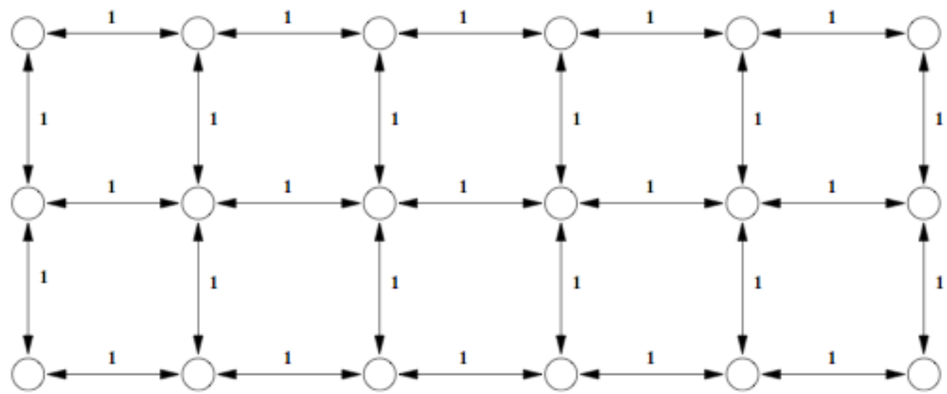
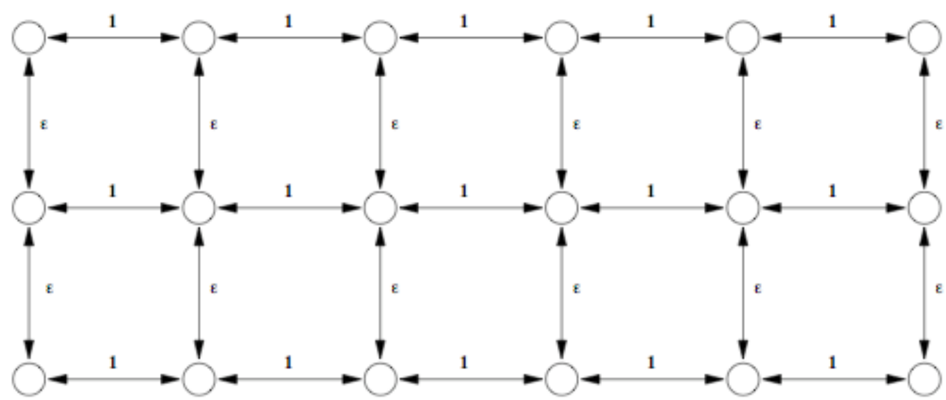
Test Problems

- uniform 2D lattice



- anisotropic 2D lattice



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Test Problems

- tandem queueing network

FIG. 5.6. *Tandem queueing network.*

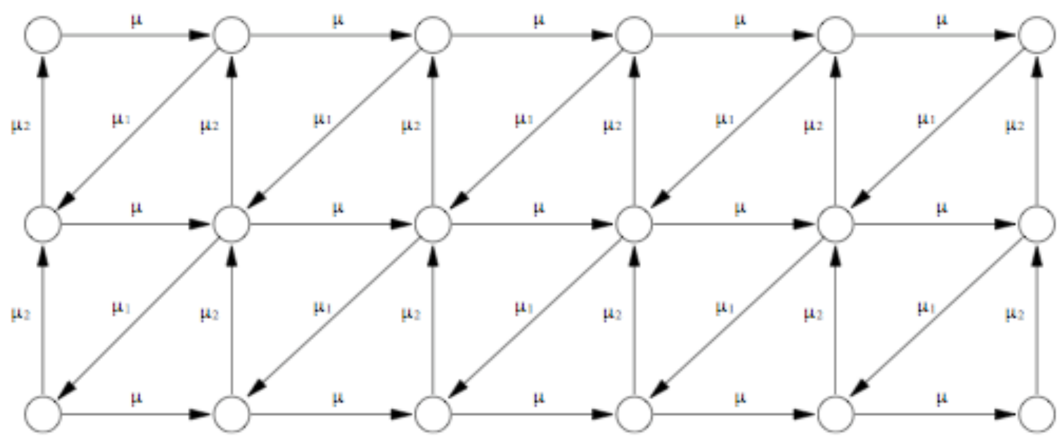

FIG. 5.7. *Graph for tandem queueing network.*

11. Numerical Results: Geometric Aggregation (size 3)

n	γ_{res}	iter	C_{op}	levels
27	0.64	34	1.32	2
81	0.88	92	1.43	3
243	0.95	> 100	1.47	4
729	0.97	> 100	1.49	5

TABLE 5.1

Uniform chain. G-AM with V-cycles and size-three aggregates. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
27	0.24	12	1.32	2
81	0.26	12	1.43	3
243	0.26	12	1.47	4
729	0.26	12	1.49	5
2187	0.26	12	1.50	6
6561	0.26	12	1.50	7

TABLE 5.2

Uniform chain. G-SAM with V-cycles and size-three aggregates. (Smoothing with lumping.)

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Numerical Results: Geometric Aggregation (size 3)

n	γ_{res}	iter	C_{op}	levels
27	0.66	37	1.32	2
81	0.88	>100	1.43	3
243	0.95	>100	1.47	4
729	0.97	>100	1.49	5

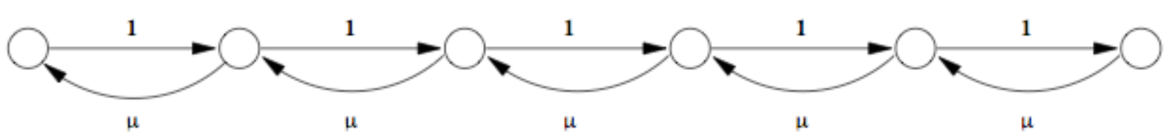
TABLE 5.3

Birth-death chain ($\mu = 0.96$). G-AM with V-cycles and size-three aggregates. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
27	0.24	12	1.32	2
81	0.26	13	1.43	3
243	0.26	13	1.47	4
729	0.26	14	1.49	5

TABLE 5.4

Birth-death chain ($\mu = 0.96$). G-SAM with V-cycles and size-three aggregates. (Smoothing with lumping.)



Numerical Results: Geometric Aggregation (size 3)

n	γ_{res}	iter	C_{op}	levels
54	0.76	56	1.43	3
162	0.92	>100	1.47	4
486	0.97	>100	1.49	5
1458	0.98	>100	1.50	6

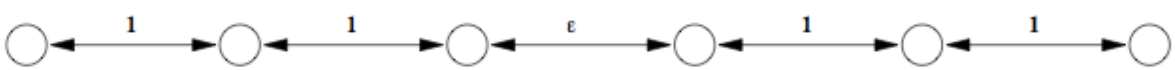
TABLE 5.5

Uniform chain with two weak links ($\epsilon = 0.001$). G-AM with V-cycles and size-three aggregates. The two weak links occur between aggregates at all levels. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
54	0.24	12	1.43	3
162	0.26	12	1.47	4
486	0.26	12	1.49	5
1458	0.26	12	1.50	6
4374	0.26	12	1.50	7

TABLE 5.6

Uniform chain with two weak links ($\epsilon = 0.001$). G-SAM with V-cycles and size-three aggregates. The two weak links occur between aggregates at all levels. (Smoothing with lumping.)



Numerical Results: Geometric Aggregation (size 3)

n	γ_{res}	iter	C_{op}	levels
27	1.00	>100	1.32	2
81	1.00	>100	1.43	3
243	0.99	>100	1.47	4
729	0.98	>100	1.49	5

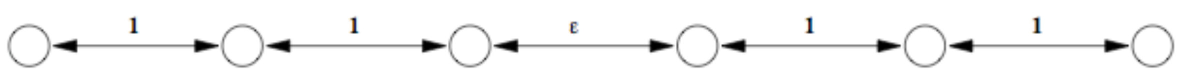
TABLE 5.7

Uniform chain with two weak links ($\epsilon = 0.001$). G-AM with V-cycles and size-three aggregates. The two weak links occur inside an aggregate on the finest level. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
27	0.99	>100	1.32	2
81	1.00	>100	1.43	3
243	1.00	>100	1.47	4
729	1.00	>100	1.49	5

TABLE 5.8

Uniform chain with two weak links ($\epsilon = 0.001$). G-SAM with V-cycles and size-three aggregates. The two weak links occur inside an aggregate on the finest level. (Smoothing with lumping.)



Numerical Results: Geometric Aggregation (3x3)

n	γ_{res}	iter	C_{op}	levels
64	0.71	44	1.11	2
100	0.85	82	1.17	3
169	0.86	98	1.15	3
400	0.88	>100	1.13	3
900	0.95	>100	1.12	4

TABLE 5.9

Uniform 2D lattice. G-AM with V-cycles and size-three aggregates. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels	γ_{res}	iter	C_{op}	levels
64	0.49	24	1.17	2	0.39	18	1.34	2
100	0.52	25	1.25	3	0.41	19	1.57	3
169	0.53	26	1.23	3	0.43	20	1.51	3
400	0.56	27	1.21	3	0.44	20	1.48	3
900	0.57	29	1.21	4	0.44	21	1.48	4
1600	0.60	29	1.23	4	0.45	21	1.51	4
2500	0.61	30	1.22	4	0.45	21	1.48	4
4900	0.62	31	1.22	4	0.45	21	1.50	4
6724	0.62	31	1.23	5	0.45	21	1.53	5

TABLE 5.10

Uniform 2D lattice. G-SAM with V-cycles (left) and W-cycles (right), using three-by-three aggregates. (Smoothing with lumping.)

Numerical Results: Geometric Aggregation (3x3)

n	γ_{res}	iter	C_{op}	levels
64	0.59	30	1.11	2
100	0.72	42	1.17	3
169	0.76	47	1.15	3
400	0.84	69	1.13	3
900	0.90	>100	1.12	4
1600	0.92	>100	1.13	4

TABLE 5.11

Anisotropic 2D lattice ($\epsilon = 1e-6$). G-AM with V-cycles and three-by-three aggregates. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
64	0.04	6	1.51	2
100	0.05	6	1.97	3
169	0.08	7	1.82	3
400	0.10	7	1.79	3
900	0.11	8	1.84	4
1600	0.10	7	1.86	4
2500	0.10	7	1.81	4
4900	0.11	8	1.84	4
6724	0.11	8	1.94	5

TABLE 5.12

Anisotropic 2D lattice ($\epsilon = 1e-6$). G-SAM with V-cycles and three-by-three aggregates. (Smoothing with lumping.)

Numerical Results: Geometric Aggregation (3x3)

n	γ_{res}	iter	C_{op}	levels
121	0.90	>100	1.20	3
256	0.91	>100	1.19	3
676	0.90	>100	1.18	3
1681	0.95	>100	1.19	4

TABLE 5.13

Tandem queueing network. G-AM with V-cycles and three-by-three aggregates. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels	γ_{res}	iter	C_{op}	levels
121	0.51	22	1.26	3	0.32	15	1.60	3
256	0.50	22	1.28	3	0.34	15	1.60	3
676	0.51	23	1.26	3	0.33	15	1.56	3
1681	0.55	28	1.27	4	0.33	15	1.61	4
3721	0.60	30	1.28	4	0.33	15	1.63	4

TABLE 5.14

Tandem queueing network. G-SAM with V-cycles (left) and W-cycles (right), using three-by-three aggregates. (Smoothing with lumping.)

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12. Numerical Results: Algebraic Aggregation

- error equation: $A\,\mathrm{diag}(\mathbf{x}_i)\,\mathbf{e}_i = 0$
- use strength of connection in $A\,\mathrm{diag}(\mathbf{x}_i)$
- define row-based strength (determine all states that strongly influence a row's state, similar to AMG)
- state that has largest value in $\mathbf{x}_i$ is seed point for new aggregate, and all unassigned states influenced by it join its aggregate
- repeat

(our Google SISC paper 2008)

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Numerical Results: Algebraic Aggregation

n	γ_{res}	iter	C_{op}	levels
27	0.69	41	1.71	3
81	0.86	81	1.85	4
243	0.96	> 100	1.96	6
729	0.97	> 100	1.99	8
2187	0.97	> 100	2.00	9

TABLE 6.1

Uniform chain. A-AM with V-cycles and distance-one aggregation. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels	γ_{res}	iter	C_{op}	levels
27	0.11	9	2.18	3	0.32	16	1.39	2
81	0.19	10	2.67	4	0.33	16	1.42	3
243	0.20	10	3.23	6	0.33	16	1.48	4
729	0.20	10	3.60	7	0.33	16	1.50	5
2187	0.21	10	3.73	8	0.34	16	1.50	6
6561	0.21	10	4.05	8	0.33	16	1.50	7

TABLE 6.2

Uniform chain. A-SAM with V-cycles using distance-one aggregation (left) and distance-two aggregation (right). (Smoothing with lumping.)

Numerical Results: effect of η

n	γ_{res}	iter	C_{op}	levels
27	0.79	56	2.24	3
81	0.93	>100	2.67	4
243	0.98	>100	3.18	6
729	0.98	>100	3.45	8
2187	0.98	>100	3.64	9

TABLE 6.3

Uniform chain. A-SAM with V-cycles using distance-one aggregation, and lumping the full value of all off-diagonal elements of $R_s\, D\, P_s$. (Smoothing with lumping.)

n	γ_{res}	iter	C_{op}	levels	γ_{res}	iter	C_{op}	levels
27	0.13	9	1.97	3	0.16	9	2.24	3
81	0.16	9	2.57	4	0.15	9	2.59	4
243	0.23	10	3.20	6	0.26	11	3.40	6
729	0.42	12	3.70	8	0.46	13	3.64	7
2187	0.61	14	3.93	9	0.62	14	3.95	9

TABLE 6.4

Uniform chain. A-SAM with V-cycles using distance-one aggregation, lumping only the off-diagonal elements of $R_s\, D\, P_s$ that cause nonnegative off-diagonal elements of A_{cs}. Lumping their full value ($\eta = 1$, left), and part of their value ($\eta = 0.75$, right).

Numerical Results: effect of η

n	γ_{res}	iter	C_{op}	levels	γ_{res}	iter	C_{op}	levels
27	0.19	10	1.96	3	0.19	10	2.24	3
81	0.15	9	2.80	4	0.20	10	2.93	4
243	0.19	10	3.15	6	0.23	11	3.16	5
729	0.20	10	3.48	7	0.24	10	3.26	6
2187	0.24	10	3.86	9	0.22	10	3.40	7

TABLE 6.5

Uniform chain. A-SAM with V-cycles using distance-one aggregation, lumping part of the value of the off-diagonal elements of $R_s\,D\,P_s$ that cause nonnegative off-diagonal elements of A_{cs}: $\eta = 0.25$, left, and, $\eta = 0.1$, right.

n	γ_{res}	iter	C_{op}	levels	γ_{res}	iter	C_{op}	levels
27	0.11	9	2.18	3	0.12	9	2.18	3
81	0.19	10	2.67	4	0.18	10	2.65	4
243	0.20	10	3.23	6	0.20	10	3.30	6
729	0.20	10	3.60	7	0.21	10	3.53	7
2187	0.21	10	3.73	8	0.19	10	3.87	8

TABLE 6.6

Uniform chain. A-SAM with V-cycles using distance-one aggregation, lumping part of the value of the off-diagonal elements of $R_s\,D\,P_s$ that cause nonnegative off-diagonal elements of A_{cs}: $\eta = 0.01$, left, and, $\eta = 1e-6$, right.

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Numerical Results: Algebraic Aggregation

n	γ_{res}	iter	C_{op}	levels
27	0.75	54	1.71	3
81	0.87	>100	1.85	4
243	0.96	>100	1.96	6
729	0.97	>100	1.99	8

TABLE 6.7

Birth-death chain ($\mu = 0.96$). A-AM with V-cycles and distance-one aggregation. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
27	0.35	17	1.32	2
81	0.35	17	1.43	3
243	0.35	18	1.47	4
729	0.35	19	1.49	5

TABLE 6.8

Birth-death chain ($\mu = 0.96$). A-SAM with V-cycles and distance-two aggregation. (Smoothing with lumping.)

Numerical Results: Algebraic Aggregation

n	γ_{res}	iter	C_{op}	levels
54	0.83	61	1.86	4
162	0.90	>100	1.91	5
486	0.96	>100	1.98	7
1458	0.97	>100	1.99	9
4374	0.98	>100	2.00	10

TABLE 6.9

Uniform chain with two weak links ($\epsilon = 0.001$). A-AM with V-cycles and distance-one aggregation. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
54	0.34	16	1.40	3
162	0.33	16	1.48	4
486	0.33	16	1.49	5
1458	0.34	16	1.50	6
4374	0.33	16	1.50	7

TABLE 6.10

Uniform chain with two weak links ($\epsilon = 0.001$). A-SAM with V-cycles and distance-two aggregation. (Smoothing with lumping.)

Numerical Results: Algebraic Aggregation

n	γ_{res}	iter	C_{op}	levels
64	0.59	30	1.60	3
100	0.80	43	1.83	4
169	0.73	53	1.79	4
400	0.88	86	1.94	6
900	0.87	>100	1.97	7

TABLE 6.11

Uniform 2D lattice. A-AM with V-cycles and distance-one aggregation. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
64	0.42	19	1.24	2
100	0.44	21	1.29	3
169	0.43	21	1.35	3
400	0.46	23	1.50	4
900	0.47	24	1.59	4
1600	0.47	24	1.68	5
2500	0.48	24	1.66	5
4900	0.47	24	1.75	5
6724	0.47	24	1.76	5

TABLE 6.12

Uniform 2D lattice. A-SAM with V-cycles and distance-two aggregation. (Smoothing with lumping.)

Numerical Results: Algebraic Aggregation

n	γ_{res}	iter	C_{op}	levels
64	0.38	18	1.65	4
100	0.43	21	1.60	4
169	0.67	33	1.90	6
400	0.69	35	1.98	7
900	0.86	67	1.99	8
1600	0.89	70	2.08	9
2500	0.92	84	2.01	10
4900	0.90	95	2.03	11
6724	0.91	>100	1.98	11

TABLE 6.13

Anisotropic 2D lattice ($\epsilon = 1e-6$). A-AM with V-cycles and distance-one aggregation. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
64	0.40	19	1.76	3
100	0.27	14	2.02	3
169	0.33	16	1.91	4
400	0.33	16	2.50	5
900	0.34	16	2.82	5
1600	0.33	16	2.94	6
2500	0.33	16	3.52	7
4900	0.33	16	3.79	7
6724	0.33	16	4.01	7

TABLE 6.14

Anisotropic 2D lattice ($\epsilon = 1e-6$). A-SAM with V-cycles and distance-two aggregation. (Smoothing with lumping.)

Numerical Results: Algebraic Aggregation

n	γ_{res}	iter	C_{op}	levels
121	0.68	40	1.89	4
256	0.88	88	2.35	7
676	0.95	>100	3.28	14
1681	0.97	>100	4.86	20
3721	0.98	>100	7.21	30

TABLE 6.15

Tandem queueing network. A-AM with V-cycles and distance-one aggregation. (No smoothing.)

n	γ_{res}	iter	C_{op}	levels
121	0.42	20	2.03	3
256	0.36	19	2.01	4
676	0.42	23	2.32	4
1681	0.42	27	2.71	5
3721	0.42	31	3.03	5

TABLE 6.16

Tandem queueing network. A-SAM with V-cycles and distance-two aggregation. (Smoothing with lumping.)

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Conclusions

- SAM: algorithm for stationary vector of slowly mixing Markov chains with near-optimal complexity
- smoothing is essential
- pretty good convergence results
- good theoretical framework (well-posedness)
- different ways of choosing R_s, P_s, lumping?
- no theory yet on optimal convergence (non-symmetric matrices)

- Questions?

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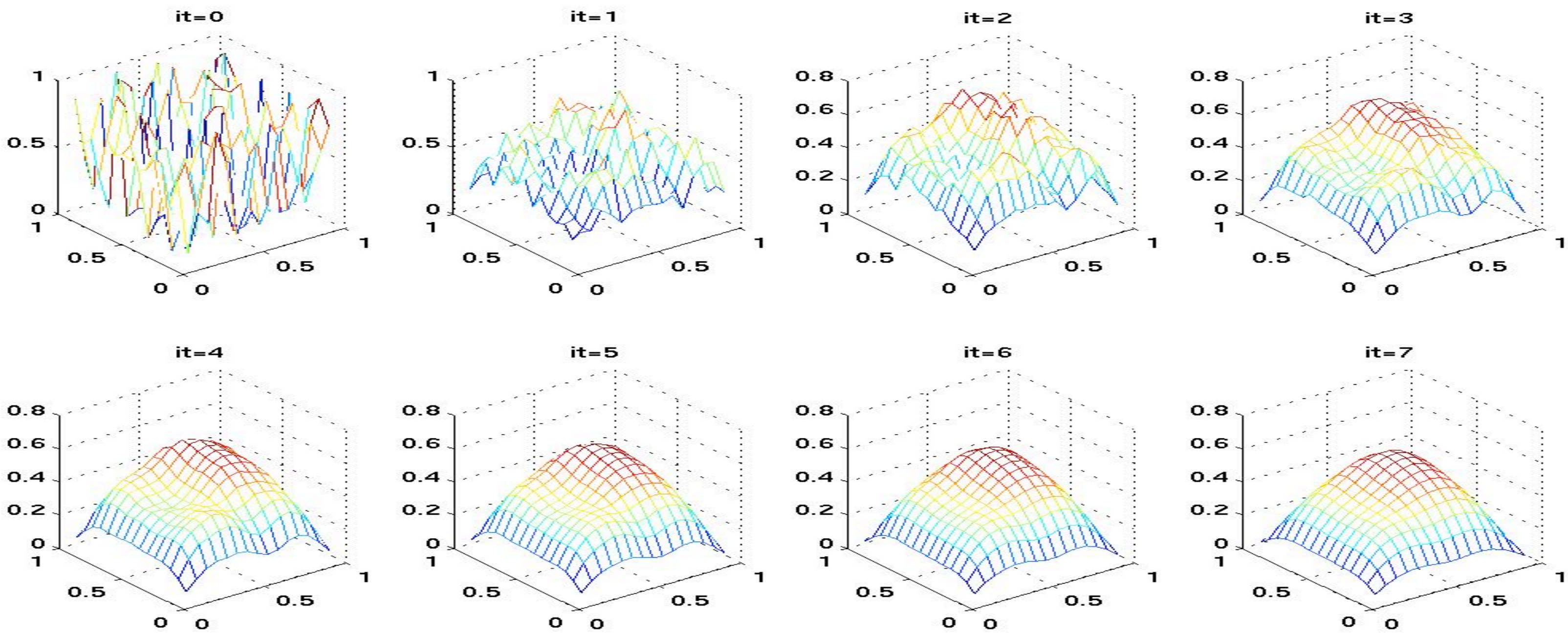
Principle of Multigrid (for PDEs)

$$-u_{xx} - u_{yy} = f(x, y) \qquad Ax = b$$



- high-frequency error is removed by relaxation (weighted Jacobi, Gauss-Seidel, ...)
- low-frequency-error needs to be removed by coarse-grid correction

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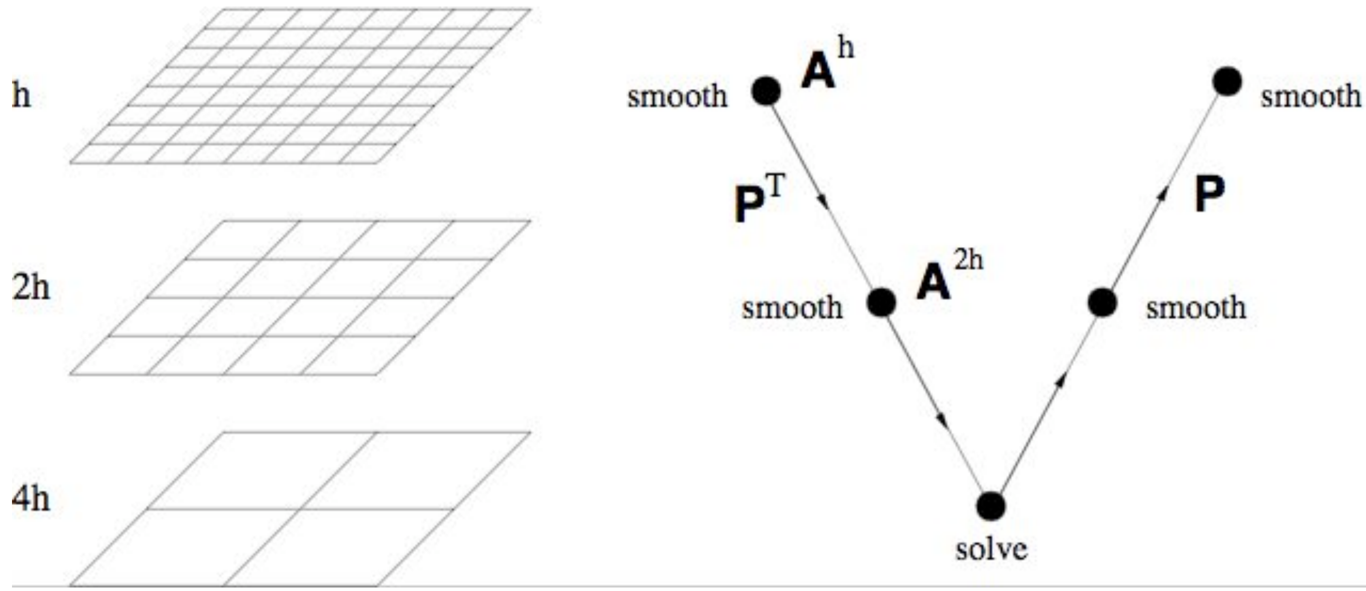
Multigrid Hierarchy: V-cycle



- multigrid V-cycle:
  - relax (=smooth) on successively coarser grids
  - transfer error using restriction ($R=P^T$) and interpolation (P)
- W=O(n)